Classnotes - MA1101

Functions of Several Variables

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Contents

Chapter 1

Differential Calculus

1.1 Regions in the plane

Let D be a subset of the plane $\mathbb{R}^2$; often called a **region**.

Let $(a, b) \in \mathbb{R}^2$ be any point.

An **ϵ-disk** around (a, b) is the set of all points $(x, y) \in \mathbb{R}^2$ whose distance from (a, b) is less than ϵ.

(a, b) is **an interior point** of D if *some* ϵ-disk around (a, b) is contained in D.

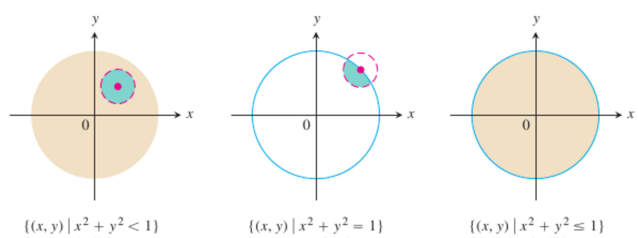
(a, b) is **a boundary point** of D if *every* ϵ-disk around (a, b) contains points from D *and* points not from D.

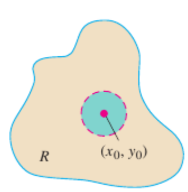
R is an **open subset** of $\mathbb{R}^2$ if all points of D are its interior points.

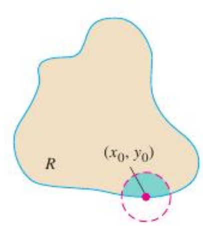
D is a **closed subset** of $\mathbb{R}^2$ if it contains all its boundary points.

$\overline{D} = D \cup$ the set of boundary points of D; It is the **closure** of D.

D is a **bounded subset** of $\mathbb{R}^2$ if D is contained in *some* ϵ-disk. (around some point)







An interior point　　　A boundary point

A subset D of $\mathbb{R}^2$ is called **connected** if any two points in the subset can be joined by a piecewise smooth curve entirely lying in D. A **domain** is an open connected subset together with some or all of its boundary points.

Let D be a region in the plane. Let $f : D \to \mathbb{R}$ be a function.
The **graph** of f is $\{(x, y, z) \in \mathbb{R}^3 : z = f(x, y),\ (x, y) \in D\}$.
The graph here is also called the **surface** $z = f(x, y)$.
The **domain** of f is D.
The **co-domain** of f is $\mathbb{R}$.
The **range** of f is $\{z \in \mathbb{R} : z = f(x, y)$ for some $(x, y) \in D\}$.

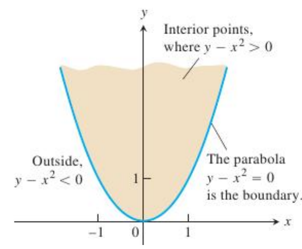
Sometimes, we do not fix the domain D but ask you to find it out.

The function $f(x, y) = \sqrt{y - x^2}$

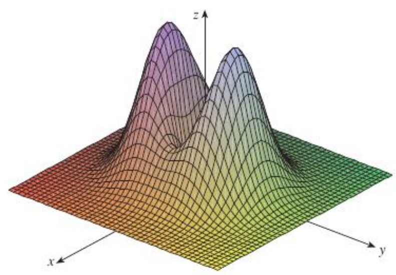
has domain $D = \{(x, y) : x^2 \leq y\}$.

Its range is the set of all non-negative reals.

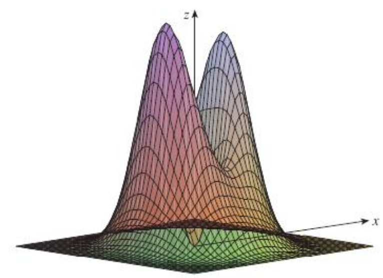
What is its graph?



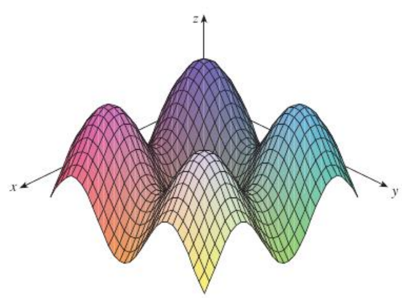
Some examples of surfaces are here:



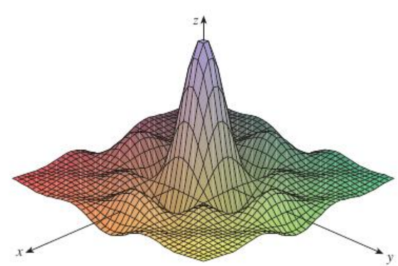
(a) $f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$



(b) $f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$



(c) $f(x, y) = \sin x + \sin y$

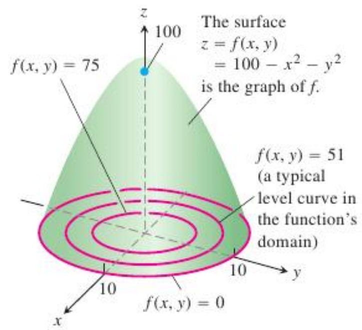


(d) $f(x, y) = \dfrac{\sin x \sin y}{xy}$

1.2 Level curves and surfaces

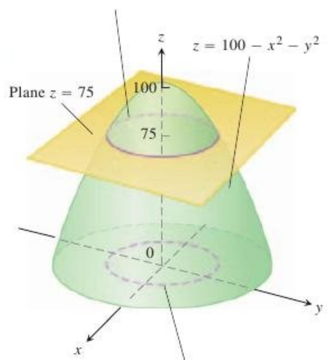
Let $f(x, y)$ be a function of two variables. That is, $f : D \to \mathbb{R}$, where D is a domain in $\mathbb{R}^2$. The **level curves** of f are the curves $f(x, y) = c$ in the xy-plane, for some constant c in the range of f.

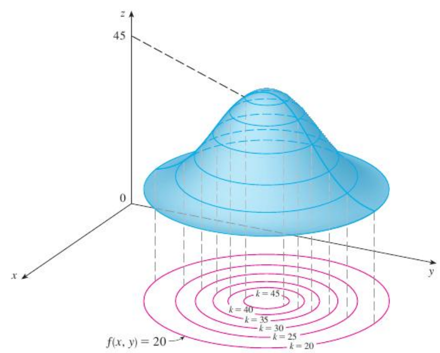
Consider the function $f(x, y) = 100 - x^2 - y^2$.
Its domain is $\mathbb{R}^2$. Its range is the interval $(-\infty, 100]$.
The level curve $f(x, y) = 0$ is $\{(x, y) : x^2 + y^2 = 100\}$.
The level curve $f(x, y) = 51$ is $\{(x, y) : x^2 + y^2 = 49\}$.



The union of all level curves, translated in z-direction suitably, is the surface $z = f(x, y)$; it is also the graph of f.

The **contour curve** is the curve $f(x, y) = c$ in the plane $z = c$.





The level curve is the projection of the contour curve on the xy-plane.
Similarly, for a function $f(x, y, z)$ of three variables, the **level surfaces** are the surfaces $f(x, y, z) = c$ for values c in the range of f.

Let $f : D \to \mathbb{R}$ be a function. Let $(a, b) \in \overline{D}$.

The **limit** of $f(x, y)$ as (x, y) approaches (a, b) is L iff given any $\epsilon > 0$, we can choose a corresponding $\delta > 0$ such that for all $(x, y) \in D$ with $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$, we have $|f(x, y) - L| < \epsilon$.

In this case, we write $\displaystyle\lim_{(x,y)\to(a,b)} f(x, y) = L$.

We also say that L is the limit of f at (a, b).

If for no real number L, the above happens, then **limit of f at (a, b) does not exist.**

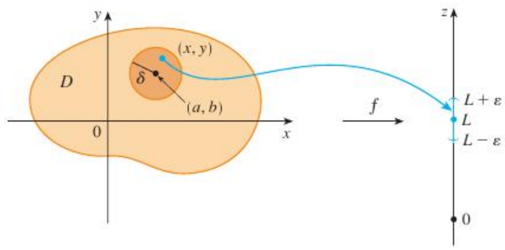
It is often difficult to show that limit of a function does not exist at a point. We will come back to this question soon. When limit exists, we write it in many alternative ways:

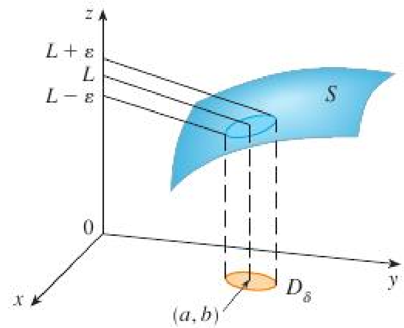
$$\text{The limit of } f(x, y) \text{ as } (x, y) \text{ approaches } (a, b) \text{ is } L.$$

$$f(x, y) \to L \text{ as } (x, y) \to (a, b).$$

$$\lim_{(x,y)\to(a,b)} f(x, y) = L.$$

$$\lim_{\substack{x\to a\\ y\to b}} f(x, y) = L.$$





The intuitive understanding of the notion of limit is as follows:

The distance between $f(x, y)$ and L can be made arbitrarily small by making the distance between (x, y) and (a, b) sufficiently small but not necessarily zero.

Example 1.1. Determine if $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2}$ exists.

Observe that the domain D of f is $\mathbb{R}^2 \setminus \{(0,0)\}$. And $f(0, y) = 0$ for $y \neq 0$; $f(x, 0) = 0$ for $x \neq 0$. We guess that if the limit exists, it would be 0. To see that it is the case, we start with any $\epsilon > 0$. We want to choose a $\delta > 0$ such that the following sentence becmes true:

$$\text{If} \quad 0 < \sqrt{x^2+y^2} < \delta, \quad \text{then} \quad \left|\frac{4xy^2}{x^2+y^2}\right| < \epsilon.$$

Since $|y^2| = y^2 \le x^2 + y^2$ and $|x^2| = x^2 \le x^2 + y^2$, we have

$$\left|\frac{4xy^2}{x^2+y^2}\right| \le 4|x| \le 4\sqrt{x^2+y^2}.$$

So, we choose $\delta = \epsilon/4$. Assume that $0 < \sqrt{x^2+y^2} < \delta$. Then

$$\left|\frac{4xy^2}{x^2+y^2} - 0\right| \le 4\sqrt{x^2+y^2} < 4\delta = \epsilon.$$

Hence

$$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = 0.$$

Example 1.2. Consider $f(x, y) = \sqrt{1 - x^2 - y^2}$ when $D = \{(x, y) : x^2 + y^2 \le 1\}$.
We guess that limit $f(x, y)$ is 1 as $(x, y) \to (0, 0)$.

To show that the guess is right, let $\epsilon > 0$. Observe that $0 \le f(x, y) \le 1$ on D.
So, if $\epsilon \ge 1$, then $|f(x, y) - 1|$ varies between 0 and 1.
That is, $|f(x, y) - 1| < \epsilon$, for (x, y) near $(0, 0)$.
Next, assume that $0 < \epsilon < 1$. Choose $\delta = \sqrt{1 - (1-\epsilon)^2}$. Let $|(x, y) - (0, 0)| < \delta$. Then

$$x^2 + y^2 < 1 - (1-\epsilon)^2 \Rightarrow 1 - x^2 - y^2 > (1-\epsilon)^2 \Rightarrow f(x, y) > 1 - \epsilon.$$

That is, $|f(x, y) - 1| = 1 - f(x, y) < \epsilon$. Therefore, $f(x, y) \to 1$ as $(x, y) \to (0, 0)$.

For a function of one variable, there are only two directions for approaching a point; from left and from right. Whereas for a function of two variables, there are infinitely many directions, and infinite number of paths on which one can approach a point. The limit refers only to the distance between (x, y) and (a, b). It does not refer to any specific direction of approach to (a, b). If the limit exists, then $f(x, y)$ must approach the same limit no matter how (x, y) approaches (a, b). Thus, if we can find two different paths of approach along which the function $f(x, y)$ has different limits, then it follows that limit of $f(x, y)$ as (x, y) approaches (a, b) does not exist.

Theorem 1.1. *Suppose that $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2. If $L_1 \neq L_2$, then the limit of $f(x, y)$ as $(x, y) \to (a, b)$ does not exist.*

Example 1.3. Consider $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ for $(x, y) \neq (0, 0)$. What is its limit at $(0, 0)$?

When $y = 0$, limit of $f(x, y)$ as $x \to 0$ is $\lim\limits_{x \to 0} \dfrac{x^2}{x^2} = \lim\limits_{x \to 0}(1) = 1$.

That is, $f(x, y) \to 1$ as $(x, y) \to (0, 0)$ along the x-axis.

When $x = 0$, limit of $f(x, y)$ as $y \to 0$ is $\lim\limits_{y \to 0} \dfrac{-y^2}{y^2} = -1$.

That is, $f(x, y) \to -1$ as $(x, y) \to (0, 0)$ along the y-axis.

Hence $\lim\limits_{(x,y) \to (0,0)} f(x, y)$ does not exist.

Example 1.4. Consider $f(x, y) = \dfrac{xy}{x^2 + y^2}$ for $(x, y) \neq (0, 0)$. What is its limit at $(0, 0)$?

Along the x-axis, $y = 0$; then limit of $f(x, y)$ as $(x, y) \to (0, 0)$ is 0.

Along the y-axis, $x = 0$; then limit of $f(x, y)$ as $(x, y) \to (0, 0)$ is 0.

Does it say that limit of $f(x, y)$ as $(x, y) \to (0, 0)$ is 0?

Along the line $y = x$, limit of $f(x, y)$ as $(x, y) \to 0$ is $\lim\limits_{x \to 0} \dfrac{x^2}{x^2 + x^2} = 1/2$.

Hence $\lim\limits_{(x,y) \to (0,0)} f(x, y)$ does not exist.

Example 1.5. Consider $f(x, y) = \dfrac{xy^2}{x^2 + y^4}$ for $(x, y) \neq (0, 0)$. What is its limit at $(0, 0)$?

If $y = mx$, for some $m \in \mathbb{R}$, then $f(x, y) = \dfrac{m^2 x}{1 + m^4 x^2}$. So, $\lim\limits_{(x,y) \to (0,0)}$ along all straight lines is 0.

If $x = y^2,\ y \neq 0$, then $f(x, y) = \dfrac{y^4}{y^4 + y^4} = 1/2$. As $(x, y) \to (0, 0)$ along $x = y^2$, $f(x, y) \to 1/2$.

Hence $\lim\limits_{(x,y) \to (0,0)} f(x, y)$ does not exist.

A question: are the following same?

$$\lim_{(x,y) \to (a,b)} f(x, y), \quad \lim_{x \to a} \lim_{y \to b} f(x, y), \quad \lim_{y \to b} \lim_{x \to a} f(x, y)$$

Example 1.6. Let $f(x,y) = \dfrac{(y-x)(1+x)}{(y+x)(1+y)}$ for $x+y \neq 0, -1 < x,\ y < 1$. Then

$$\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{y\to 0} \frac{y}{y(1+y)} = 1.$$

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0} \frac{-x(1+x)}{x} = -1.$$

$$\text{Along } y = mx, \quad \lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x(m-1)(1+x)}{x(1+m)(1+mx)} = \frac{m-1}{m+1}.$$

For different values of m, we get the last limit value different. So, limit of $f(x,y)$ as $(x,y) \to (0,0)$ does not exist. But the two iterated limits exist and they are not equal.

Example 1.7. Let $f(x,y) = x\sin\dfrac{1}{y} + y\sin\dfrac{1}{x}$ for $x \neq 0,\ y \neq 0$. Then

$$\lim_{x\to 0} y\sin\frac{1}{x} \ \text{ and } \ \lim_{y\to 0} x\sin\frac{1}{y} \ \text{ do not exist.}$$

So, neither $\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y)$ exists not $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y)$ exists.

However, $|f(x,y) - 0| \le |x| + |y| = \sqrt{x^2} + \sqrt{y^2} \le 2\sqrt{x^2+y^2} = 2|(x,y)|$. That is,

If $|(x,y) - (0,0)| < \epsilon/2$, then $|f(x,y) - 0| < \epsilon$. Therefore,

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

That is, the two iterated limits do not exist, but the limit exists.

Hence existence of the limit of $f(x,y)$ as $(x,y) \to (a,b)$ and the two iterated limits have no connection.

The usual operations of addition, multiplication etc have the expected effects as the following theorem shows. Its proof is analogous to the single variable limits.

Theorem 1.2. *Let* $L, M, c \in \mathbb{R}$; $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$; $\lim\limits_{(x,y)\to(a,b)} g(x,y) = M$. *Then*

1. Constant Multiple*:* $\lim\limits_{(x,y)\to(a,b)} cf(x,y) = cL.$
2. Sum*:* $\lim\limits_{(x,y)\to(a,b)} (f(x,y) + g(x,y)) = L + M.$
3. Product*:* $\lim\limits_{(x,y)\to(a,b)} (f(x,y)\, g(x,y)) = LM.$
4. Quotient*: If* $M \neq 0$ *and* $g(x,y) \neq 0$ *in an open disk around the point* (a,b), *then* $\lim\limits_{(x,y)\to(a,b)} (f(x,y)/g(x,y)) = L/M$
5. Power*: If* $r \in \mathbb{R},\ L^r \in \mathbb{R}$ *and* $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$, *then* $\lim\limits_{(x,y)\to(a,b)} (f(x,y))^r = L^r.$

1.3 Continuity

Let $f(x, y)$ be a real valued function on a domain $D \subseteq \mathbb{R}^2$. We say that $f(x, y)$ is **continuous** at a point $(a, b) \in D$ if

1. $f(a, b)$ is well defined.
2. $\lim\limits_{(x,y)\to(a,b)} f(x, y)$ exists.
3. $\lim\limits_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

The function $f(x, y)$ is said to be **continuous on** D if $f(x, y)$ is continuous at all points in D.

Therefore, constant multiples, sum, difference, product, quotient, and rational powers of continuous functions are continuous whenever they are well defined.

Polynomials in two variables are continuous functions.

Rational functions, i.e., ratios of polynomials are continuous functions provided they are well defined.

Example 1.8. $f(x, y) = \begin{cases} \frac{3x^2y}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$ is continuous on $\mathbb{R}^2$.

At any point other than the origin, $f(x, y)$ is a rational function; therefore, it is continuous. To see that $f(x, y)$ is continuous at the origin, let $\epsilon > 0$ be given. Take $\delta = \epsilon/3$. Assume that $\sqrt{x^2 + y^2} < \delta = \epsilon/3$. Then

$$\Big|\frac{3x^2y}{x^2 + y^2} - f(0, 0)\Big| \leq \Big|\frac{3(x^2 + y^2)y}{x^2 + y^2}\Big| \leq 3|y| \leq 3\sqrt{x^2 + y^2} < \epsilon.$$

Example 1.9. $f(x, y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$ is continuous on $\mathbb{R}^2$. Why?

At all nonzero points, it is continuous, being a rational function. For the point $(0, 0)$, let $\epsilon > 0$ be given. Choose $\delta = \sqrt{\epsilon}$. Notice that $xy \leq x^2 + y^2$ and $x^2 - y^2 \leq x^2 + y^2$.
For all (x, y) with $\sqrt{x^2 + y^2} < \delta$, we have

$$|f(x, y) - 0| \leq \frac{(x^2 + y^2)(x^2 + y^2)}{x^2 + y^2} < \delta^2 = \epsilon.$$

Hence $\lim\limits_{(x,y)\to(0,0)} f(x, y) = 0 = f(0, 0)$.

Example 1.10. $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ is continuous on $D = \mathbb{R}^2 \setminus \{(0, 0)\}$.

$f(x, y)$ is not continuous at $(0, 0)$ since it is not defined at $(0, 0)$.

Also, $f(x, y)$ is not continuous at $(0, 0)$ since $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist. See Example 1.3.

Therefore, the function $g(x, y)$ defined on $\mathbb{R}^2$ by the following is not continuous at $(0, 0)$.

$$g(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

As in the single variable case, *composition of continuous functions is continuous*:

Let $f : D \to \mathbb{R}$ be continuous at (a, b) with $f(a, b) = c$. Let $g : I \to \mathbb{R}$ be continuous at $c \in I$ for some interval I in $\mathbb{R}$. Then $g(f(x, y))$ from D to $\mathbb{R}$ is continuous at (a, b). Proof of this fact is left to you as an exercise.

