# THE UNCERTAINTY PRINCIPLE FOR FOURIER TRANSFORM AND HEISENBERG UNIQUENESS PAIRS

DEBKUMAR GIRI

**Abstract.** In 2011, the concept of the Heisenberg uniqueness pair (HUP) was introduced as a variation of the uncertainty principle related to the Fourier transform. This fundamental idea was presented in a fundamental article by Hedenmalm and Montes-Rodríguez [6]. According to [6], let $\Gamma$ represent a smooth curve (or a union of a finite number of smooth curves) in $\mathbb{R}^2$, and let $\Lambda$ denote any subset of $\mathbb{R}^2$.

Let $\mathcal{X}(\Gamma)$ be the space of all finite complex-valued Borel measures $\mu$ in $\mathbb{R}^2$ that are supported on $\Gamma$ and are absolutely continuous with respect to arc length on $\Gamma$. For any point $(\xi, \eta) \in \mathbb{R}^2$, the Fourier transform of $\mu$ is defined as $\widehat{\mu}(\xi, \eta) = \int_\Gamma e^{\pi i(x\xi + y\eta)} d\mu(x, y)$. The pair $(\Gamma, \Lambda)$ is referred to as a Heisenberg uniqueness pair (HUP) if, for any measure $\mu \in \mathcal{X}(\Gamma)$ that satisfies $\widehat{\mu}(\xi, \eta) = 0$ for all $(\xi, \eta) \in \Lambda$, it follows that $\mu$ is identically zero.

In this talk, we will discuss the Heisenberg uniqueness pairs related to a system of four parallel lines and a hyperbola. Additionally, we will present an explicit formulation of certain pre-annihilator spaces associated with hyperbola. We will also discuss the weak-star density of the linear span of the trigonometric functions given by:

$$\left\{e_{m,n}(x, y) := e^{\pi i(mx+ny)},\ e^{<\beta>}_{m,n}(x, y) := e^{\pi i\beta(m/x+n/y)};\ m, n \in \mathbb{Z}\right\}$$

for $\beta > 0$. We aim to extend the results obtained by Hedenmalm and Montes-Rodríguez [Ann. of Math. 173 (2011)] as well as Canto-Martín, Hedenmalm, and Montes-Rodríguez [J. Eur. Math. Soc. 16 (2014)] in the plane. They have extensively studied the weak-star completeness of the *hyperbolic trigonometric system* in $L^\infty(\mathbb{R})$. This is the dual formulation of the Heisenberg uniqueness pair for (hyperbola, certain lattice-cross in the plane).

## References

[1] F. Canto-Martín, H. Hedenmalm and A. Montes-Rodríguez, *Perron-Frobenius operators and the Klein-Gordon equation,* J. Eur. Math. Soc. (JEMS) 16 (2014), no. 1, 31-66.

[2] Somnath Ghosh and Debkumar Giri, *Density of exponentials and Perron-Frobenius operators*, Adv. Math. 457 (2024), Paper No. 109932, 35pp.

[3] Debkumar Giri and Ramesh Manna, *Revisit on Heisenberg uniqueness pairs for the hyperbola*, Math. Z. 306 (2024), no. 3, Paper No. 39, 15 pp.

[4] Debkumar Giri and Rama Rawat, *Heisenberg uniqueness pairs for the hyperbola*, Bull. Lond. Math. Soc. 53 (2021), no. 1, 16-25.

[5] Debkumar Giri and R. K. Srivastava, *Heisenberg uniqueness pairs for some algebraic curves in the plane*, Adv. Math. 310 (2017), 993-1016.

[6] H. Hedenmalm and A. Montes-Rodríguez, *Heisenberg uniqueness pairs and the Klein-Gordon equation,* Ann. of Math. (2) 173 (2011), no. 3, 1507-1527.