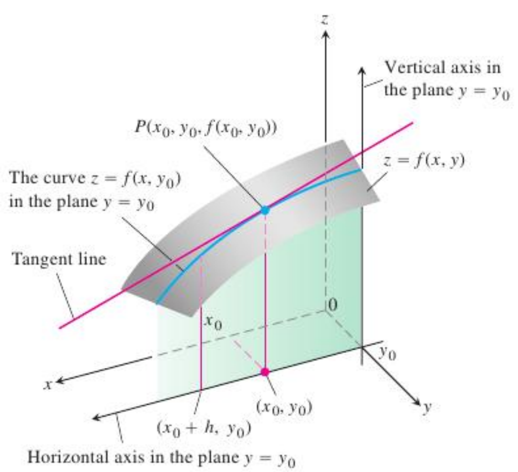
For example, e^{x-y} is continuous at all points in the plane.
$\cos \dfrac{xy}{1+x^2}$ and $\ln(1 + x^2 + y^2)$ are continuous on $\mathbb{R}^2$.
At which points is $\tan^{-1}(y/x)$ continuous?

The function y/x is continuous everywhere except when $x = 0$.
The function $\tan^{-1}$ is continuous everywhere on $\mathbb{R}$.
So, $\tan^{-1}(y/x)$ is continuous everywhere except at $x = 0$.

The function $\frac{1}{x^2+y^2+z^2-1}$ is continuous everywhere except on the sphere $x^2 + y^2 + z^2 = 1$, where it is not defined.

1.4 Partial Derivatives

Let $f(x, y)$ be a real valued function defined on a domain $D \subseteq \mathbb{R}^2$. Let $(a, b) \in D$.



If C is the curve of intersection of the surface $z = f(x, y)$ with the plane $y = b$, then the slope of the tangent line to C at $(a, b, f(a, b))$ is the partial derivative of $f(x, y)$ with respect to x at (a, b). In the figure take $x_0 = a, y_0 = b$. A formal definition of the partial derivative follows.

The **partial derivative of** $f(x, y)$ **with respect to** x **at the point** (a, b) is

$$f_x(a,b) = \frac{\partial f}{\partial x}\Big|_{(a,b)} = \frac{df(x,b)}{dx}\Big|_{x=a} = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h},$$

provided this limit exists. Notice that $f(x, b)$ must be continuous at $x = a$.

The **partial derivative of $f(x, y)$ with respect to y at the point** (a, b) is

$$f_y(a,b) = \frac{\partial f}{\partial y}\Big|_{(a,b)} = \frac{df(a,y)}{dy}\Big|_{y=b} = \lim_{k\to 0} \frac{f(a,b+k) - f(a,b)}{k},$$

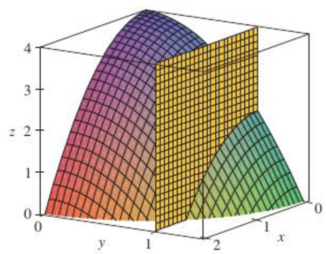
provided this limit exists. Again, $f(a, y)$ must be continuous at $y = b$.

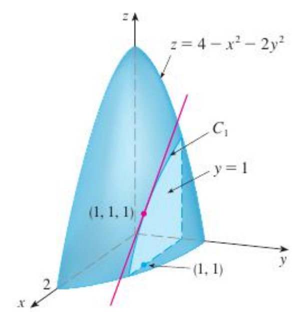
Example 1.11. Find $f_x(1, 1)$ where $f(x, y) = 4 - x^2 - 2y^2$.

$$f_x(1,1) = \lim_{h\to 0} \frac{(4-(1+h)^2-2)-(4-1-2)}{h} = \lim_{h\to 0} \frac{-2h-h^2}{h} = -2.$$

That is, treat y as a constant and differentiate with respect to x.

$$f_x(1,1) = f_x(x,y)\big|_{(1,1)} = -2x\big|_{(1,1)} = -2.$$





The vertical plane $y = 1$ crosses the paraboloid in the curve $C_1 : z = 2 - x^2, y = 1$. The slope of the tangent line to this parabola at the point $(1, 1, 1)$ (which corresponds to $(x, y) = (1, 1)$) is $f_x(1, 1) = -2$.

Example 1.12. Find f_x and f_y, where $f(x, y) = y \sin(xy)$.

Treating y as a constant and differentiating with respect to x, we get f_x. Similarly, f_y.

$$f_x(x,y) = y\cos(xy)\,y, \quad f_y(x,y) = yx\cos(xy) + \sin(xy).$$

Example 1.13. Find $\partial z/\partial x$ and $\partial z/\partial y$ where $z = f(x, y)$ is defined by $x^3 + y^3 + z^3 - 6xyz = 1$.

Differentiate $x^3 + y^3 + z^3 - 6xyz - 1 = 0$ with respect to x treating y as a constant:

$$3x^2 + 0 + 3z^2\frac{\partial z}{\partial x} - 6y\left(z + x\frac{\partial z}{\partial x}\right) - 0 = 0.$$

Solving this for $\partial z/\partial x$, we have

$$\frac{\partial z}{\partial x}(3z^2 - 6xy) + (3x^2 - 6yz) = 0, \quad \text{that is,}$$

$$\frac{\partial z}{\partial x} = -\frac{x^2 - 2yz}{z^2 - 2xy}.$$

Similarly,

$$\frac{\partial z}{\partial y} = -\frac{y^2 - 2xz}{z^2 - 2xy}.$$

Example 1.14. The plane $x = 1$ intersects the surface $z = x^2 + y^2$ in a parabola. Find the slope of the tangent to the parabola at the point $(1, 2, 5)$.

The asked slope is $\partial z/\partial y$ at $(1, 2)$. It is

$$\frac{\partial(x^2 + y^2)}{\partial y}(1, 2) = (2y)(1, 2) = 4.$$

Alternatively, the parabola is $z = x^2 + y^2, x = 1$ OR, $z = 1 + y^2$. So, the slope at $(1, 2, 5)$ is

$$\frac{dz}{dy}\Big|_{y=2} = \frac{d(1 + y^2)}{dy}\Big|_{y=2} = (2y)|_{y=2} = 4.$$

For a function $f(x, y)$, partial derivatives of second order are:

$$\begin{aligned} f_{xx} &= (f_x)_x = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2}. \\ f_{xy} &= (f_x)_y = \frac{\partial f_x}{\partial y} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x}. \\ f_{yx} &= (f_y)_x = \frac{\partial f_y}{\partial x} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y}. \\ f_{yy} &= (f_y)_y = \frac{\partial}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y^2}. \end{aligned}$$

Similarly, higher order partial derivatives are defined. For example,

$$f_{xxy} = \frac{\partial}{\partial y}\frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial^3 f}{\partial y \partial x \partial x}.$$

Observe that $f_x(a, b)$ is not the same as $\lim\limits_{(x,y)\to(a,b)} f_x(x, y)$. To see this, let

$$f(x, y) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \le 0. \end{cases}$$

Then $f_x(x, y) = 0$ for all $x > 0$. Also, $f_x(x, y) = 0$ for all $x < 0$. Now, $\lim\limits_{(x,y)\to(0,0)} f_x(x, y) = 0$. But $f_x(0, 0)$ does not exist. Reason?

$$f_x(0, 0) = \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h\to 0} \frac{1 \text{ or } 0}{h} \quad \text{does not exist}$$

On the other hand, $f_x(a, b)$ can exist though $\lim\limits_{(x,y)\to(a,b)} f_x$ does not.
However, if $f_x(x, y)$ is continuous at (a, b), then

$$f_x(a, b) = \lim_{(x,y)\to(a,b)} f_x(x, y).$$

Similarly, f_{xy} need not be equal to f_{yx}. See the following example.

Example 1.15. Consider $f(x,y) = \dfrac{xy(x^2-y^2)}{x^2+y^2}$ for $(x,y) \neq (0,0)$, and $f(0,0) = 0$.

$$\begin{aligned}
f(x,0) &= f(0,y) = f(0,0) = 0. \\
f_x(x,0) &= f_y(0,y) = f_{xx}(0,0) = f_{yy}(0,0) = 0. \\
f_x(0,y) &= \lim_{h\to 0} \frac{f(h,y)-f(0,y)}{h} = -y,\ f_y(x,0) = \lim_{k\to 0}\frac{f(x,k)-f(x,0)}{k} = x. \\
f_{xy}(0,0) &= \lim_{k\to 0}\frac{f_x(0,k)-f_x(0,0)}{k} = \lim_{k\to 0}\frac{-k-0}{k} = -1. \\
f_{yx}(0,0) &= \lim_{h\to 0}\frac{f_y(h,0)-f_y(0,0)}{h} = \lim_{h\to 0}\frac{h-0}{h} = 1.
\end{aligned}$$

That is, $f_{xy} \neq f_{yx}$.

But continuity of both of f_{xy} and f_{yx} implies their equality.

Theorem 1.3. (Clairaut) *Let $D \subseteq \mathbb{R}^2$ be a domain. Let $f : D \to \mathbb{R}$. Suppose that f_{xy} and f_{yx} are continuous on D. Then $f_{xy} = f_{yx}$.*

Proof: Let $(a,b) \in D$. Let $h \neq 0$. Write $g(x) = f(x, b+h) - f(x,b)$. Then

$$\Delta f := g(a+h) - g(a) = [f(a+h, b+h) - f(a+h, b)] - [f(a, b+h) - f(a,b)].$$

By MVT, we have c between a and $a+h$ such that

$$\Delta f = g'(c)h = h[f_x(c, b+h) - f_x(c,b)].$$

Again, by MVT (on f_x with the second variable), we have d between b and $b+h$ such that

$$\Delta f = h \cdot h \cdot f_{xy}(c,d) = h^2 f_{xy}(c,d).$$

Due to continuity of f_{xy}, we have

$$\lim_{h\to 0}\frac{\Delta f}{h^2} = \lim_{(c,d)\to(a,b)} f_{xy}(c,d) = f_{xy}(a,b).$$

Write

$$\Delta f = [f(a+h, b+h) - f(a, b+h)] - [f(a+h, b) - f(a,b)]$$

and apply MVT twice as above to get $f_{yx}(a,b) = \lim_{h\to 0}\frac{\Delta f}{h^2}$. But the two limits with $(\Delta f)/h^2$ are equal. So, $f_{xy}(a,b) = f_{yx}(a,b)$. □

In one variable, $f'(t)$ exists at $t = a$ implies that $f(t)$ is continuous at $t = a$. We have seen similarly that existence of $f_x(a,b)$ and $f_y(a,b)$ guarantees continuity of $f(x,b)$ and of $f(a,y)$ at (a,b). But for $f(x,y)$, even both $f_x(x,y)$ and $f_y(x,y)$ exist at (a,b), the function $f(x,y)$ need not be continuous at (a,b). See the following example.

Example 1.16. Let $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

Here, $f(x,0) = 0 = f(0,y)$. So, $f_x(0,0) = 0 = f_y(0,0)$. And limit of $f(x,y)$ as $(x,y) \to (0,0)$ does not exist. Hence $f(x,y)$ is not continuous at $(0,0)$.

Further, we find that $f_{xx}(x,0) = 0 = f_{yy}(0,y)$. What about $f_{xy}(0,0)$?

$$f_x(0,y) = \lim_{h\to 0} \frac{f(h,y) - f(0,y)}{h} = \lim_{h\to 0} \frac{y}{h^2 + y^2} = \frac{1}{y}.$$

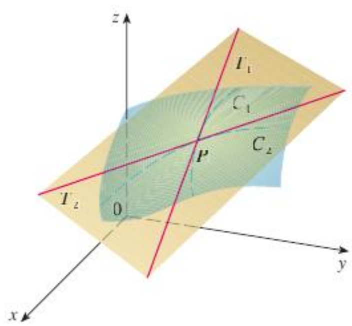
$f_x(0,y)$ is not continuous at $y = 0$.

Notice that the second partial derivatives $f_{xy}(0,0)$ and $f_{yx}(0,0)$ do not exist.

1.5 Increment Theorem

In order to see the connection between continuity of a function and the partial derivatives, the associated geometry may help.

Let S be the surface $z = f(x,y)$, where f_x, f_y are continuous on the domain D of f. Let $(a,b) \in D$. Let C_1 and C_2 be the curves of intersection of the planes $x = a$ and of $y = b$ with S.



Let T_1 and T_2 be tangent lines to the curves C_1 and C_2 at the point $P(a,b,f(a,b))$. The **tangent plane** to the surface S at P is the plane containing T_1 and T_2.

The tangent plane to S at P consists of all possible tangent lines at P to the curves C that lie on S and pass through P. This plane approximates S at P most closely.

Write the z-coordinate of P as c. Then $P = (a,b,c)$. Equation of any plane passing through P is $z - c = A(x-a) + B(y-b)$. When $y = b$, the tangent plane represents the tangent to the intersected curve at P. Thus, $A = f_x(a,b)$, the slope of the tangent line. Similarly, $B = f_y(a,b)$. Hence equation of the tangent plane to the surface $z = f(x,y)$ at the point $P(a,b,c)$ on S is

$$z - c = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

provided that f_x, f_y are continuous at (a,b).

Example 1.17. Find the equation of the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at $(1,1,3)$.

Here, $z_x = 4x, z_y = 2y$. So, $z_x(1,1) = 4, z_y(1,1) = 2$. Then the equation of the tangent plane is $z - 3 = 4(x-1) + 2(y-1)$. It simplifies to $z = 4x + 2y - 3$.

The tangent plane gives a linear approximation to the surface at that point. Why?
Write the equation as $f(x,y) - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$. Then

$$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

This formula holds true for all points $(x, y, f(x,y))$ on the tangent plane at $(a, b, f(a,b))$. For approximating $f(x,y)$ for (x,y) close to (a,b), we may take

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

The RHS is called the *standard linear approximation* of $f(x,y,z)$.
Writing in the increment form,

$$f(a+h, b+k) \approx f(a,b) + f_x(a,b)h + f_y(a,b)k.$$

This gives rise to the **total increment** $f(a+h, b+k) - f(a,b)$.

The total increment can be written in a more suggestive form. Towards this, we proceed as follows:

$$\Delta f = f(a+h, b+k) - f(a+h, b) + f(a+h) - f(a,b).$$

By MVT, there exist $c \in [a, a+h]$ and $d \in [b, b+k]$ such that

$$\begin{aligned} f(a+h,b) - f(a,b) &= h[f_x(c,b) - f_x(a,b)] + hf_x(a,b) \\ f(a+h,b+k) - f(a+h,b) &= k[f_y(a+h,d) - f_y(a,b)] + kf_y(a,b) \end{aligned}$$

Write $\epsilon_1 = f_x(d,b) - f_x(a,b)$ and $\epsilon_2 = f_y(a+h,c) - f_y(a,b)$. When both $h \to 0, k \to 0$, we see that $c \to a$ and $d \to b$. Since f_x and f_y are assumed to be continuous, we have $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$. Then the total increment can be written as

$$\Delta f = f(a+h, b+k) - f(a,b) = hf_x(a,b) + kf_y(a,b) + \epsilon_1 h + \epsilon_2 k,$$

where $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as both $h \to 0, k \to 0$.

We also write the increments h, k in x, y as $\Delta x,\ \Delta y$ respectively.

From the above rewriting of Δf it is also clear that $f(x,y)$ is a continuous function. Let us note down what we have proved.

Theorem 1.4. (Increment Theorem) *Let D be a domain in $\mathbb{R}^2$. Let $f : D \to R$ be such that both f_x and f_y are continuous on D. Then $f(x,y)$ is continuous on D and the total increment $\Delta f = f(a+\Delta x, b+\Delta y)$ at $(a,b) \in D$ can be written as*

$$\Delta f = f_x(a,b)\Delta x + f_y \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

where $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as both $\Delta x \to 0$ and $\Delta y \to 0$.

Recall that for a function g of one variable, its differential is defined as $dg = g'(t)dt$.

Let $f(x,y)$ be a given function. The **differential** of f, also called the **total differential**, is

$$df = f_x(x,y)dx + f_y(x,y)dy.$$

Here, $dx = \Delta x$ and $dy = \Delta y$ are the increments in x and y, respectively. The equation above represents a linear approximation to the *total increment* Δf.

Example 1.18. The dimensions of a rectangular box are measured to be 75cm, 60cm, and 40 cm, and each measurement is correct to within 0.2cm. Use differentials to estimate the largest possible error when the volume of the box is calculated from these measurements.

The volume of the box is $V = xyz$. So,

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz.$$

Given that $|\Delta x|, |\Delta y|, |\Delta z| \leq 0.2$cm, the largest error in cubic cm is

$$|\Delta V| \approx |dV| = 60 \times 40 \times 0.2 + 40 \times 75 \times 0.2 + 75 \times 60 \times 0.2 = 1980.$$

Notice that the relative error is $1980/(75 \times 60 \times 40)$ which is about 1%.

Remark: Let D be a domain in $\mathbb{R}^2$. A function $f : D \to \mathbb{R}$ is called *differentiable* at a point $(a, b) \in D$ if the total increment $\Delta z = f(a+\Delta x, b+\Delta y) - f(a, b)$ in f with respect to increments $\Delta x, \Delta y$ in x, y, can be written as

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as both $\Delta x \to 0$ and $\Delta y \to 0$.

The following statements state the connection between differentiability, continuity and the partial derivatives.

- Let D be a domain in $\mathbb{R}^2$. Let $f : D \to R$ be such that both f_x and f_y exist on D and at least one of them is continuous at $(a, b) \in D$. Then f is differentiable at (a, b).
- Let D be a domain in $\mathbb{R}^2$. Let $f : D \to R$ be differentiable at $(a, b) \in D$. Then f is continuous at (a, b).

Notice that the first statement strengthens the increment theorem. Instead of increasing the load on terminology, we will continue with the increment theorem. Note that whenever we assume that f_x and f_y are continuous, you may replace this with the weaker assumption: " $f(x, y)$ is differentiable".

Remember that we formulate and discuss our results for a function $f(x, y)$ of two variables. Analogously, all the notions and the results can be formulated for a function $f(x_1, \dots, x_n)$ of n variables for $n \geq 2$.

1.6 Chain Rules

We apply the increment theorem to partially differentiate composite functions.

Theorem 1.5. (Chain Rule 1) *Let $x(t)$ and $y(t)$ be differentiable functions. Let $f(x, y)$ be such that f_x and f_y are continuous. Then*

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Proof: By Theorem 2.2, $f(x, y)$ is a differentiable function. Use the increment Δf at a point P to obtain

$$\frac{\Delta f}{\Delta t} = \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t} + \epsilon_1 \frac{\Delta x}{\Delta t} + \epsilon_2 \frac{\Delta y}{\Delta t}.$$

As $\Delta t \to 0$, we have $\Delta x \to 0, \Delta y \to 0, \epsilon_1 \to 0, \epsilon_2 \to 0$. Then the result follows. □

For example, if $z = xy$ and $x = \sin t, y = \cos t$, then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}x'(t) + \frac{\partial z}{\partial y}y'(t) = \cos^2 t - \sin^2 t.$$

Check: $z(t) = \sin t \cos t = \frac{1}{2}\sin 2t$. So, $z'(t) = \cos 2t = \cos^2 t - \sin^2 t$.

Theorem 1.6. (Chain Rule 2) *Let $f(x, y)$ be a function, where f_x and f_y are continuous. Suppose $x = x(s, t)$ and $y = y(s, t)$ are functions such that $x_s,\ x_t,\ y_s$ and y_t are also continuous. Then*

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}, \quad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}.$$

Proof of this follows a similar line to that of Chain Rule - 1. The pattern is clearer if you use the subscript notation:

$$f_s = f_x x_s + f_y y_s, \quad f_t = f_x x_t + f_y y_t.$$

Example 1.19. Let $z = e^x \sin y,\ x = st^2,\ y = s^2 t$. Then

$$\frac{\partial z}{\partial s} = (e^x \sin y)t^2 + (e^x \cos y)2st = te^{st^2}(t\sin(s^2 t) + 2s\cos(s^2 t)).$$

$$\frac{\partial z}{\partial t} = (e^x \sin y)2st + (e^x \cos y)s^2 = se^{st^2}(2t\sin(s^2 t) + s\cos(s^2 t)).$$

Substitute expressions for x and y to get $z = z(s, t)$ and then check that the results are correct.

Example 1.20. Given that $z = f(x, y)$ has continuous second order partial derivatives and that $x = r^2 + s^2,\ y = 2rs$, find z_{rr}.

We have $x_r = 2r,\ y_r = 2s$. Then

$$\begin{aligned}
z_r &= 2rz_x + 2sz_y.\\
z_{xr} &= z_{xx}x_r + z_{xy}y_r = 2rz_{xx} + 2sz_{xy}.\\
z_{yr} &= z_{yx}x_r + z_{yy}y_r = 2rz_{yx} + 2sz_{yy}.\\
z_{rr} &= \frac{\partial z_r}{\partial r} = \frac{\partial}{\partial r}(2rz_x + 2sz_y) = 2z_x + 2rz_{xr} + 2sz_{yr}\\
&= 2z_x + 2r(2rz_{xx} + 2sz_{xy}) + 2s(2rz_{yx} + 2sz_{yy})\\
&= 2z_x + 4r^2 z_{xx} + 8rsz_{xy} + 4s^2 z_{yy}.
\end{aligned}$$

Functions can be differentiated implicitly. If F is defined within a sphere S containing a point (a, b, c), where $F(a, b, c) = 0,\ F_z(a, b, c) \neq 0$, and F_x, F_y, F_z are continuous inside the sphere, then the equation $F(x, y, z) = 0$ defines a function $z = f(x, y)$ in a sphere containing (a, b, c) and contained in the sphere S. Moreover, the function $z = f(x, y)$ can now be differentiated partially with $z_x = -F_x/F_z,\ z_y = -F_y/F_z$.

It is easier to differentiate implicitly than remembering the formula.

Example 1.21. Find z_x and z_y if $x^3 + y^3 + z^3 + 6xyz = 1$.

We differentiate 'the equation' with respect to x and y as follows:

$$3x^2 + 3z^2 z_x + 6y(z + xz_x) = 0 \Rightarrow z_x = -\frac{(x^2 + 2yz)}{z^2 + 2xy}.$$

$$3y^2 + 3z^2 z_y + 6x(z + xz_y) = 0 \Rightarrow z_y = -\frac{(y^2 + 2xz)}{z^2 + 2xy}.$$

Example 1.22. Find $\dfrac{dy}{dx}$ if $y = y(x)$ is given by $y^2 = x^2 + \sin(xy)$.

$$2y\frac{dy}{dx} - 2x - \cos(xy)(y + x\frac{dy}{dx}) = 0 \Rightarrow \frac{dy}{dx} = \frac{2x + y\cos(xy)}{2y - x\cos(xy)}.$$

Example 1.23. Find w_x if $w = x^2 + y^2 + z^2$ and $z = x^2 + y^2$.

As it looks,

$$\frac{\partial w}{\partial x} = 2x.$$

However, since $z = x^2 + y^2$, we have $w = x^2 + y^2 + (x^2 + y^2)^2$. Then

$$\frac{\partial w}{\partial x} = 2x + 4x^3 + 4xy^2.$$

Notice that, here we take z as the dependent variable and x, y as independent variables. But suppose we know that x and z are the independent variables and y is the dependent variable. Then the second equation says that $y^2 = z - x^2$. Then $w = x^2 + (z - x^2) + z^2 = z + z^2$. Thus

$$\frac{\partial w}{\partial x} = 0.$$

The correct procedure to get $\partial w/\partial x$ is :

1. w must be dependent variable and x must be independent variable.
2. Decide which of the other variables are dependent or independent.
3. Eliminate the dependent variables from w using the constraints.
4. Then take the partial derivative $\partial w/\partial x$.

Example 1.24. Given that $w = x^2 + y^2 + z^2$ and $z(x, y)$ satisfies $z^3 - xy + yz + y^3 = 1$, evaluate $\partial w/\partial x$ at $(2, -1, 1)$.

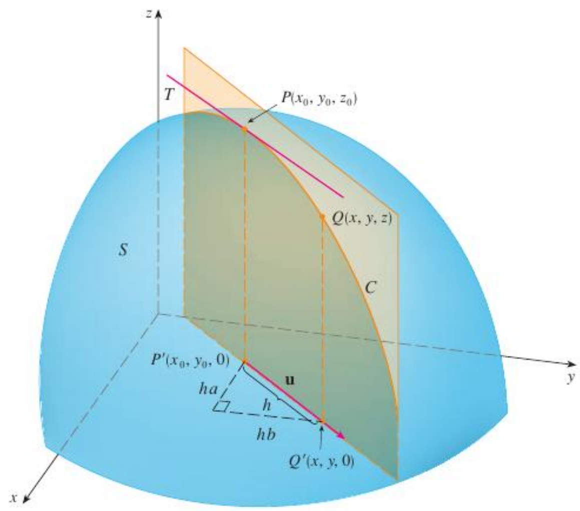
It is now clear that z, w are dependent variables and x, y are independent variables. So,

$$\frac{\partial w}{\partial x} = 2x + 2z\frac{\partial z}{\partial x}, \quad 3z^2\frac{\partial z}{\partial x} - y + y\frac{\partial z}{\partial x} = 0.$$

These two together give $\frac{\partial w}{\partial x} = 2x + \frac{2yz}{y+3z^2}$. Evaluating it at $(2, -1, 1)$ gives $\frac{\partial w}{\partial x}(2, -1, 1) = 3$.

1.7 Directional Derivative

Recall that if $f(x,y)$ is a function, then $f_x(x_0,y_0)$ is the rate of change in f with respect to change in x, at (x_0,y_0), that is, in the direction $\hat{i}$. Similarly, $f_y(x_0,y_0)$ is the rate of change at (x_0,y_0) in the direction $\hat{j}$. How do we find the rate of change of $f(x,y)$ at (x_0,y_0) in the direction of any unit vector $\hat{u}$?



Consider the surface S with the equation $z = f(x,y)$. Let $z_0 = f(x_0,y_0)$. The point $P(x_0,y_0,z_0)$ lies on S. The vertical plane that passes through P in the direction of $\hat{u}$ (containing $\hat{u}$) intersects S in a curve C. The slope of the tangent line T to C at P is the rate of change of z in the direction of $\hat{u}$.

Let $f(x,y)$ be a function defined in a domain D. Let $(x_0,y_0) \in D$. The **directional derivative** of $f(x,y)$ in the direction of a unit vector $\hat{u} = a\hat{i} + b\hat{j}$ at (x_0,y_0) is given by

$$(D_u f)(x_0,y_0) = \left(\frac{df}{ds}\right)_u \Big|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h}.$$

Example 1.25. Find the derivative of $z = x^2+y^2$ at $(1,2)$ in the direction $\hat{u} = (1/\sqrt{2})\hat{i}+(1/\sqrt{2})\hat{j}$.

$$D_u z(1,2) = \lim_{h\to 0} \frac{f\big(1+h/\sqrt{2}, 2+h/\sqrt{2}\big) - f(1,2)}{h} = \lim_{h\to 0} \frac{2h/\sqrt{2} + 2\cdot 2h/\sqrt{2}}{h} = \frac{6}{\sqrt{2}}.$$

Notice that $f_x(1,2)(1/\sqrt{2}) + f_y(1,2)(1/\sqrt{2}) = (2+2(2))\cdot(1/\sqrt{2}) = 6/\sqrt{2}$.

Theorem 1.7. *If $f(x,y)$ is a function of x and y having continuous partial derivatives f_x and f_y, then f has a directional derivative at (x,y) in any direction $\hat{u} = a\hat{i} + b\hat{j}$; and it is given by $D_u f(x,y) = f_x(x,y)a + f_y(x,y)b$.*

Proof: Let (x_0,y_0) be a point in the domain of definition of $f(x,y)$. Define the function $g(\cdot)$ by $g(h) = f(x_0+ah, y_0+bh)$. Then $g(h)$ is a continuous function of h. Now,

$$g'(h) = f_x \frac{dx}{dh} + f_y \frac{dy}{dh} = f_x\, a + f_y\, b.$$

Then $g'(0) = f_x(x_0, y_0) + f_y(x_0, y_0)$. Since f_x, f_y are continuous,

$$g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = D_u f(x_0, y_0).$$

Hence $D_u f(x_0, y_0) = g'(0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b$. $\square$

Example 1.26. Find the directional derivative of $f(x, y) = x^3 - 3xy + 4y^2$ in the direction of the line that makes an angle of $\pi/6$ with the x-axis.

Here, the direction is given by the unit vector $\hat{u} = \cos(\pi/6)\hat{\imath} + \sin(\pi/6)\hat{\jmath} = \frac{\sqrt{3}}{2}\hat{\imath} + \frac{1}{2}\hat{\jmath}$. Thus

$$D_u f(x, y) = \frac{\sqrt{3}}{2} f_x + \frac{1}{2} f_y = \frac{\sqrt{3}}{2}(3x^2 - 3y) + \frac{1}{2}(-3x + 8y) = \frac{1}{2}\Big[3\sqrt{3}x^2 - 3x + (8 - 3\sqrt{3})y\Big].$$

The formula for the directional derivative in the direction of the unit vector $\hat{u} = a\hat{\imath} + b\hat{\jmath}$ can be written as

$$D_u f = f_x a + f_y b = (f_x\hat{\imath} + f_y\hat{\jmath}) \cdot (a\hat{\imath} + b\hat{\jmath}).$$

The vector operator $\nabla := \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath}$ is called the **gradient** and the **gradient of** $f(x, y)$ is

$$\nabla f := \text{grad } f := \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath}.$$

Therefore, $D_u f = \text{grad } f \cdot \hat{u}$. That is, at (x_0, y_0), the directional derivative is given by

$$D_u f|_{(x_0,y_0)} = \text{grad } f|_{(x_0,y_0)} \cdot \hat{u}.$$

For example, for the function $f(x, y) = xe^y + \cos(xy)$, grad $f|_{(2,0)} = \hat{\imath} + 2\hat{\jmath}$. Thus, the directional derivative of f in the direction of $3\hat{\imath} - 4\hat{\jmath}$ is grad $f|_{(1,2)} \cdot ((3/5)\hat{\imath} - (4/5)\hat{\jmath}) = -1$.

However, remember that in order that this formula is applicable, we have assumed that the function $f(x, y)$ has continuous partial derivatives $f_x,\ f_y$ at (x_0, y_0).

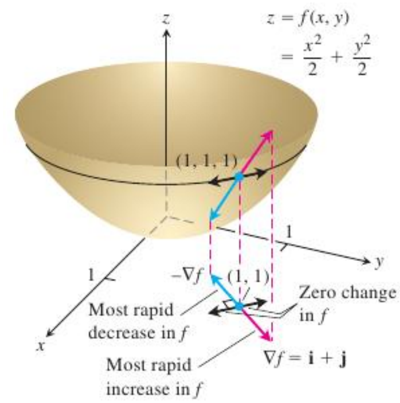
Theorem 1.8. *Let $f(x, y)$ have continuous partial derivatives f_x and f_y. The maximum value of the directional derivative $D_u f(x, y)$ is $|\text{grad } f|$ and it occurs when $\hat{u}$ has the same direction as that of* grad f.

This is obvious since $D_u f = \text{grad } f \cdot \hat{u}$ says that the directional derivative is the scalar projection of the gradient in the direction of $\hat{u}$.

Proof: $D_u f = \text{grad } f \cdot \hat{u} = |\text{grad } f|\,|\hat{u}|\cos\theta = |\text{grad } f|\cos\theta$, where θ is the angle between grad f and $\hat{u}$. Since maximum of $\cos\theta$ is 1, maximum of $D_u f$ is $|\text{grad } f|$. The maximum occurs when $\theta = 0$, that is, when the directions of grad f and $\hat{u}$ coincide. $\square$

This also says the following:

$f(x, y)$ increases most rapidly in the direction of its gradient.
$f(x, y)$ decreases most rapidly in the opposite direction of its gradient.
$f(x, y)$ remains constant in any direction orthogonal to its gradient.



Example 1.27. Find the directions in which the function $f(x, y) = x^2 + y^2$ changes most, least, and not at all, at the point $(1, 1)$.

$$\text{grad } f = f_x\hat{\imath} + f_y\hat{\jmath} = 2x\hat{\imath} + 2y\hat{\jmath}. \quad (\text{grad } f)(1, 1) = 2\hat{\imath} + 2\hat{\jmath}.$$

Thus the function $f(x, y)$ increases most at $(1, 1)$ in the direction $(\hat{\imath} + \hat{\jmath})/\sqrt{2}$. It decreases most at $(1, 1)$ in the direction $-(\hat{\imath}+\hat{\jmath})/\sqrt{2}$. And it does not change at $(1, 1)$ in the directions $\pm(\hat{\imath} - \hat{\jmath})/\sqrt{2}$.

1.8 Normal to Level Curve and Tangent Planes

Let $z = f(x, y)$ be a given surface. A level curve to this surface is a curve $f(x, y) = c$ for any constant c. On this level curve, the function $f(x, y)$ is a constant, namely, c in the range of $f(x, y)$. Suppose $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$ is a parametrization of this level curve.
Differentiating, we have $\frac{d}{dt}f(x(t), y(t)) = 0$. Or,

$$f_x\frac{dx}{dt} + f_y\frac{dy}{dt} = \text{grad } f \cdot \frac{d\vec{r}(t)}{dt} = 0.$$

Since $d\vec{r}/dt$ is the tangent to the curve, $\text{grad } f$ is the normal to the level curve. That is,

At any point (x_0, y_0) in the domain of the differentiable function $f(x, y)$, its gradient $\text{grad } f$ is the normal to the level curve that passes through (x_0, y_0).

In higher dimensions, if $f(x_1, \ldots, x_n)$ is a function of n independent variables defined on $D \subseteq \mathbb{R}^n$, then its gradient at any point is

$$\text{grad } f = \Big(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\Big).$$

The directional derivative at any point $\vec{x}$ in the direction of a unit vector $\hat{u} = (u_1, \ldots, u_n)$ is

$$D_u f = \lim_{h\to 0}\frac{f(\vec{x} + h\hat{u}) - f(\vec{x})}{h} = \text{grad } f \cdot \hat{u} = f_{x_1}u_1 + \cdots + f_{x_n}u_n.$$

The algebraic rules for the gradient are as follows:

1. Constant multiple: $\text{grad } (kf) = k(\text{grad } f)$ for $k \in \mathbb{R}$.
2. Sum: $\text{grad } (f + g) = \text{grad } f + \text{grad } g$.

3. Difference: $\text{grad}\,(f-g) = \text{grad}\, f - \text{grad}\, g$.
4. Product: $\text{grad}\,(fg) = f(\text{grad}\, g) + g(\text{grad}\, f)$.
5. Quotient: $\text{grad}\,\dfrac{f}{g} = \dfrac{g(\text{grad}\, f) - f(\text{grad}\, g)}{g^2}$.

In 3d, let $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$ be a smooth curve on the level surface $f(x,y,z) = c$. Then $f(x(t), y(t), z(t)) = c$ for all t. Differentiating this we get

$$\text{grad}\, f \cdot \vec{r}\,'(t) = 0.$$

Look at all such smooth curves that pass through a point P on the level surface. The velocity vectors $\vec{r}\,'(t)$ to all these smooth curves are orthogonal to the gradient at P.

Let $f(x,y,z)$ have continuous partial derivatives f_x, f_y, and f_z. The **tangent plane** at $P(x_0, y_0, z_0)$ on the level surface $f(x,y,z) = c$ is the plane through P which is orthogonal to $\text{grad}\, f$ at P. Its equation is

$$f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0) = 0.$$

The **normal line** to the level surface $f(x,y,z) = c$ at $P(x_0, y_0, z_0)$ is the line through P parallel to $\text{grad}\, f$. Its parametric equation is

$$x = x_0 + f_x(x_0,y_0,z_0)\, t,\ y = y_0 + f_y(x_0,y_0,z_0)\, t,\ z = z_0 + f_z(x_0,y_0,z_0)\, t.$$

The equation of the tangent plane to the surface $z = f(x,y)$ at (a,b) can be obtained as follows:

Write the surface as $F(x,y,z) = 0$, where $F(x,y,z) = f(x,y) - z$. Then $F_x = f_x,\ F_y = f_y,$ $F_z = -1$. Then the equation of the tangent plane is

$$f_x(a,b)(x-a) + f_y(a,b)(y-b) - (z - f(a,b)) = 0.$$

Example 1.28. Find the tangent plane and the normal line of the surface $x^2 + y^2 + z - 9 = 0$ at the point $(1,2,4)$.

First, check that the point $(1,2,4)$ lies on the surface. Next, $f_x(1,2,4) = 2,\ f_y(1,2,4) = 4$ and $f_z(1,2,4) = 1$. The tangent plane is given by

$$2(x-1) + 4(y-2) + (z-4) = 0.$$

The normal line at $(1,2,4)$ is given by

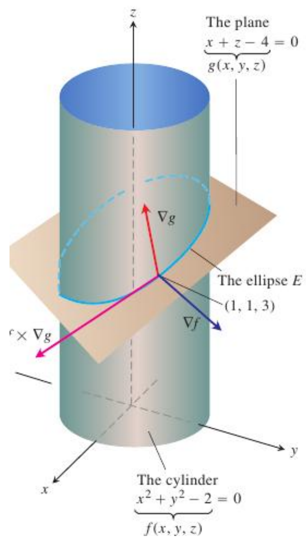
$$x = 1 + 2t,\ y = 2 + 4t,\ z = 4 + t.$$

Example 1.29. Find the tangent plane to the surface $z = x\cos y - ye^x$ at the origin.

$f_x(0,0) = 1,\ f_y(0,0) = -1$. The tangent plane is

$$x - y - z = 0.$$

Example 1.30. Find the tangent line to the curve of intersection of the surfaces $f(x,y,z) := x^2 + y^2 - 2 = 0$ and $g(x,y,z) := x + z - 4 = 0$ at the point $(1,1,3)$.



The tangent line is orthogonal to both $\text{grad } f$ and $\text{grad } g$ at $(1, 1, 3)$. So, it is parallel to

$$\text{grad } f \times \text{grad } g = (2\hat{i} + 2\hat{j}) \times (\hat{i} + \hat{k}) = 2\hat{i} - 2\hat{j} - 2\hat{k}.$$

Thus the tangent line is $x = 1 + 2t,\ y = 1 - 2t,\ z = 3 - 2t.$

1.9 Taylor's Theorem

For a function of one variable, a polynomial approximation is given by the Taylor's formula. Observe that it is a generalization of the Mean value theorem.

Theorem 1.9. (Taylor's Formula for one variable) *Let $n \in \mathbb{N}$. Suppose that $f^{(n)}(x)$ is continuous on $[a, b]$ and is differentiable on (a, b). Then there exists a point $c \in (a, b)$ such that*

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + \frac{f^{(n+1)}(c)}{(n + 1)!}(x - a)^{n+1}.$$

Proof: For $x = a$, the formula holds. So, let $x \in (a, b]$. For any $t \in [a, x]$, let

$$p(t) = f(a) + f'(a)(t - a) + \frac{f''(a)}{2!}(t - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(t - a)^n.$$

Here, we treat x as a certain point, not a variable; and t as a variable. Write

$$g(t) = f(t) - p(t) - \frac{f(x) - p(x)}{(x - a)^{n+1}}\,(t - a)^{n+1}.$$

We see that $g(a) = 0,\ g'(a) = 0,\ g''(a) = 0,\ \ldots, g^{(n)}(a) = 0,$ and $g(x) = 0.$

By Rolle's theorem, there exists $c_1 \in (a, x)$ such that $g'(c_1) = 0$. Since $g(a) = 0$, apply Rolle's theorem once more to get a $c_2 \in (a, c_1)$ such that $g''(c_2) = 0$.

Continuing this way, we get a $c_{n+1} \in (a, c_n)$ such that $g^{(n+1)}(c_{n+1}) = 0$.

Since $p(t)$ is a polynomial of degree at most n, $p^{(n+1)}(t) = 0$. Then

$$g^{(n+1)}(t) = f^{(n+1)}(t) - \frac{f(x) - p(x)}{(x-a)^{n+1}}\,(n+1)!.$$

Evaluating at $t = c_{n+1}$ we have $f^{(n+1)}(c_{n+1}) - \dfrac{f(x) - p(x)}{(x-a)^{n+1}}\,(n+1)! = 0$. That is,

$$\frac{f(x) - p(x)}{(x-a)^{n+1}} = \frac{f^{(n+1)}(c_{n+1})}{(n+1)!}.$$

Consequently, $g(t) = f(t) - p(t) - \dfrac{f^{(n+1)}(c_{n+1})}{(n+1)!}\,(t-a)^{n+1}$.

Evaluating it at $t = x$ and using the fact that $g(x) = 0$, we get

$$f(x) = p(x) + \frac{f^{(n+1)}(c_{n+1})}{(n+1)!}\,(x-a)^{n+1}.$$

Since x is an arbitrary point in $(a, b]$, this completes the proof. $\square$

We have a similar result for functions of several variables.

Theorem 1.10. (Taylor) *Let D be a domain in $\mathbb{R}^2$; (a, b) be an interior point of D. Let $f : D \to \mathbb{R}$ have continuous partial derivatives of order up to $n+1$ in some open disk D_0 centered at (a, b) and contained in D. Then for any $(a+h, b+k) \in D_0$, there exists $\theta \in [0, 1]$ such that*

$$\begin{aligned} f(a+h, b+k) &= f(a,b) + \sum_{m=1}^{n} \frac{1}{m!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^m f(a,b) \\ &+ \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(a+\theta h, b+\theta k) \end{aligned}$$

For example, $m = 2$ on the right gives $\frac{1}{2!}(h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy})$.

Proof: Let $\phi(t) = f(a+th, b+tk)$. Then,

$\phi'(t) = f_x(a+th, b+tk)h + f_y(a+th, b+tk)k = (h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y})f(a+th, b+tk).$

$\phi^{(2)}(t) = (f_{xx}h + f_{xy}k)h + (f_{yx}h + f_{yy}k)k = (h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y})^2 f(a+th, b+tk).$

By induction, it follows that

$$\phi^{(m)}(t) = \big(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\big)^m f(a+th, b+tk).$$

Using Taylor's formula for the single variable function $\phi(t)$, we have

$$\phi(1) = \phi(0) + \sum_{m=1}^{n} \frac{\phi^{(m)}(0)}{m!} + \frac{\phi^{(n+1)}(\theta)}{(n+1)!}.$$

for some $\theta \in [0, 1]$. Substituting the expression for $\phi^{(n+1)}(\theta)$, we get the required result. $\square$

Example 1.31. Let $f(x, y) = x^2 + xy - y^2,\ a = 1,\ b = -2$.
Here, $f(1, -2) = -5,\ f_x(1, -2) = 0,\ f_y(1, -2) = 5,\ f_{xx} = 2,\ f_{xy} = 1,\ f_{yy} = -2$. Then

$$f(x, y) = -5 + 5(y+2) + \frac{1}{2}\big[2(x-1)^2 + 2(x-1)(y+2) - 2(y+2)^2\big].$$

This becomes exact, since third (and more) order derivatives are 0.

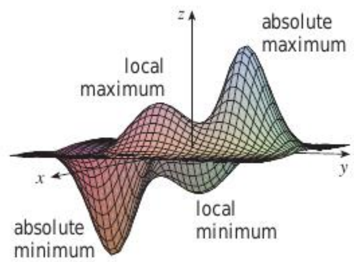
1.10 Extreme Values

We extend the notions of local maxima and local minima to a function of two variables.

Let D be a domain in $\mathbb{R}^2$; $(a, b) \in D$; and let $f : D \to \mathbb{R}$.

$f(x, y)$ has a **local maximum** at (a, b) if $f(x, y) \le f(a, b)$ for all (x, y) near (a, b).
That is, for all (x, y) in some open disk centered at (a, b), $f(x, y) \le f(a, b)$.
The number $f(a, b)$ is then called a **local maximum value** of $f(x, y)$; and the point (a, b) is called a **point of local maximum** of $f(x, y)$.

If for all $(x, y) \in D$, $f(x, y) \le f(a, b)$ then f has an **absolute maximum** at (a, b).
The number $f(a, b)$ is called the **absolute maximum value** of f; and the point (a, b) is called a **point of absolute maximum** of $f(x, y)$.



Replace all $\le$ by $\ge$; then call all those **minimum** instead of *maximum*.

Let D be a domain in $\mathbb{R}^2$; $f : D \to R$. Let $(a, b) \in D$. The function $f(x, y)$ has a **local extremum** at (a, b) if $f(x, y)$ has a local maximum or a local minimum at (a, b).

An interior point (a, b) of D is a **critical point** of $f(x, y)$ if either $f_x(a, b) = 0 = f_y(a, b)$ or at least one of $f_x(a, b)$ or $f_y(a, b)$ does not exist.

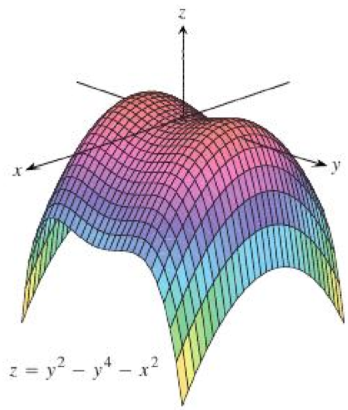
Theorem 1.11. *Let D be a domain in $\mathbb{R}^2$; $f : D \to R$. Let (a, b) be an interior point of D. If $f(x, y)$ has a local extremum at (a, b), then (a, b) is a critical point of $f(x, y)$.*

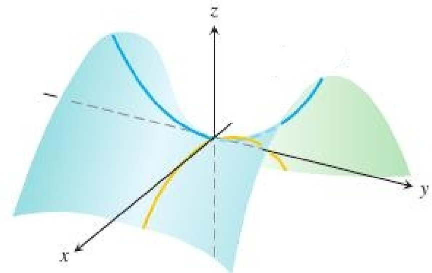
Proof: Suppose f has a local maximum at an interior point (a, b) of D. Suppose $f_x(a, b)$ exists. The function $g(x) = f(x, b)$ has a local maximum at $x = a$. Then $g'(a) = 0$. That is, $f_x(a, b) = 0$. Similarly, consider $h(y) = f(a, y)$ and conclude that $f_y(a, b) = 0$. Give similar argument if f has a local minimum at (a, b). $\square$

Geometrically, it says that if at an interior point (a, b), there exists a tangent plane to the surface $z = f(x, y)$, and if this point (a, b) happens to be an extremum point, then there exists a horizontal tangent plane to the surface at (a, b).

Let D be a domain in $\mathbb{R}^2$. Let $f : D \to \mathbb{R}$ have continuous partial derivatives f_x and f_y. Let (a, b) be a critical point of $f(x, y)$. The point $(a, b, f(a, b))$ on the surface is called a **saddle point** of $f(x, y)$ if in every open disk centered at (a, b) and contained in D, there are points (x_1, y_1), (x_2, y_2) such that $f(x_1, y_1) < f(a, b) < f(x_2, y_2)$.

At a saddle point, the function has neither a local maximum nor a local minimum; the surface crosses its tangent plane.





For a function $f(x, y)$, its **Hessian** is defined by

$$H(f) := \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2.$$

Suppose that the function $f(x, y)$ has second order continuous partial derivatives in an open disk centered at a point (a, b) inside its domain of definition. If $H(f)(a, b) > 0$, then the surface $z = f(x, y)$ curves the same way in all directions near (a, b).

We will not prove this geometrical fact. We rather prove one of its corollaries which will help us in determining the local maxima and local minima.

Theorem 1.12. *Let $f : D \to \mathbb{R}$ have continuous first and second order partial derivatives in an open disk centered at $(a, b) \in D$. Suppose (a, b) is a critical point of $f(x, y)$.*

1. *If $H(f)(a, b) > 0$ and $f_{xx}(a, b) < 0$, then $f(x, y)$ has a local maximum at (a, b).*
2. *If $H(f)(a, b) > 0$ and $f_{xx}(a, b) > 0$, then $f(x, y)$ has a local minimum at (a, b).*
3. *If $H(f)(a, b) < 0$ then $f(x, y)$ has a saddle point at (a, b).*
4. *If $H(f)(a, b) = 0$, then nothing can be said, in general.*

Proof: Let $(a + h, b + k)$ be in an open disk centered at (a, b) and contained in D. By Taylor's formula,

$$f(a + h, b + k) = (f + hf_x + kf_y)\Big|_{(a,b)} + \frac{1}{2}(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy})\Big|_{(a+\theta h, b+\theta k)}$$

Since (a, b) is a critical point of f, $f_x(a, b) = 0 = f_y(a, b)$. Then

$$f(a + h, b + k) - f(a, b) = \frac{1}{2}(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy})\Big|_{(a+\theta h, b+\theta k)} \tag{1.1}$$

(1) Let $H(f)(a, b) > 0$ and $f_{xx}(a, b) < 0$. Multiply both sides of Equation 1.1 by $2f_{xx}$, add and subtract $(f_{xy})^2k^2$, and rearrange to get (All of $f_{xx},\ f_{xy},\ f_{yy}$ are evaluated at $(a + \theta h, b + \theta k)$.)

$$2f_{xx}[f(a + h, b + k) - f(a, b)] = (hf_{xx} + kf_{xy})^2 + (f_{xx}f_{yy} - (f_{xy})^2)k^2.$$

By continuity of functions involved, $f_{xx}(a+\theta h, b+\theta k) < 0$. The RHS is positive. Therefore, $f(a+h, b+k) - f(a,b) < 0$. That is, (a,b) is a local maximum point.

(2) Let $H(f)(a,b) > 0$ and $f_{xx}(a,b) > 0$, By continuity again, $f_{xx}(a+\theta h, b+\theta k) > 0$. So, $f(a+h, b+k) - f(a,b) > 0$. That is, (a,b) is a local minimum point.

(3) Let $H(f)(a,b) < 0$. We want to show that $f(a+h, b+k) - f(a,b)$ has opposite signs at different points in any small disk around (a,b). We break this case into three sub-cases:

$$\text{(3A)}\ \ f_{xx}(a,b) \neq 0. \quad \text{(3B)}\ \ f_{yy}(a,b) \neq 0, \quad \text{(3C)}\ \ f_{xx}(a,b) = f_{yy}(a,b) = 0.$$

(3A) Let $H(f)(a,b) < 0$ and $f_{xx}(a,b) \neq 0$.
First, set $h = t, k = 0$ in Equation 1.1 and evaluate the following limit:

$$\lim_{t \to 0} \frac{f(a+h, b+k) - f(a,b)}{t^2} = \lim_{t \to 0} \frac{t^2 f_{xx}(a+t, b)}{2t^2} = \frac{f_{xx}(a,b)}{2}.$$

Next, set $h = -t f_{xy}(a,b),\ k = t f_{xx}(a,b)$. Use Equation (1.1) to obtain

$$\lim_{t \to 0} \frac{f(a+h, b+k) - f(a,b)}{t^2} = \lim_{t \to 0} \frac{1}{2}(f_{xy}^2 f_{xx} - 2 f_{xx} f_{xy}^2 + f_{xx}^2 f_{yy}) = \frac{f_{xx}(a,b)}{2} H(f)(a,b).$$

Since $H(f)(a,b) < 0$, these two limits have opposite signs. Due to continuity, $f(a+h, b+k) - f(a,b)$ will have opposite signs in any neighborhood of (a,b).

(3B) Let $H(f)(a,b) < 0$ and $f_{yy}(a,b) \neq 0$. This is similar to (3A).

(3C) Let $H(f)(a,b) < 0$ and $f_{xx}(a,b) = f_{yy}(a,b) = 0$.

First, set $h = k = t$. Use Equation (1.1) to get

$$\lim_{t \to 0} \frac{f(a+h, b+k) - f(a,b)}{t^2} = \lim_{t \to 0} \frac{1}{2}(f_{xx} + 2f_{xy} + f_{yy})|_{(a+t, b+t)} = f_{xy}(a,b).$$

Next, set $h = t,\ k = -t$. Using Equation (1.1) again, we have

$$\lim_{t \to 0} \frac{f(a+h, b+k) - f(a,b)}{t^2} = \lim_{t \to 0} \frac{1}{2}(f_{xx} - 2f_{xy} + f_{yy})|_{(a+t, b+t)} = -f_{xy}(a,b).$$

As in (3A), $f(a+h, b+k) - f(a,b)$ will have opposite signs in any neighborhood of (a,b). $\square$

Notice that the case $H(f)(a,b) > 0$ and $f_{xx}(a,b) = 0$ is not possible. Moreover, Under the condition that $H(f)(a,b) > 0$, both $f_{xx}(a,b)$ and $f_{yy}(a,b)$ have the same sign. Thus, in Theorem 1.12, the sign condition on $f_{xx}(a,b)$ can be replaced by the corresponding sign condition on $f_{yy}(a,b)$. It also says that if $f_{xx}(a,b)$ and $f_{yy}(a,b)$ have the opposite signs, then the critical point (a,b) is a saddle point of $f(x,y)$.

Example 1.32. Find the extreme values of $f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$.

Domain of f is $\mathbb{R}^2$ having no boundary points. f is differentiable. Its extreme values are all local extrema. The critical points are those where both f_x and f_y vanish. Now,

$$f_x = y - 2x - 2, \quad f_y = x - 2y - 2.$$

The critical points satisfy $f_x = 0 = f_y$. That is, $x = y = -2$.

$f_{xx}(-2,-2) = -2,\ f_{xy}(-2,-2) = 1,\ f_{yy}(-2,-2) = -2$.

Then $H(f)(-2,-2) = 3 > 0,\ f_{xx} < 0$.

Thus, f has local maximum at $(-2,-2)$.

Here also f has absolute maximum and the maximum value is $f(-2,-2) = 8$.

Example 1.33. Investigate $f(x,y) = x^4 + y^4 - 4xy + 1$ for extreme values.

The function has continuous first and second partial derivatives everywhere.

The critical points are at (x,y) where $f_x = 4x^3 - 4y = 0 = f_y = 4y^3 - 4x$.

That is, when $x^3 = y$ and $y^3 = x$. Giving $x^9 = x$ which has solutions $x = 0, 1, -1$ in $\mathbb{R}$. The corresponding y values are $0, 1, -1$.

Now, $f_{xx} = 12x^2,\ f_{xy} = -4,\ f_{yy} = 12y^2$. Thus $H(f) = 144x^2y^2 - 16$.

At $x = 0, y = 0$, $H(f) = -16$. Thus f has a saddle point at $(0,0)$.

At $x = 1, y = 1$, $H(f) > 0,\ f_{xx} > 0$. Thus f has a local minimum at $(1,1)$.

At $x = -1, y = -1$, $H(f) > 0,\ f_{xx} > 0$. Thus f has a local minimum at $(-1,-1)$.

The local minimum values are $f(1,1) = -1$ and $f(-1,-1) = -1$. Both are absolute minima.

Example 1.34. Find absolute extrema of $f(x,y) = 2 + 2x + 2y - x^2 - y^2$ defined on the triangular region bounded by the straight lines $x = 0,\ y = 0$, and $x + y = 9$.

1. The critical points are solutions of $f_x = 2 - 2x = 0 = f_y = 2 - 2y$. That is, $x = 1, y = 1$.
This accounts for the interior points of the domain.

2. Draw the picture. The vertices of the triangle are $A(0,0),\ B(0,9),\ C(9,0)$. These are possible extremum points. This accounts for the vertices which are on the boundary.

3. Next, we should consider the boundary in detail.

3(a). On the line segment AB, f is given by $(y = 0)$:
$g(x) = f(x,0) = 2 + 2x - x^2$ for $0 \le x \le 9$. Now, $g'(x) = 0 \Rightarrow x = 1$.
Thus $(1,0)$ is a possible extremum point.

3(b). Similarly, on the line segment AC, f is given by $(x = 0)$:
$g(y) = f(0,y) = 2 + 2y - y^2$ for $0 \le y \le 9$. Then $g'(y) = 0 \Rightarrow y = 1$.
Thus, a possible extremum point is $(0,1)$.

3(c). On the line segment BC, f is given by $(x + y = 9)$:
$g(x) = f(x, 9-x) = 2 + 2x + 2(9-x) - x^2 - (9-x)^2 = -61 + 18x - 2x^2$ for $0 \le x \le 9$.
$g'(x) = 0 \Rightarrow 18 - 4x = 0 \Rightarrow x = 9/2,\ y = 9 - x = 9/2$.
Thus $(9/2, 9/2)$ is a possible extremum point.

The values at these possible extrema are

$$f(1,1) = 4, f(0,0) = 2, f(0,9) = -61, f(9,0) = -61, f(1,0) = 3, f(0,1) = 3, f(9/2,9/2) = -41/2.$$

Therefore, $f(x,y)$ has absolute minimum at $(0,9)$ and $(9,0)$ and its minimum value is -61.

It has absolute maximum at $(1, 1)$ and its maximum value is 4.

Example 1.35. Maximize the volume of a box of length x, width y and height z subject to the condition that $x + 2y + 2z = 108$.

$V = xyz = (108 - 2y - 2z)yz$. Take $f(y, z) = (108 - 2y - 2z)yz$. Then

$$f_y = (108 - 4y - 2z)z, \quad f_z = (108 - 2y - 4z)y.$$

The equations $f_y = 0 = f_z$ imply that

$$(z = 0 \text{ or } 108 - 4y - 2z = 0) \quad \text{and} \quad (y = 0 \text{ or } 108 - 2y - 4z = 0)$$

We have four possibilities:

(a) $z = 0,\ y = 0$.

(b) $z = 0,\ 108 - 2y - 4z = 0 \Rightarrow z = 0,\ y = 54$.

(c) $108 - 4y - 2z = 0,\ y = 0 \Rightarrow z = 54,\ y = 0$.

(d) $108 - 4y - 2z = 0,\ 108 - 2y - 4z = 0$. Subtracting, $-2y + 2z = 0 \Rightarrow y = z \Rightarrow z = 18,\ y = 18$.

Therefore, the critical points (y, z) are $(0, 0)$, $(0, 54)$, $(54, 0)$ and $(18, 18)$.

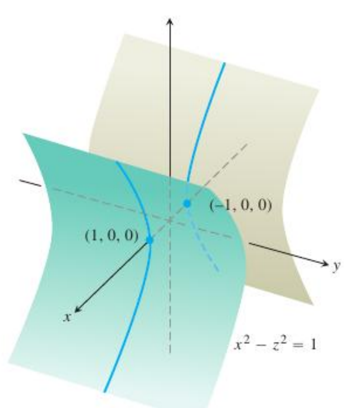
At the first three points, $f(y, z)$ is 0, which is clearly not the maximum value of $f(y, z)$. The only possibility is $(18, 18)$. To see that this a point where $f(y, z)$ is maximum, consider

$$f_{yy} = -4z,\ f_{yz} = 108 - 4y - 2z - 2z = 108 - 4y - 4z,\ f_{zz} = -4y.$$

At $(18, 18)$, $f_{yy} < 0$, and $H(f) = f_{yy} f_{zz} - f_{yz}^2 = 16 \times 18 \times 18 - 16(-9)^2 > 0$.

Hence the volume of the box is maximum when its length is $108 - 36 - 36 = 36$, width is 18 and height is 18 units. The maximum volume is 11664 cubic units.

Example 1.36. Find the points closest to the origin on the hyperbolic cylinder $x^2 - z^2 = 1$.



We seek a point (x, y, z) that minimizes $f(x, y, z) = x^2 + y^2 + z^2$ subject to $x^2 - z^2 = 1$.
As earlier, taking $z^2 = x^2 - 1$, we seek (x, y) that minimizes

$$g(x, y) := f(x, y, \pm\sqrt{x^2 - 1}) = x^2 + y^2 + x^2 - 1 = 2x^2 + y^2 - 1$$

Now, $g_x = 4x,\ g_y = 2y$. Equating them to zero gives $x = 0$ and $y = 0$. But $x = 0$ does not correspond to any point on the surface $x^2 - z^2 = 1$. So, the method fails!

Instead of eliminating z, suppose we eliminate x. In that case, we seek to minimize

$$g(y, z) := f(\pm\sqrt{1 + z_2}, y, z) = 1 + z^2 + y^2 + z^2 = y^2 + 2z^2 + 1.$$

Then $g_y = 0 = g_z$ implies that $2y = 0 = 4z$. The point so obtained is $y = 0,\ z = 0$. This corresponds to the points $(\pm 1, 0, 0)$ on the surface.

Now, of course, we can proceed as earlier for minimizing $g(y, z)$.

Here, $g_{yy} = 2,\ g_{yz} = 0,\ g_{zz} = 4$.

At $y = 0, z = 0$, we have $H(g)(0, 0) = g_{yy}g_{zz} - g_{yz}^2 = 8 > 0$.

Since $g_{yy}(0, 0) > 0$, we conclude that $g(y, z)$ has a local minimum at $(0, 0)$.

These points $(\pm 1, 0, 0)$ of local minima give the minimum value of the distance $f(x, y, z)$ as 1.

But how do we know eliminating which variable would result in a solution?

We would rather look for alternative ways of solving extremum problems with constraints.

1.11 Lagrange Multipliers

Our requirement is to minimize or maximize a certain function $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$. The constraint represents a surface in three dimensional space. Let S be a surface given by $g(x, y, z) = 0$. Let $f(x, y, z)$ have an extreme value at $P(x_0, y_0, z_0)$ on the surface S. Let C be a curve given by $\vec{r}(t) = x(y)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$ that lies on S and passes through P. Suppose for $t = t_0$, we get the point P, that is, $P = \vec{r}(t_0)$.

The composite function $h(t) = f \circ g = f(x(t), y(t), z(t))$ represents the values that f takes on C. Since f has an extreme value at $P(t = t_0)$, the function $h(t)$ has an extreme value at $t = t_0$. Then $h'(t_0) = 0$. That is,

$$0 = h'(t_0) = f_x(P)x'(t_0) + f_y(P)y'(t_0) + f_z(P)z'(t_0) = (\text{grad } f)(P) \cdot \vec{r}\,'(t_0).$$

For every such curve C, $(\text{grad } g)(P)$ is orthogonal to $\vec{r}\,'(t_0)$. Thus, $(\text{grad } f)(P)$ is parallel to $(\text{grad } g)(P)$. If $(\text{grad } g)(P) \neq 0$, then

$$(\text{grad } f + \lambda \,\text{grad } g)(x_0, y_0, z_0) = 0 \ \text{ for some } \lambda \in \mathbb{R}.$$

Breaking into components, we have, at (x_0, y_0, z_0)

$$f_x + \lambda g_x = 0,\ f_y + \lambda g_y = 0,\ f_z + \lambda g_z = 0,\ g = 0.$$

Similar equations hold when there are more than one constraint.

Example 1.36 Contd.: We see that

$$f(x, y, z) = x^2 + y^2 + z^2, \quad g(x, y, z) = x^2 - z^2 - 1.$$

The necessary equations at a possible extremum point (x_0, y_0, z_0) are

$$f_x + \lambda g_x = 2x + \lambda 2x = 0, \quad f_y + \lambda g_y = 2y = 0,$$
$$f_z + \lambda g_z = 2z - \lambda 2z = 0, \quad g = x^2 - z^2 - 1 = 0.$$

It gives $x_0 = 0$ or $\lambda = -1$; $y_0 = 0$; $z_0 = 0$ or $\lambda = 1$.

From these options, $x_0 = 0$ is not possible for any z since $x^2 - z^2 = 1$. $\lambda = 1$ gives $x = 0$, which is again not possible. We are left with $\lambda = -1, y_0 = 0, z_0 = 0$. Now, $x_0^2 - z_0^2 - 1 = 0$ gives $x_0 = \pm 1$. The corresponding points are $(\pm 1, 0, 0)$. $f(x, y, z)$ at these extremum points has value 1. Since $f(x, y, z)$ is unbounded above, it does not have a maximum. Therefore, $f(x, y, z)$ at these points is minimum. Thus the points closest to the origin on the cylinder are $(\pm 1, 0, 0)$.

Notice that if we set $F(x, y, z, \lambda) := f(x, y, z) + \lambda g(x, y, z) = 0$, then

$$F_x = f_x + \lambda g_x = 0,\ F_y = f_y + \lambda g_y = 0,\ F_z = f_z + \lambda g_z = 0.$$

Moreover, $g(x, y, z) = 0$ also comes from $F_\lambda = 0$.

We can now formulate the method of solving a constrained optimization problem.

Requirement: Find extrema of the function $f(x_1, \ldots, x_n)$ subject to the conditions

$$g_1(x_1, \ldots, x_n) = 0,\ \cdots,\ g_m(x_1, \ldots, x_n) = 0.$$

Method: Set the **auxiliary function**:

$$F(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) := f(x_1, \ldots, x_n) + \lambda_1 g_1(x_1, \ldots, x_n) + \cdots \lambda_m g_m(x_1, \ldots, x_n).$$

Equate to zero the partial derivatives of F with respect to $x_1, \ldots, x_n,\ \lambda_1, \ldots, \lambda_m$. It results in $m + n$ equations in $x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m$.

Determine $x_1, \ldots, x_n\ \lambda_1, \ldots, \lambda_m$ from these equations.

The required extremum points may be found from among these values of $x_1, \ldots, x_n,\ \lambda_1, \ldots, \lambda_m$.

Remember that the method succeeds under the condition that such extreme values exist where $\operatorname{grad} g_j \neq 0$ for any j.

Example 1.37. Find the maximum value of $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $g(x, y, z) := x - y + z - 1 = 0$ and the cylinder $h(x, y, z) := x^2 + y^2 - 1 = 0$.

The auxiliary function is

$$F(x, y, z, \lambda, \mu) := f + \lambda g + \mu h = x + 2y + 3z + \lambda(x - y + z - 1) + \mu(x^2 + y^2 - 1).$$

Setting $F_x = F_y = F_y = F_\lambda = F_\mu = 0$, for (x_0, y_0, z_0), we have

$$1 + \lambda + 2x_0\mu = 0,\ 2 - \lambda + 2y_0\mu = 0,\ 3 + \lambda = 0,\ x_0 - y_0 + z_0 - 1 = 0,\ x_0^2 + y_0^2 - 1 = 0.$$

We obtain: $\lambda = -3,\ x_0 = 1/\mu,\ y_0 = -5/(2\mu),\ 1/\mu^2 + 25/(4\mu^2) = 1$. That is, $\mu^2 = 29/4$. Then the extreme points are

$$x_0 = \pm 2/\sqrt{29},\ y_0 = \mp 5/\sqrt{29},\ z_0 = 1 \pm 7/\sqrt{29}.$$

The corresponding values of $f(x_0, y_0, z_0)$ show that the maximum value of f is $3 + \sqrt{29}$.

Notice that if $\mu = 0$, then $1 + \lambda = 0 = 2 - \lambda$ leads to inconsistency. Also the conditions that $\operatorname{grad} g \neq 0$ and $\operatorname{grad} h \neq 0$ are satisfied automatically for the given constraints.

1.12 Review Problems

Probelm 1.1: Where is the function $f(x,y)=\frac{2xy}{x^2+y^2}$ continuous? What are the limits of f at the points of discontinuity?

$f(x,y)$ is defined everywhere in the plane except at the origin. When $(x,y)\neq(0,0)$, the functions $g(x)=2xy$ and $h(x,y)=x^2+y^2$ are continuous. Hence $f(x,y)$ is continuous everywhere except at the origin.
The only point of discontinuity is possibly the origin. We show that as $(x,y)\to(0,0)$, $f(x,y)$ has no limit. On the contrary, suppose $f(x,y)$ has the limit L at $(0,0)$. Then

$$L=\lim_{y=x,x\to 0} f(x,y)=\lim_{x\to 0}\frac{2x^2}{2x^2}=1$$

and also

$$L=\lim_{y=-x,x\to 0} f(x,y)=\lim_{x\to 0}\frac{-2x^2}{2x^2}=-1$$

It is a contradiction.

Problem 1.2: Find the total increment Δz and the total differential dz of the function $z=xy$ at $(2,3)$ for $\Delta x=0.1,\ \Delta y=0.2$.

At $(2,3)$ with $\Delta x=0.1,\ \Delta y=0.2$, we have

$$\Delta z=(x+\Delta x)(y+\Delta y)-xy=y\Delta x+x\Delta y+\Delta x\Delta y=3\times 0.1+2\times 0.2+0.1\times 0.2=0.72.$$

$$dz=z_x dx+z_y dy=ydx+xdy=y\Delta x+x\Delta y.=3\times 0.1+2\times 0.2=0.7.$$

Problem 1.3: It is known that in computing the coordinates of a point (x,y,z,t) certain (small) errors such as $\Delta x,\Delta y,\Delta z,\Delta t$ might have been committed. Find the maximum absolute error so committed when we evaluate a function $f(x,y,z,t)$ at that point.

Let $\Delta u=f(x+\Delta x,y+\Delta y,z+\Delta z,t+\Delta t)-f(x,y,z,t)$. We want to find $\max\Delta u$. By Taylor's formula,

$$\Delta u=(f_x\Delta x+f_y\Delta y+f_z\Delta z+f_t\Delta t)(a,b,c,d)$$

where (a,b,c,d) lies on the line segment joining (x,y,z,t) to $(x+\Delta x,y+\Delta y,z+\Delta z,t+\Delta t)$. Therefore,

$$|\Delta u|\le|f_x|\,|\Delta x|+|f_y|\,|\Delta y|+|f_z|\,|\Delta z|+|f_t|\,|\Delta t|.$$

Problem 1.4: The hypotenuse c and the side a of a right angled triangle ABC determined with maximum absolute errors $|\Delta c|=0.2,\ |\Delta a|=0.1$ are, respectively, $c=75,\ a=32$. Determine the angle A and determine the maximum absolute error ΔA in the calculation of the angle A.

$A(a,c)=\sin^{-1}\dfrac{a}{c}$ gives $\dfrac{\partial A}{\partial a}=\dfrac{1}{\sqrt{c^2-a^2}},\ \dfrac{\partial A}{\partial c}=\dfrac{-a}{c\sqrt{c^2-a^2}}$. Then
$|\Delta A|\le\frac{1}{\sqrt{(75)^2-(32)^2}}\times 0.1+\frac{32}{75\sqrt{(75)^2-(32)^2}}\times 0.2=0.00273.$

Therefore $\sin^{-1}\frac{32}{75}-0.00273\le A\le\sin^{-1}\frac{32}{75}+0.00273.$

Problem 1.5: Let $f(x,y,z)=x^2+y^2+z^2$. Find $\left(\frac{\partial f}{\partial s}\right)_v(1,1,1)$, where $\vec{v}=2\hat{i}+\hat{j}+3\hat{k}$.

The unit vector in the direction of $\vec{v}$ is $\hat{u} = \frac{2}{\sqrt{14}}\hat{i} + \frac{1}{\sqrt{14}}\hat{j} + \frac{3}{\sqrt{14}}\hat{k}$. The gradient of f at $(1,1,1)$ is grad $f(1,1,1) = (f_x\hat{i} + f_y\hat{j} + f_z\hat{k})(1,1,1) = 2\hat{i} + 2\hat{j} + 2\hat{k}$. Then

$$\Big(\frac{\partial f}{\partial s}\Big)_v(1,1,1) = (\text{grad } f \cdot \hat{u})(1,1,1) = \frac{12}{\sqrt{14}}.$$

Problem 1.6: Find a point in the plane where the function $f(x,y) = \frac{1}{2} - \sin(x^2+y^2)$ has a local maximum.

We see that at $(0,0)$, the function has a maximum value of $\frac{1}{2}$. To prove this, consider the neighborhood $B = \{(x,y) : x^2+y^2 \leq \pi/9\}$ of $(0,0)$. Now, for any point $(a,b) \in B$ other than $(0,0)$, we have

$$f(a,b) = \frac{1}{2} - \sin(a^2+b^2) \leq \frac{1}{2} = f(0,0).$$

Problem 1.7: Decompose a given positive number a into three parts to make their product maximum.

Let $a = x + y + (a-x-y)$, for $0 \leq x, y, a-x-y \leq a$. Then x and y can take values from the region D bounded by the straight lines $x = 0, y = 0$ and $x + y = a$. The function to be maximized is

$$f(x,y) = xy(a-x-y)$$

defined from D to $\mathbb{R}$. The partial derivatives of f are continuous everywhere on D. They are

$$f_x = y(a-2x-y), \; f_y = x(a-x-2y).$$

The critical points satisfy $y(a-2x-y) = 0, \; x(a-x-2y) = 0$.

The solutions of these equations give:

$$P_1 = (0,0), \; P_2 = (0,a), \; P_3 = (a,0), \; P_4 = (\frac{a}{3}, \frac{a}{3}).$$

Of these, the points P_1, P_2, P_3 are on the boundary of D, where the value of $f(x,y)$ is zero. The only interior point is P_4, where the value of $f(x,y,z) = \frac{a^3}{27}$, which is the maximum value of $f(x,y,z)$. Comparing $f(P_1), \; f(P_2), \; f(P_3), f(P_4)$, we get the required decomposition of a as $a = \frac{a}{3} + \frac{a}{3} + \frac{a}{3}$.

Problem 1.8: Test for maxima-minima the function $z = x^3 + y^3 - 3xy$.

The function is differentiable everywhere. Thus the critical points are obtained by solving

$$z_x = 3x^2 - 3y = 0, \; z_y = 3y^2 - 3x = 0.$$

These are $P_1 = (1,1)$ and $P_2 = (0,0)$.

The second derivatives are $z_{xx} = 6x, \; z_{xy} = -3, \; z_{yy} = 6y$.

For P_1, $H(P_1) = (z_{xx}z_{yy} - z_{xy}^2)(P_1) = 36 - 9 = 27 > 0, \; z_{xx}(P_1) = 6 > 0$. Thus, P_1 is a minimum point and $z_{min} = -1$.

For P_2, $H(P_2) = (z_{xx}z_{yy} - z_{xy}^2)(P_2) = -9 < 0$. Hence P_2 is a saddle point.

Problem 1.9: Find the maximum of $w = xyz$ given that $xy + zx + yz = a$ for a given positive number a, and $x > 0,\ y > 0,\ z > 0$.

The auxiliary function is

$$F(x, y, z, \lambda) = xyz + \lambda(xy + zx + yz - a).$$

Equating its partial derivatives to zero, we have

$$yz + \lambda(y + z) = 0,\ xz + \lambda(x + z) = 0,\ xy + \lambda(x + y) = 0.$$

Multiply the first by x, the second by y, and subtract to obtain:

$$\lambda x(y + z) - \lambda y(x + z) = 0 \Rightarrow \lambda z(x - y) = 0.$$

If $\lambda = 0$, then $xy + \lambda(x + y) = 0$ would imply $x = 0$ or $y = 0$. But $x > 0$ and $y > 0$. So, $\lambda \neq 0$. Also, $z > 0$. Therefore, $x = y$. Similarly, using the second and third equations, we get $y = z$. Therefore, $x = y = z$. Then

$$xy + zx + yz = a \ \text{ gives } \ x = y = z = \sqrt{a/3}.$$

The corresponding value of w cannot be minimum, since by reducing x, y close to 0, and taking z close to a so that $xy + zx + yz = a$ is satisfied, w can be made as small as possible. Hence w has a maximum at $(\sqrt{a/3}, \sqrt{a/3}, \sqrt{a/3})$. The maximum value of w is $(a/3)^3$.

Problem 1.10: Determine the maximum value of $z = (x_1 \cdots x_n)^{1/n}$ provided that $x_1 + \cdots + x_n = a$, where a is a given positive number.

Maximizing z is equivalent to maximizing $f(x_1, \ldots, x_n) = z^n = x_1 x_2 \cdots x_n$. Set up the auxiliary function

$$F(x_1, \ldots, x_n, \lambda) = x_1 x_2 \cdots x_n + \lambda(x_1 + \cdots x_n - a).$$

Equate the partial derivatives F_{x_i} to zero to obtain

$$x_1 \cdots x_{i-1} x_{i+1} \cdots x_n + \lambda = 0 \ \text{ for } \ i = 1, 2, \ldots, n.$$

Notice that $\lambda \neq 0$. Then multiplying by x_i, we see that $\lambda x_i = x_1 x_2 \cdots x_n$ for each i. Therefore, $x_1 = x_2 = \cdots = x_n = a/n$. In that case, $f = (a/n)^n$ and $z = a/n$. This value is not a minimum value of z since z can be made arbitrarily small by choosing x_1 close to 0. Thus, the maximum of z is a/n.

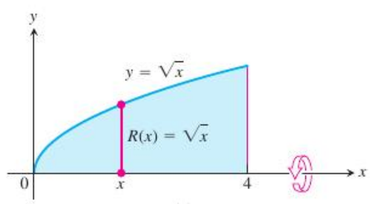
This gives an alternative proof that the geometric mean of n positive numbers is no more than the arithmetic mean of those numbers.

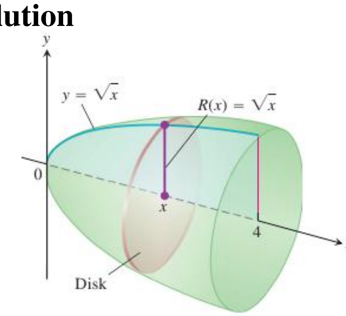
Chapter 2

Multiple Integrals

2.1 Volume of a solid of revolution

The solid obtained by rotating a plane region about a straight line in the same plane is called a **solid of revolution**. The line is called the **axis of revolution**





Suppose the region is bounded above by the curve $y = f(x)$ and below by the x-axis, where $a \leq x \leq b$. To find the volume of the solid so generated, we divide the interval $[a, b]$ into n equal parts. Let the partition be

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

On the ith subinterval we approximate the slice of the solid by $\pi[f(x_i^*)]^2(x_i - x_{i-1})$ for a point $x_i^* \in [x_{i-1}, x_i]$. Reason: the slice is a portion of a cylinder whose cross section with a plane vertical to its axis is a circle. Then the volume of the solid of revolution is approximated by the sum

$$\sum_{i=1}^{n} \pi[f(x_i^*)]^2(x_i - x_{i-1}).$$

Then the volume of the solid of revolution is the limit of the above sum where $n \to \infty$. Observe that the cross sectional area for $x \in [a, b]$ is $A(x) = \pi(f(x))^2$. If $A(x)$ is a continuous function of x, then the limit of the above sum is the required volume; that is,

$$V = \int_a^b A(x)\, dx = \int_a^b \pi[f(x)]^2\, dx.$$

If the axis of revolution is a straight line other than the x-axis, similar formulas can be obtained for the volume.

Example 2.1. The region between the curve $y = \sqrt{x},\ 0 \le x \le 4$ and the x-axis is revolved around x-axis. Find the volume of the solid of revolution.

As shown in the above figure, the required volume is

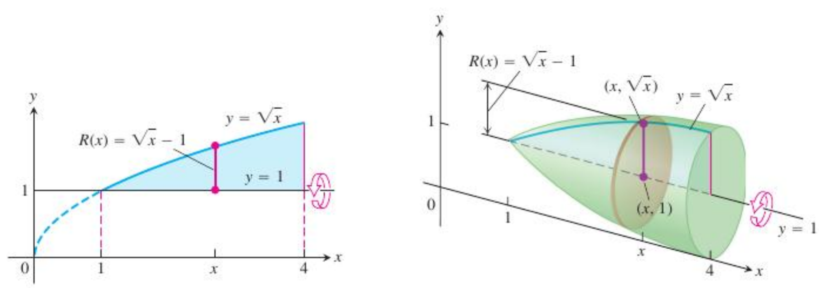
$$V = \int_0^4 \pi(\sqrt{x})^2\,dx = \int_0^4 \pi\,x\,dx = \pi\Big[\frac{x^2}{2}\Big]_0^4 = 8\pi.$$

Example 2.2. Find the volume of the sphere $x^2 + y^2 + z^2 = a^2,\ a > 0$.

We think of the sphere as the solid of revolution of the region bounded by the upper semi-circle $x^2 + y^2 = a^2,\ y \ge 0$. Here, $-a \le x \le a$. The curve is thus $y = \sqrt{a^2 - x^2}$. Then the volume of the sphere is

$$V = \int_{-a}^{a} \pi(\sqrt{a^2 - x^2})^2\,dx = \int_{-a}^{a} \pi(a^2 - x^2)\,dx = \pi\Big[a^2x - \frac{x^3}{3}\Big]_{-a}^{a} = \frac{4}{3}\pi a^3.$$

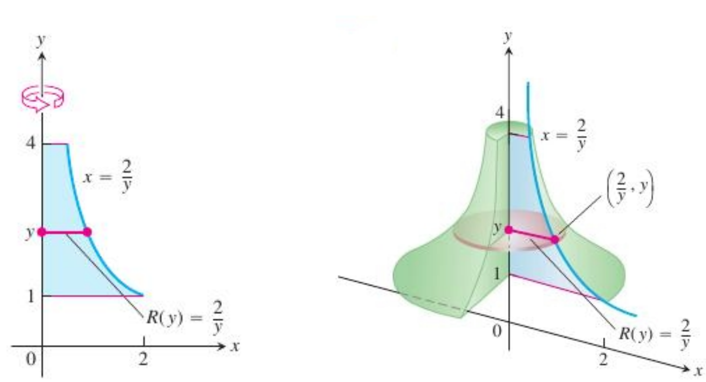
Example 2.3. Find the volume of the solid obtained by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1,\ x = 4$ about the line $y = 1$.



The required volume is

$$V = \int_1^4 \pi[R(x)]^2\,dx = \int_1^4 \pi(\sqrt{x} - 1)^2\,dx = \int_1^4 \pi(x - 2\sqrt{x} + 1)\,dx = \frac{7\pi}{6}.$$

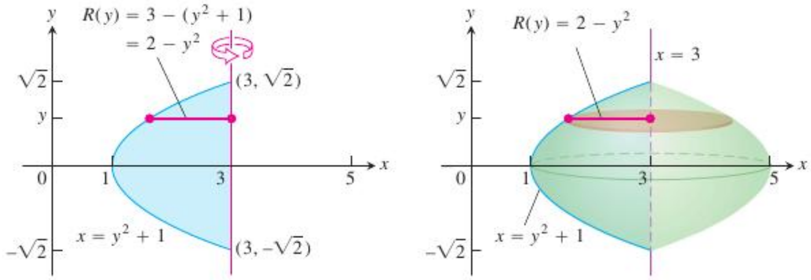
Example 2.4. Find the volume of the solid generated by revolving the region between the y-axis and the curve $xy = 2,\ 1 \le y \le 4$, about the y-axis.



The volume is

$$V = \int_1^4 \pi[R(y)]^2\,dy = \pi\int_1^4 \frac{4}{y^2}\,dy = 3\pi.$$

Example 2.5. Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.

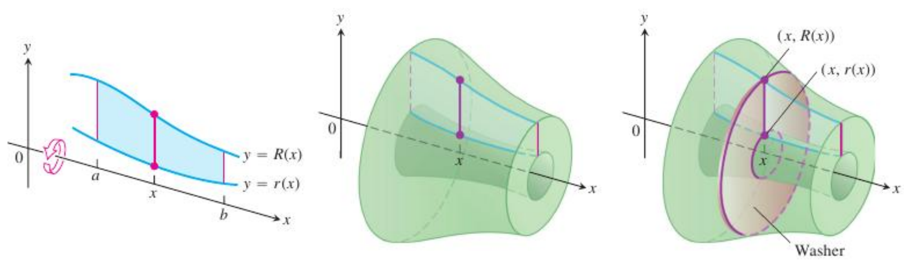


Notice that the cross sections are perpendicular to the axis of revolution: $x = 3$.

The volume is

$$V = \int_{-\sqrt{2}}^{\sqrt{2}} \pi[R(y)]^2 dy = \int_{-\sqrt{2}}^{\sqrt{2}} \pi[2 - y^2]^2 dy = \frac{64\pi\sqrt{2}}{15}.$$

If the region which revolves does not border the axis of revolution, then there are **holes** in the solid.



In this case, we subtract the volume of the hole to obtain the volume of the solid of revolution. Look at the figure. In this case, the volume of the the solid of revolution is given by

$$V = \int_a^b A(x)\, dx = \int_a^b \pi[(R(x))^2 - (r(x))^2]\, dx.$$

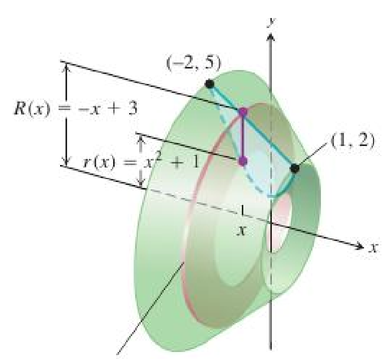
Example 2.6. The region bounded by the curve $y = x^2 + 1$ and the line $x + y = 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

The outer radius of the washer is $R(x) = -x + 3$ and the inner radius is $r(x) = x^2 + 1$. The limits of integration are obtained by finding the points of intersection of the given curves:

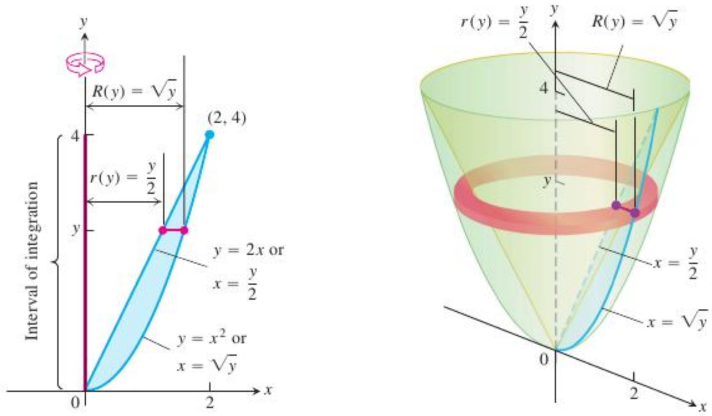
$$x^2 + 1 = -x + 3 \Rightarrow x = -2,\ 1.$$

The required volume is

$$V = \int_{-2}^{1} \pi[(-x + 3)^2 - (x^2 + 1)^2]\, dx = \frac{117\pi}{5}.$$



Example 2.7. Find the volume of the solid obtained by revolving the region bounded by the curves $y = x^2$ and $y = 2x$, about the y-axis.

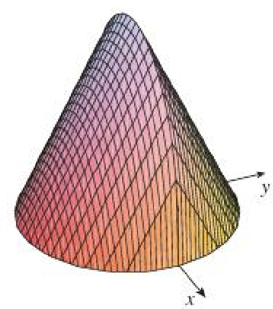


The given curves intersect at $y = 0$ and $y = 4$. The required volume is

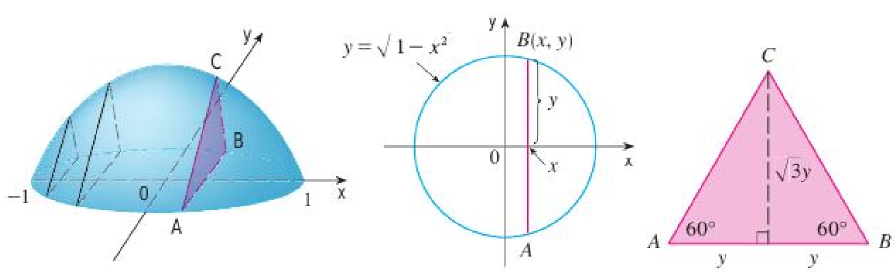
$$V = \int_0^4 [(R(y))^2 - (r(y))^2]\, dy = \int_0^4 \pi[(\sqrt{y})^2 - (y/2)^2]\, dy = \frac{8\pi}{3}.$$

Example 2.8.

In the figure is shown a solid with a circular base of radius 1. Parallel cross sections perpendicular to the base are equilateral triangles. Find the volume of the solid.



Take the base of the solid as the disk $x^2+y^2 \leq 1$. The solid, its base, and a typical triangle at a distance x from the origin are shown in the figure below.



The point B lies on the circle $y = \sqrt{1-x^2}$. So, the length of AB is $2\sqrt{1-x^2}$. Since the triangle is equilateral, its height is $\sqrt{3}\sqrt{1-x^2}$. The cross sectional area is

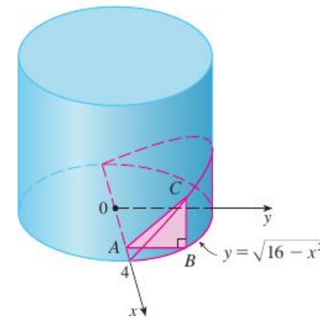
$$A(x) = \frac{1}{2}\, 2\,\sqrt{1-x^2}\,\sqrt{3}\,\sqrt{1-x^2} = \sqrt{3}\,(1-x^2).$$

Thus, the volume of the solid is

$$V = \int_{-1}^{1} A(x)\, dx = \int_{-1}^{1} \sqrt{3}\,(1-x^2)\, dx = \frac{4}{\sqrt{3}}.$$

Example 2.9.

A wedge is cut out of a circular cylinder of radius 4 by two planes. One plane is perpendicular to the axis of the cylinder. The other intersects the first at an angle of $30°$ along a diameter of the cylinder. Find the volume of the wedge.



If we place the x-axis along the diameter where the planes meet, then the base of the solid is a semicircle with equation

$$y = \sqrt{16 - x^2}, \ -4 \le x \le 4.$$

A cross-section perpendicular to the x-axis at a distance x from the origin is the triangle ABC, whose base is $y = \sqrt{16 - x^2}$; its height is $|BC| = y \tan 30° = \sqrt{16-x^2}/\sqrt{3}$. Thus the cross sectional area is

$$A(x) = \frac{1}{2}\sqrt{16-x^2}\,\frac{\sqrt{16-x^2}}{\sqrt{3}} = \frac{16-x^2}{2\sqrt{3}}.$$

Then the required volume of the wedge is

$$V = \int_{-4}^{4} A(x)\,dx = \int_{-4}^{4} \frac{16-x^2}{2\sqrt{3}}\,dx = \frac{128}{3\sqrt{3}}.$$

Example 2.10. Find the volume of the solid generated by revolving about the x-axis the region bounded by the curve $y = 4/(x^2+4)$ and the lines $x = 0, \ x = 2, \ y = 0$.

The volume is $V = \displaystyle\int_0^2 \pi \frac{16}{(x^2+4)^2}\,dx.$

Substitute $x = 2\tan t$. $dx = 2\sec^2 t\,dt$, $(x^2+4)^2 = 16\sec^4 t$ for $0 \le t \le \pi/4$. So,

$$V = \int_0^{\pi/4} 16\pi \frac{2\sec^2 t}{16\sec^4 t}\,dt = \int_0^{\pi/4} 2\pi\cos^2 t\,dt = \pi\Big(\frac{\pi}{4} + \frac{1}{2}\Big).$$

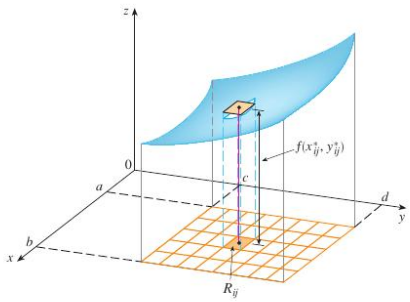
2.2 Approximating Volume

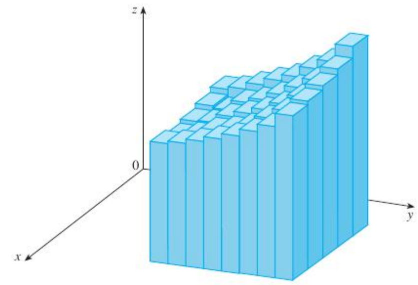
We now consider solids which are not necessarily solids of revolution. First, we take a typical simpler case, when a given solid has all plane faces except one, which is a portion of a surface given by a function $f(x, y)$.

Let $f(x, y)$ be defined on the rectangle $R: \ a \le x \le b, \ c \le y \le d$.
For simplicity, take $f(x, y) \ge 0$. The graph of f is the surface $z = f(x, y)$. We approximate the volume of the solid

$$S: \ \{(x, y, z) : (x, y) \in R, \ 0 \le z \le f(x, y)\}$$

by partitioning R and then adding up the volumes of the solid rods:





So, consider a **partition** of R as

$$P: \; R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \text{ for } 1 \le i \le m, \; 1 \le j \le n, \; a = x_0, \; b = x_m, \; c = y_0, \; d = y_n.$$

Denote by $A(R_{ij})$ the area of the rectangle R_{ij}; Denote by $\|P\| = \max A(R_{ij})$, the **norm** of P.

Choose **sample points** $(x_i^*, y_j^*) \in R_{ij}$. An approximation to the volume of S is the **Riemann sum**

$$S_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) A(R_{ij}).$$

If limit of S_{mn} exists as $\|P\| \to 0$, then this limit is called the **double integral** of $f(x, y)$. It is denoted by $\iint_R f(x, y) dA$. Whenever the integral exists, it is also enough to consider **uniform partitions**, that is, $x_i - x_{i-1} = (b - a)/m = \Delta x$ and $y_j - y_{j-1} = (d - c)/n = \Delta y$. In this case, we write $A(R_{ij}) = \Delta A = \Delta x \Delta y$. Then

$$\iint_R f(x, y) dA = \lim_{\|P\| \to 0} S_{mn} = \lim_{m \to \infty} \lim_{n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) \Delta A.$$

Since $f(x, y) \ge 0$, the value of this integral is the volume of the solid S bounded by the rectangle R and the surface $z = f(x, y)$.

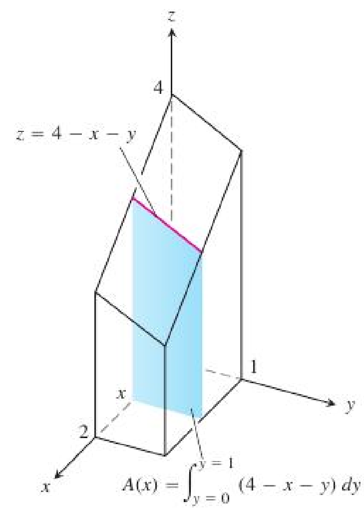
When the integral of $f(x, y)$ exists, we say that f is **Riemann integrable** or just **integrable**.

Riemann sum is well defined even if f is not a positive function. However, the double integral computes the signed volume. Analogous to the single variable case, we have the following result; we omit its proof.

Theorem 2.1. *Each continuous function defined on a closed bounded rectangle is integrable.*

Volumes of solids can also be calculated by using iterated integrals.

Example 2.11. Find the volume V of the solid raised over the rectangle $R: \; [0, 1] \times [0, 2]$ and bounded above by the plane $z = 4 - x - y$, we proceed as follows (similar to solids of revolution):



Suppose $A(x)$ is the cross sectional are at x. Then $V = \int_0^1 A(x)dx$. Now, $A(x) = \int_0^2 (4-x-y)dy$. Thus, $V = \int_0^1 \int_0^2 (4-x-y)dydx$. Therefore,

$$\iint_R (4-x-y)dA = \int_0^1 \int_0^2 (4-x-y)dydx.$$

The expression on the left is a double integral and on the right is an **iterated integral**.

Theorem 2.2. (Fubini) *Let R be the rectangle $[a,b] \times [c,d]$. Let $f : R \to \mathbb{R}$ be a continuous function. Then*

$$\iint_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy.$$

Example 2.12. Evaluate $\iint_R (1-6x^2y)dA$, where $R = [0,2] \times [-1,1]$.

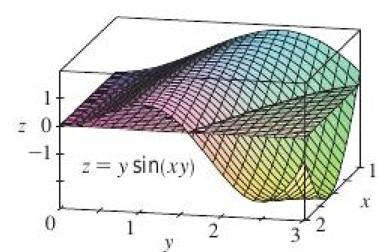
$$\iint_R (1-6x^2y)dA = \int_{-1}^1 \int_0^2 (1-6x^2y)dxdy = \int_{-1}^1 (2-16y)dy = 4.$$

Also, reversing the order of integration, we have

$$\iint_R (1-6x^2y)dA = \int_0^2 \int_{-1}^1 (1-6x^2y)dydx = \int_0^2 2dx = 4.$$

Example 2.13. Evaluate $\iint_R y\sin(xy)dA$, where $R = [1,2] \times [0,\pi]$.

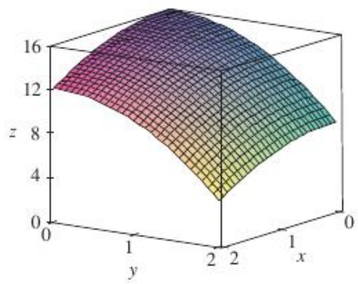
$\iint_R y\sin(xy)dA = \int_0^\pi \int_1^2 y\sin(xy)\,dxdy = \int_0^\pi (-\cos 2y + \cos y)dy = 0.$



The volume of the solid above R and below the surface $z = y\sin(xy)$ is the same as the volume below R and above the surface. Therefore, the net volume is zero.

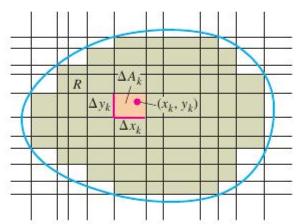
Example 2.14. Find the volume of the solid bounded by the elliptic paraboloid $x^2+2y^2+z=16$, planes $x=2$ and $y=2$, and the three coordinate planes.

Let R be the rectangle $[0,2]\times[0,2]$. The solid is above R and below the surface defined by $z=f(x,y)=16-x^2-2y^2$, where f is defined on R.

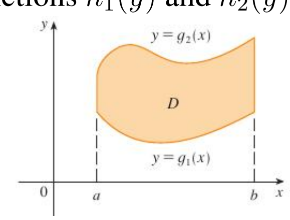


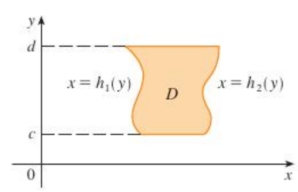
$$V=\iint_R(16-x^2-2y^2)dA=\int_0^2\int_0^2(16-x^2-2y^2)dxdy=48.$$

The double integrals can be extended to functions defined on non-rectangular regions. Essentially, the approach is the same as earlier. We partition the region into smaller rectangles, form the Riemann sum, take its limit as the norm of the partition goes to zero.



The double integral of f over such a bounded region R can also be evaluated using iterated integrals. Look at R bounded by two continuous functions $g_1(x)$ and $g_2(x)$; or, as a region bounded by two continuous functions $h_1(y)$ and $h_2(y)$.



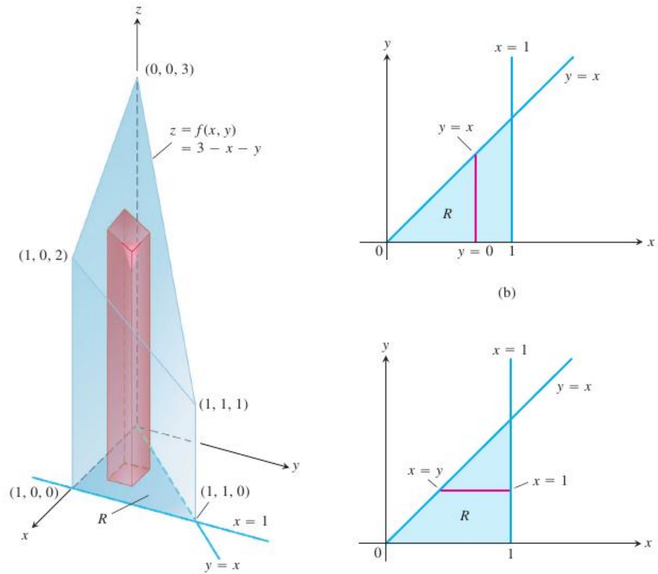


Theorem 2.3. *Let $f(x,y)$ be a continuous real valued function on a region R.*

1. *If R is given by $a\le x\le b,\ g_1(x)\le y\le g_2(x)$, where $g_1,g_2:[a,b]\to\mathbb{R}$ are continuous, then*
$$\iint_R f(x,y)dA=\int_a^b\int_{g_1(x)}^{g_2(x)}f(x,y)dydx.$$

2. *If R is given by $c\le y\le d,\ h_1(y)\le x\le h_2(y)$, where $h_1,h_2:[c,d]\to\mathbb{R}$ are continuous, then*
$$\iint_R f(x,y)dA=\int_c^d\int_{h_1(y)}^{h_2(y)}f(x,y)dxdy.$$

Example 2.15. Find the volume of the prism whose base is the triangle in the xy-plane bounded by the lines $y = 0,\ x = 1$ and $y = x$, and whose top lies in the plane $z = 3 - x - y$.

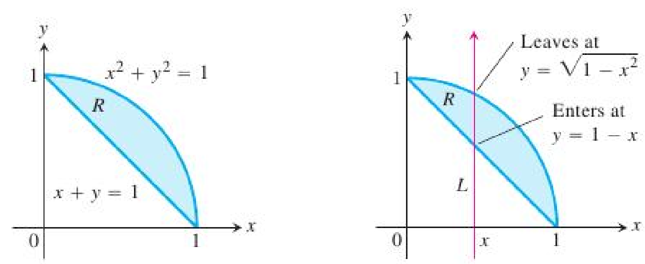


$$V = \int_0^1 \int_0^x (3 - x - y)dydx = \int_0^1 (3x - 3x^2/2)dx = 1.$$

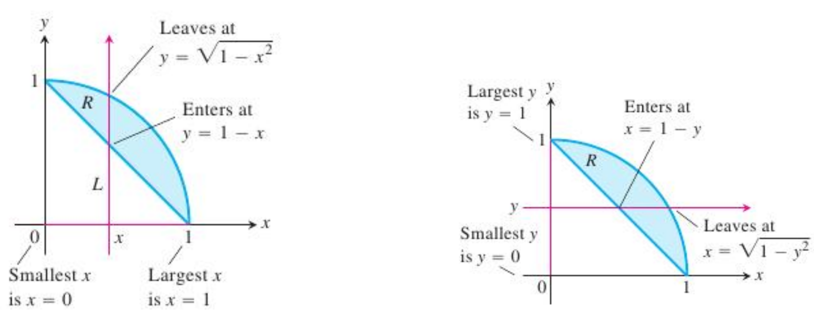
Also,

$$V = \int_0^1 \int_y^1 (3 - x - y)dxdy = \int_0^1 (5/2 - 4y + 3y^2/2)dy = 1.$$

Suppose R is the region bounded by the line $x + y = 1$ and the portion of the circle $x^2 + y^2 = 1$ in the first quadrant. Sketch it and then find the limits:



Write the appropriate integrals.



$$\iint_R f(x, y)dA = \int_0^1 \int_{1-x}^{\sqrt{1-x^2}} f(x, y)dydx = \int_0^1 \int_{1-y}^{\sqrt{1-y^2}} f(x, y)dxdy.$$

For evaluating a double integral as an iterated integral, choose some order: first x, next y. If it does not work, or if it is complicated, you may have to choose the reverse order.

Example 2.16. Evaluate $\iint_R \frac{\sin x}{x} dA$, where R is the triangle in the xy-plane bounded by the lines $y = 0,\ x = 1$, and $y = x$.

Here, the triangular region R can be expressed as $\{(x, y) : 0 \le y \le 1, y \le x \le 1\}$. So,

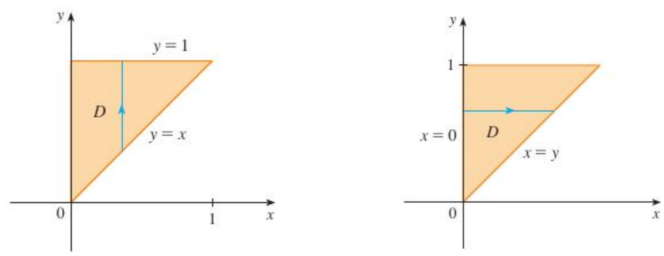
$$\iint_R \frac{\sin x}{x} dA = \int_0^1 \int_y^1 \frac{\sin x}{x} dxdy.$$

We are stuck. No way to proceed further. On the other hand, we express the same R in a different way: $\{(x, y) : 0 \le y \le x, 0 \le x \le 1\}$. Then

$$\iint_R \frac{\sin x}{x} dA = \int_0^1 \int_0^x \frac{\sin x}{x} dydx = \int_0^1 \Big(\frac{\sin x}{x} \int_0^x dy\Big) dx = \int_0^1 \sin x dx = -\cos(1) + 1.$$

Example 2.17. Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2) dydx$.

Write $D :\ \ 0 \le x \le 1,\ x \le y \le 1$. We plan to change the order of integration.



$$\begin{aligned}
\int_0^1 \int_x^1 \sin(y^2) dydx &= \iint_D \sin(y^2) dA \\
&= \int_0^1 \int_0^y \sin(y^2) dxdy \\
&= \int_0^1 y \sin(y^2) dy = \frac{1}{2}(1 - \cos(1).
\end{aligned}$$

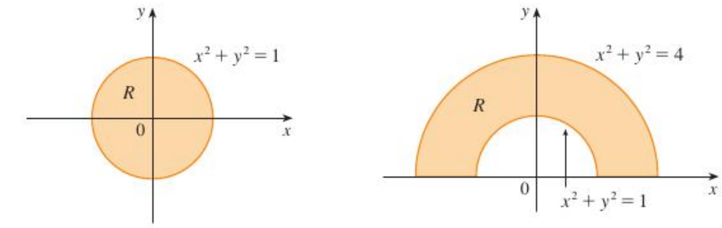
Properties of double integrals with respect to addition, multiplication etc. are as follows.

Theorem 2.4. *Let $f(x, y)$ and $g(x, y)$ be continuous on a domain D. Let c be a constant.*

1. (Constant Multiple)*:* $\iint_D cf(x, y)dA = c \iint_D f(x, y)dA$.
2. (Sum-Difference)*:* $\iint_D [f(x, y) \pm g(x, y)]dA = \iint_D f(x, y)dA \pm \iint_D g(x, y)dA$.
3. (Additivity)*:* $\iint_{D\cup R} f(x, y)dA = \iint_D f(x, y)dA + \iint_R f(x, y)dA$,
   provided $f(x, y)$ is continuous on a domain R also, and D and R are non-overlapping.
4. (Domination)*: If $f(x, y) \le g(x, y)$ in D, then $\iint_D f(x, y)dA \le \iint_D g(x, y)dA$.*
5. (Area)*: $\iint_D 1\, dA = \Delta(D) =$ Area of D.*
6. (Boundedness)*: If $m \le f(x, y) \le M$ in D, then $m\Delta(D) \le \iint_D f(x, y)dA \le M\Delta(D)$.*

2.3 Riemann Sum in Polar coordinates

Suppose R is one of the following regions in the plane:

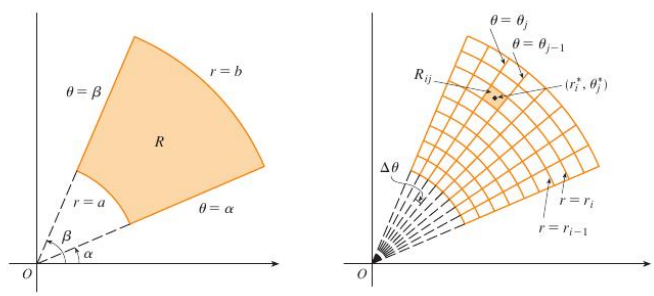


(a) $R = \{(r, \theta) \mid 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$ (b) $R = \{(r, \theta) \mid 1 \leq r \leq 2, 0 \leq \theta \leq \pi\}$

It is easy to describe such regions in polar coordinates. Using polar coordinates, we define a **polar rectangle** as a region given in the form:

$$R = \{(r, \theta) : a \leq r \leq b,\ \alpha \leq \theta \leq \beta, \beta - \alpha \leq 2\pi\}.$$

We can divide a polar rectangle into polar subrectangles as in the following:



$$R_{ij} = \{(r, \theta) : r_{i-1} \leq r \leq r_i,\ \theta_{j-1} \leq \theta \leq \theta_j\}.$$

Suppose f is a real valued function defined on a polar rectangle R. Let P be a partition of R into smaller polar rectangles R_{ij}. The area of R_{ij} is

$$\Delta(R_{ij}) = \frac{1}{2}(r_i^2 - r_{i-1}^2)(\theta_j - \theta_{j-1}).$$

Take a uniform grid dividing r into m equal parts and θ into n equal parts. Write $r_i - r_{i-1} = \Delta r$ and $\theta_j - \theta_{j-1} = \Delta\theta$. Also write the mid-point of r_{i-1} and r_i as $r_i^* = \frac{1}{2}(r_i + r_{i-1})$, similarly, $\theta_j^* = \frac{1}{2}(\theta_{j-1} + \theta_j)$. Then the Riemann sum for $f(x, y)$ in Cartesian coordinates can be written as

$$S = \sum_{i=1}^{m} \sum_{j=1}^{n} f(r_i^*, \theta_j^*)\Delta(R_{ij}) = \sum_{i=1}^{m} \sum_{j=1}^{n} f(r_i^*, \theta_j^*) r_i^* \Delta r \Delta\theta.$$

Therefore, if $f(r, \theta)$ is continuous on the polar rectangle R, then

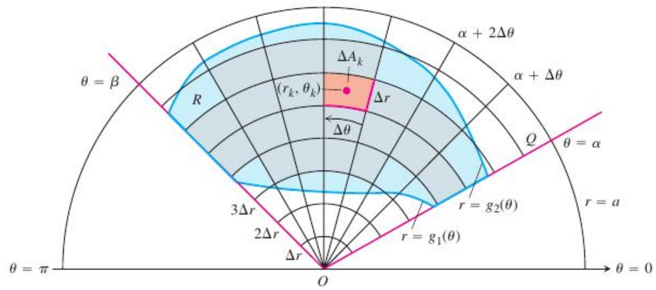
$$\iint_R f(r, \theta) dA = \iint_R f(r, \theta) r\, dr\, d\theta$$

If $f(x, y)$ is continuous on the polar rectangle R, then converting this into polar form, we have

$$\iint_R f(x, y) dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r\, dr\, d\theta.$$

The double integral in polar form can be generalized to functions defined on regions other than polar rectangles. Let f be a continuous function defined over a region bounded by the rays

$\theta = \alpha,\ \theta = \beta$ and the continuous curves $r = g_1(\theta),\ r = g_2(\theta)$.



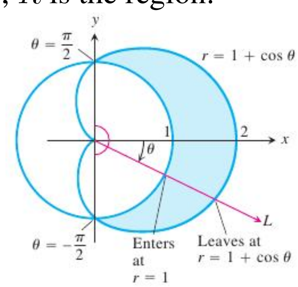
Then

$$\iint_R f(r,\theta)dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r,\theta)\, r\, dr\, d\theta.$$

Caution: Do not forget the r on the right hand side.

Example 2.18. Find the limits of integration for integrating $f(r,\theta)$ over the region R that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $x^2 + y^2 = 1$.

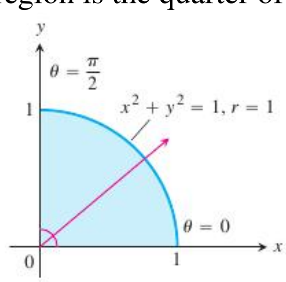
Better write the circle as $r = 1$. Now, R is the region:



$$\iint_R f(r,\theta)dA = \int_{-\pi/2}^{\pi/2} \int_1^{1+\cos\theta} f(r,\theta) r\, dr\, d\theta.$$

Example 2.19. Evaluate $I = \int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy dx$.

The limits of integration say that the region is the quarter of the unit disk in the first quadrant:



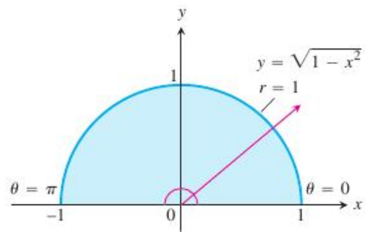
The region in polar coordinates is $R:\ \ 0 \le r \le 1,\ 0 \le \theta \le \pi/2$.
Changing to polar coordinates, we have $x = r\cos\theta, y = r\sin\theta$ and then

$$I = \int_0^1 \int_0^{\pi/2} r^2\, r\, dr\, d\theta = \int_0^{\pi/2} \frac{1}{4}\, d\theta = \frac{\pi}{8}.$$

Example 2.20. Evaluate $I = \int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} e^{x^2+y^2}\,dydx.$

The region is the upper semi-unit-disk, whose polar description is

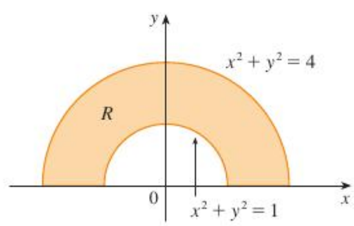
$$R = \{(r,\theta) : 0 \le r \le 1,\ 0 \le \theta \le \pi\}.$$



Then $I = \iint_R e^{x^2+y^2}dA$. Using integration in polar form,

$$I = \int_0^{\pi}\int_0^{1} e^{r^2} r\,dr\,d\theta = \int_0^{\pi}\left[\frac{1}{2}e^{r^2}\right]_0^1 d\theta = \int_0^{\pi}\frac{e-1}{2}\,d\theta = \frac{\pi}{2}(e-1).$$

Example 2.21. Evaluate $\iint_R (3x+4y^2)dA$, where R is the region in the upper half plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.



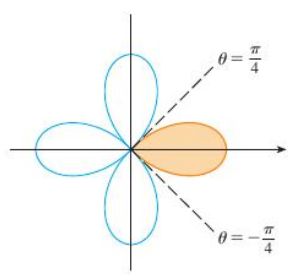
$R = \{(r,\theta) : 1 \le r \le 2,\ 0 \le \theta \le \pi\}$. Therefore,

$$\iint_R (3x+4y^2)dA = \int_0^{\pi}\int_1^{2}(3r\cos\theta + 4r^2\sin^2\theta)r\,dr\,d\theta$$

$$= \int_0^{\pi}\left[r^3\cos\theta + r^4\sin^2\theta\right]_1^2 d\theta = \int_0^{\pi}(7\cos\theta + 15\sin^2\theta)\,d\theta = \frac{15\pi}{2}.$$

Example 2.22. Find the area enclosed by one of the four leaves of the curve $r = \cos(2\theta)$.

The region is $R = \{(r,\theta) : -\pi/4 \le \theta \le \pi/4,\ 0 \le r \le \cos(2\theta)\}$.

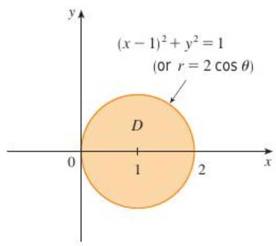


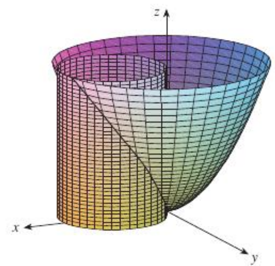
Then the required area is

$$\iint_R dA = \int_{-\pi/4}^{\pi/4}\int_0^{\cos(2\theta)} r\,dr\,d\theta = \int_{-\pi/4}^{\pi/4}\frac{\cos^2(2\theta)-1}{2}\,d\theta = \int_{-\pi/4}^{\pi/4}\frac{cos(4\theta)-1}{4}\,d\theta = \frac{\pi}{8}.$$

Example 2.23. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

The solid lies above the disk D whose boundary has equation $x^2+y^2 = 2x$, or in polar coordinates, $r^2 = 2r\cos\theta$, or $r = 2\cos\theta$.





The disk $D = \{(r,\theta) : -\pi/2 \le \theta \le \pi/2,\ 0 \le r \le 2\cos\theta\}$.

Then the required volume V is given by

$$V = \iint_D (x^2+y^2)dA = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^2\, r\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} 4\cos^4\theta\, d\theta = \int_{-\pi/2}^{\pi/2}(3+\cos 4\theta+4\cos 2\theta)\, d\theta = \frac{3\pi}{2}.$$

2.4 Triple Integral

Let $f(x, y, z)$ be a real valued function defined on a bounded region D in $\mathbb{R}^3$. As earlier we divide the region into smaller cubes enclosed by planes parallel to the coordinate planes. The set of these smaller cubes is called a partition P. The norm of the partition is the maximum volume enclosed by any smaller cube. Then form the Riemann sum S and take its limit as the cubes become smaller and smaller. If the limit exists, we say that the limit is the triple integral of the function over the domain D.

$$\iiint_D f(x,y,z)dV = \lim_{\|P\|\to 0}\sum f(x_i^*, y_j^*, z_k^*)(x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1}),$$

where (x_i^*, y_j^*, z_k^*) is a point in the (i, j, k)-th cube in the partition.

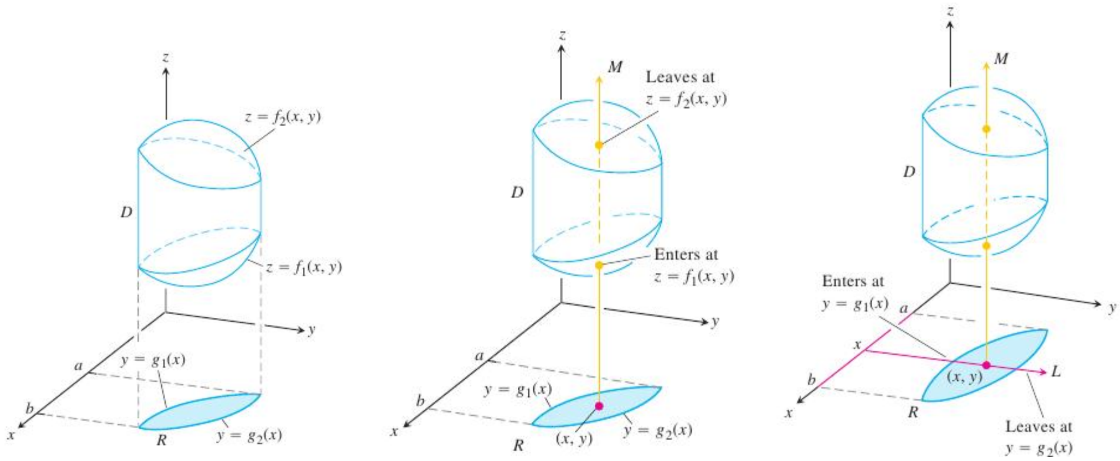
As earlier, Fubuni's theorem says that for continuous functions, if the region D can be written as

$$D = \{(x,y,z) : a \le x \le b,\ g_1(x) \le y \le g_2(x),\ h_1(x,y) \le z \le h_2(x,y)\},$$

then the triple integral can be written as an iterated integral:

$$\iiint_D f(x,y,z)dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)dz\, dy\, dx.$$

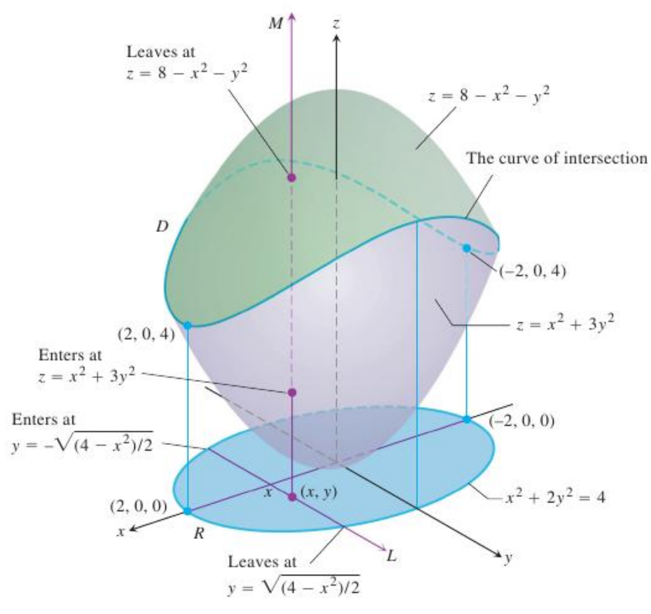
To find the limits of integration, we first sketch the region D along with its shadow on the xy-plane. Next, we find the z-limits, then y-limits and then x-limits.



Observe that the volume of D is $\iiint_D 1\, dV$.

All properties for double integrals hold analogously for triple integrals.

Example 2.24. Find the volume of the solid enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.



Eliminating z from the two equations, we get the projection of the solid on the xy-plane, which is $x^2 + 2y^2 = 4$. This gives the limits of integration for y as $\mp\sqrt{(4 - x^2)/2}$. Clearly, $-2 \leq x \leq 2$.

Therefore,

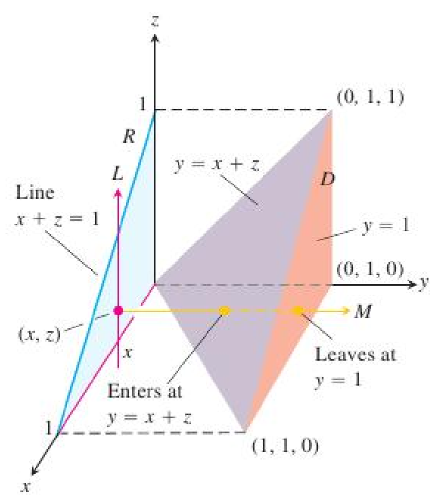
$$\begin{aligned}
V &= \iiint_D dV = \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx \\
&= \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (8-2x^2-4y^2)\,dy\,dx \\
&= \int_{-2}^{2}\Big[(8-2x^2)y - \frac{4}{3}y^3\Big]_{y=-\sqrt{(4-x^2)/2}}^{y=\sqrt{(4-x^2)/2}} dx \\
&= \int_{-2}^{2}\Big[8\Big(\frac{4-x^2}{2}\Big)^{3/2} - \frac{8}{3}\Big(\frac{4-x^2}{2}\Big)^{3/2}\Big]\,dx \\
&= \frac{4\sqrt{2}}{3}\int_{-2}^{2}(4-x^2)^{3/2}\,dx = 8\pi\sqrt{2}.
\end{aligned}$$

Notice that changing the order of integration involves expressing the domain by choosing different order of the limits of values in the axes.

Example 2.25. Write the integral of $f(x,y,z)$ over a tetrahedron with vertices at $(0,0,0)$, $(1,1,0)$, $(0,1,0)$, and $(0,1,1)$ as an iterated integral.

First, sketch the region D to see the limits geometrically. The right hand side bounding surface of D lies in the plane $y=1$. The left hand side bounding surface lies in the plane $y=z+x$. The projection of D on the zx-plane is R. The upper boundary of R is the line $z=1-x$. The lower boundary of R is the line $z=0$.

To find the y-limits for D, we consider a typical point (x,z) in R and a line through this point parallel to y-axis. It enters D at $y=x+z$ and leaves at $y=1$.
To find the z-limits for D, we find that the line L through (x,z) parallel to z-axis enters R at $z=0$ and leaves R at $z=1-x$.



Finally, as L sweeps across R the value of x varies from $x=0$ to $x=1$.

Therefore, $D=\{(x,y,z): 0\le x\le 1, 0\le z\le 1-x, x+z\le y\le 1\}$.

Thus the triple integral of a function $f(x, y, z)$ over D is given by

$$\iiint_D f(x, y, z)\, dV = \int_0^1 \int_0^{1-x} \int_{x+z}^1 f(x, y, z)\, dy\, dz\, dx.$$

If we interchange the orders of y and z, then first we consider limits for z and then of y. In this case, we project D on the xy-plane. A line parallel to z-axis through (x, y) in the xy-plane enters D at $z = 0$ and leaves D through the upper plane $z = y - x$.

For the y-limits, on the xy-plane, where $z = 0$, the sloped side of D crosses the plane along the line $y = x$. A line through (x, y) parallel to y-axis enters the xy-plane at $y = x$ and leaves at $y = 1$. The x-limits are as earlier.

Therefore $D = \{(x, y, z) : 0 \le x \le 1, x \le y \le 1, 0 \le z \le y - x\}$.

The same triple integral is rewritten as the following iterated integral:

$$\iiint_D f(x, y, z)\, dV = \int_0^1 \int_x^1 \int_0^{y-x} f(x, y, z)\, dz\, dy\, dx.$$

Example 2.26. Evaluate $\int_0^1 \int_0^z \int_0^y e^{(1-x)^3}\, dx\, dy\, dz$ by changing the order of integration.

Here, the domain is $D = \{(x, y, z) : 0 \le z \le 1, 0 \le y \le z, 0 \le x \le y\}$. Sketch the region. Its projection on the yz-plane is the triangle bounded by the lines $y = 0, z = 1$ and $z = y$. That is, the projection is $\{(y, z) : 0 \le z \le 1, 0 \le y \le z\}$. Its projection on the xy-plane is the triangle bounded by the lines $x = 0, y = 1$ and $y = x$, which is also expressed as $\{(x, y) : 0 \le y \le 1, 0 \le x \le y\}$. Its projection on the zx-plane is the triangle bounded by the lines $z = 0, x = 1$ and $x = z$, that is, $\{(z, x) : 0 \le x \le 1, x \le z \le 1\}$.

We plan to change the order of integration from $dxdydz$ to $dzdydx$. All of x, y, z take values from $[0, 1]$, so the x-limits are 0 and 1. Next, $x \le y$ says that the y-limits are x and 1. Since $y \le z$, the z-limits are y and 1.

Therefore, $D = \{(x, y, z) : 0 \le x \le 1, x \le y \le 1, y \le z \le 1\}$.

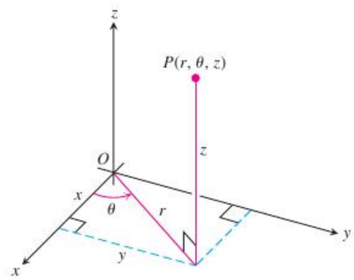
$$\int_0^1 \int_0^z \int_0^y e^{(1-x)^3}\, dx\, dy\, dz = \int_0^1 \int_x^1 \int_y^1 e^{(1-x)^3}\, dz\, dy\, dx$$

$$= \int_0^1 \int_x^1 (1-y) e^{(1-x)^3}\, dy\, dx = \int_0^1 \frac{(1-x)^2}{2} e^{(1-x)^3}\, dx$$

$$= -\int_{(1-0)^3}^0 \frac{e^t}{6}\, dt = \frac{e-1}{6}. \qquad \text{with } t = (1-x)^3$$

2.5 Triple Integral in Cylindrical coordinates

Cylindrical coordinates express a point P in space as a triple (r, θ, z), where (r, θ) is the polar representation of the projection of P on the xy-plane.

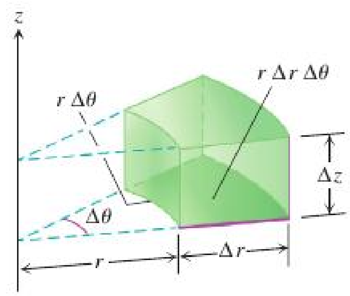


If P has Cartesian representation (x, y, z) and cylindrical representation (r, θ, z), then

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z,\ r^2 = x^2 + y^2,\ \tan\theta = y/x.$$

In cylindrical coordinates,
$r = a$ describes a cylinder with axis as z-axis.
$\theta = \alpha$ describes a plane containing the z-axis.
$z = b$ describes a plane perpendicular to z-axis.

The Riemann sum of $f(r, \theta, z)$ uses a partition of D into cylindrical wedges:

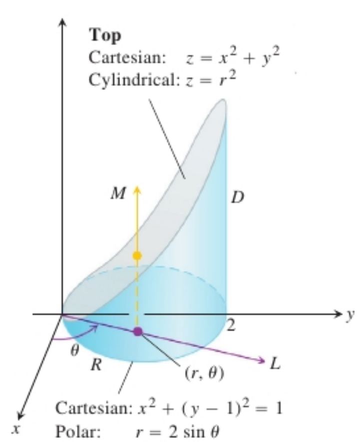


The volume element $dV = r\,dr\,d\theta dz$. Thus the triple integral is

$$\iiint_D f(r, \theta, z)dV = \iiint_D f(r, \theta, z) r\,dr\,d\theta\,dz.$$

Its conversion to iterated integrals uses a similar technique of determining the limits of integration.

Example 2.27. Find the limits of integration in cylindrical coordinates for integrating a function $f(r, \theta, z)$ over the region D bounded below by the plane $z = 0$, laterally by the circular cylinder $x^2 + (y - 1)^2 = 1$, and above by the paraboloid $z = x^2 + y^2$.



The projection of D onto the xy-plane gives the disk R enclosed by the circle $x^2 + (y-1)^2 = 1$. It simplifies to $x^2 + y^2 = 2y$. Its polar form is $r^2 = 2r\sin\theta$ or, $r = 2\sin\theta$.

A line through a point $(r, \theta) \in R$ enters D at $z = 0$ and leaves D at $z = x^2 + y^2 = r^2$.

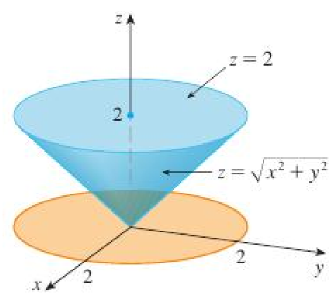
A line in the (r, θ)-plane through the origin enters R at $r = 0$ and leaves R at $r = 2\sin\theta$.

As this line sweeps through R it enters R at $\theta = 0$ and leaves at $\theta = \pi$. Hence

$$\iiint f(r, \theta, z) dV = \int_0^{\pi} \int_0^{2\sin\theta} \int_0^{r^2} f(r, \theta, z) r\, dz\, dr\, d\theta.$$

Example 2.28. Evaluate $I = \displaystyle\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2 + y^2)\, dz\, dy\, dx.$

The z-limits show that the solid is bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above bythe plane $z = 2$. Its projection on the xy-plane is the disk $x^2 + y^2 = 4$. The limits for y also confirm this. A sketch of the solid looks as follows:



Since the projection of the solid on the xy-plane is a disk; cylindrical coordinates will be easier.

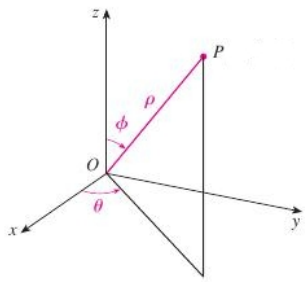
The projected disk gives the limits as $0 \le \theta \le 2\pi,\ 0 \le r \le 2$ whereas $\sqrt{x^2 + y^2} = r \le z \le 2$. Thus

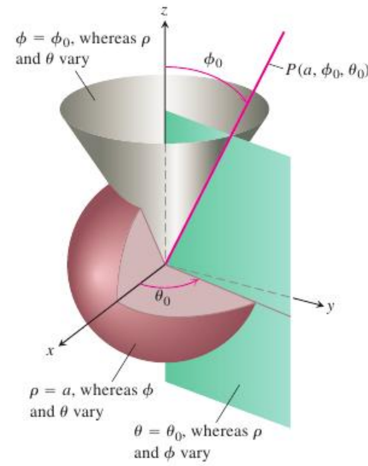
$$\begin{aligned} I &= \iiint_D (x^2 + y^2) dV = \int_0^{2\pi} \int_0^2 \int_r^2 r^2\, r\, dz\, dr\, d\theta \\ &= \int_0^{2\pi} \int_0^2 (r^3(2-r)\, dr\, d\theta = \int_0^{2\pi} \left(2\frac{2^4}{4} - \frac{2^5}{5}\right) d\theta = \int_0^{2\pi} \frac{8}{5}\, d\theta = \frac{16}{5}\pi. \end{aligned}$$

2.6 Triple Integral in Spherical coordinates

Spherical coordinates express a point P in space as a triple (ρ, ϕ, θ), where ρ is the distance of P from the origin O, ϕ is the angle between z-axis and the line OP, and θ is the angle between the projected line of OP on the xy-plane and the x-axis. This θ is the same as the 'cylindrical' θ. Moreover, $\rho \ge 0,\ 0 \le \phi \le \pi$, and $0 \le \theta \le 2\pi$. If $P(x, y, z)$ has spherical representation (ρ, ϕ, θ), then

$$x = \rho \sin\phi \cos\theta,\ y = \rho \sin\phi \sin\theta,\ z = \rho \cos\phi,\ r = \rho \sin\phi,\ \rho = \sqrt{x^2 + y^2 + z^2}.$$



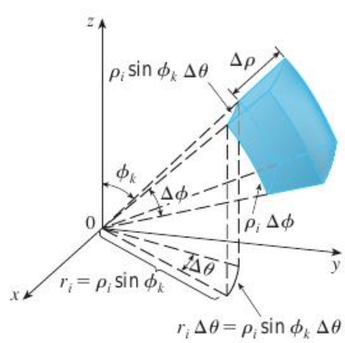


In spherical coordinates,
$\rho = a$ describes a sphere centered at origin.
$\phi = \phi_0$ describes a cone with axis as z-axis.
$\theta = \theta_0$ describes the plane containing z-axis and OP.

When computing triple integrals over a region D in spherical coordinates, we partition the region into n spherical wedges. The size of the kth spherical wedge, which contains a point $(\rho_k, \phi_k, \theta_k)$, is given by the changes $\Delta\rho_k, \Delta\phi_k, \Delta\theta_k$ in ρ, ϕ, θ.

Such a spherical wedge has one edge a circular arc of length $\rho_k \Delta\phi_k$, another edge a circular arc of length $\rho_k \sin\phi_k \Delta\theta_k$ and thickness $\Delta\rho_k$. The volume of such a spherical wedge is approximately a rectangular box with dimensions ρ_k, $\rho_k \times \Delta\phi_k$ (arc of a circle with radius ρ_k and angle ϕ_k, and $\rho_k \sin\phi_k \times \Delta\theta_k$ (arc of a circle with radius $\rho_k \sin\phi_k$ and angle θ_k). Thus

$$\Delta V_k = \rho_k^2 \sin\phi_k \Delta\rho_k \Delta\phi_k \Delta\theta_k.$$

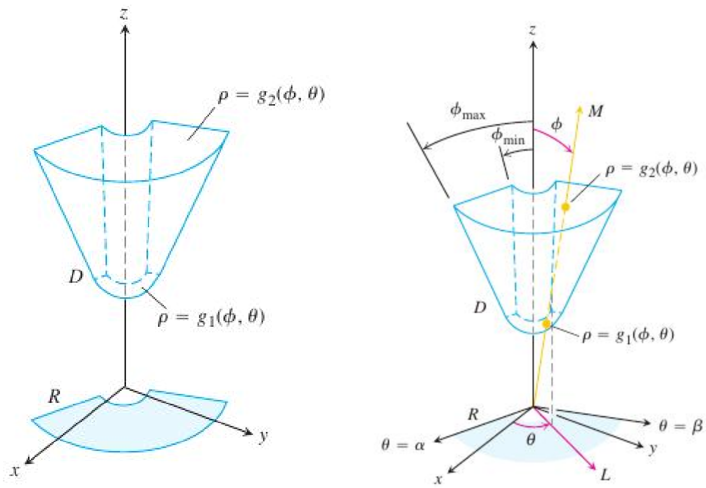


The corresponding Riemann sum is $S = \sum_{k=1}^{n} f(\rho_k, \phi_k, \theta_k)\rho_k^2 \sin\phi_k \Delta\rho_k \Delta\phi_k \Delta\theta_k$. Accordingly,

$$\iiint_D f(\rho, \phi, \theta) dV = \iiint_D f(\rho, \phi, \theta)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

The procedure in computing a triple integral in spherical coordinates is similar to that in cylindrical coordinates:

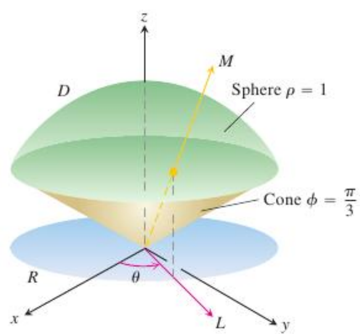
Sketch the region D and its projection on the xy-plane. Then find the ρ limit, ϕ limit and θ limit.



$$\iiint_D f(\rho,\phi,\theta)dV = \int_\alpha^\beta \int_{\phi-min}^{\phi-max} \int_{g_1(\phi,\theta}^{g_2(\phi,\theta)} f(\rho,\phi,\theta)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

Example 2.29. Find the volume of the solid D cut from the ball $\rho \le 1$ by the cone $\phi = \pi/3$.

Draw a ray M through D from the origin making an angle ϕ with the z-axis. Draw also its projection L on the xy-plane. The line L makes an angle θ with the x-axis. Let R be the projected region of D in the xy-plane.



M enters D at $\rho = 0$ and leaves D at $\rho = 1$.
Angle ϕ runs through 0 to $\pi/3$, since D is bounded by the cone $\phi = \pi/3$.
L sweeps through R as θ varies from 0 to 2π. Thus

$$\begin{aligned} V &= \iiint_D \rho^2 \sin\phi \, dV = \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^2 \sin\phi \;\, d\rho \, d\phi \, d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/3} \frac{1}{3} \sin\phi \, d\phi \, d\theta = \int_0^{2\pi} \left[\frac{-\cos\phi}{3}\right]_0^{\pi/3} = \frac{1}{6}\, 2\pi = \frac{\pi}{3}. \end{aligned}$$

Example 2.30. Evaluate $I = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} e^{(x^2+y^2+z^2)^{3/2}} dz \, dy \, dx.$

Notice that $I = \iiint_D e^{(x^2+y^2+z^2)^{3/2}} dV$, where D is the unit ball.

Writing in spherical coordinates, $I = \iiint_D e^{\rho^3} dV$. Then converting to iterated integral,

$$I = \int_0^1 \int_0^\pi \int_0^{2\pi} e^{\rho^3} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

Since the integrand is a product of separate functions of ρ, of ϕ, of θ,

$$I = \int_0^1 e^{\rho^3}\rho^2\,d\rho \int_0^\pi \sin\phi\,d\phi \int_0^{2\pi} d\theta = \left[\frac{e^{\rho^3}}{3}\right]_0^1 \Big[-\cos\phi\Big]_0^\pi (2\pi) = \frac{4\pi}{3}(e-1).$$

2.7 Change of Variables

The change of coordinate system from Cartesian to Cylindrical or to Spherical are examples of change of variables. Let us consider what happens when a different type of change of variables occurs.

Suppose f maps a region D in $\mathbb{R}^2$ onto a region R in $\mathbb{R}^2$ in a one-one manner. For convenience, we say that D is a region in the uv-plane and R is a region in the xy-plane; and f maps (u,v) to (x,y). Then f can be thought of as a pair of maps: (f_1, f_2). That is, $x = f_1(u,v)$ and $y = f_2(u,v)$. We often show this dependence implicitly by writing

$$x = x(u,v), \quad y = y(u,v).$$

Example 2.31. What is the image of $D = \{(u,v) : 0 \le u \le 1, 0 \le v \le 1\}$ under the the map given by $x = u^2 - v^2,\ y = 2uv$?

Let us see the boundaries of the square $D : 0 \le u \le 1, 0 \le v \le 1$.

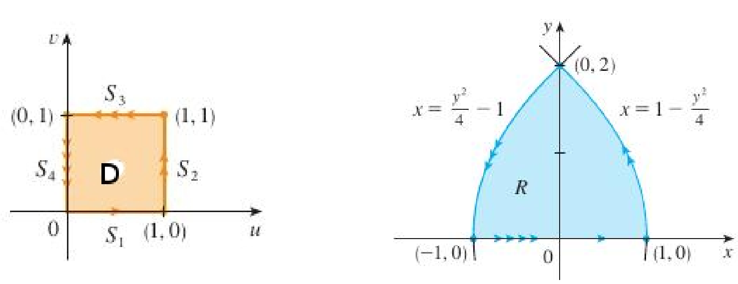
The lower boundary is the line segment $0 \le u \le 1, v = 0$. It is transformed to the line segment $x = u^2, y = 0$ or in the xy-plane it is the line segment $0 \le x \le 1, y = 0$.

The left boundary of D is the line segment $u = 0, 0 \le v \le 1$. It is transformed to $x = -v^2, y = 0$. This is the line segment joining $(0,0)$ to $(-1,0)$ in the xy-plane.

The upper boundary line of D is the line segment $0 \le u \le 1, v = 1$. This is transformed to $x = u^2 - 1, y = 2u$. Eliminating u from these equations, we get the arc of the curve $x = \frac{y^2}{4} - 1$ joining the points $(-1,0)$ to $(0,2)$ in the xy-plane.

The right hand side boundary of D is the line segment $u = 1$ and v varying from 1 to 0. This is transformed to $x = 1 - v^2, y = 2v$. Eliminating v from these equations we have the arc of the curve $x = 1 - \frac{y^2}{4}$ joining the points $(0,2)$ to $(1,0)$.

The interior of D is mapped onto the interior of the so obtained region R in the xy-plane whose boundary are the line segments and the arcs. This transformation is shown is the picture below.



If $(u,v) \mapsto (x,y)$, then how does area of a small rectangle change?

A typical small rectangle with sides Δu and Δv has corners at the points

$$A_1 = (a,b),\ A_2 = (a+\Delta u, b),\ A_3 = (a, b+\Delta v),\ A_4 = (a+\Delta u, b+\Delta v).$$

Let the images of A_k under $(u,v) \mapsto (x,y)$ be $B_k = (a_k, b_k)$ for $k = 1,2,3,4$. Then

$a_1 = x(a,b)$
$a_2 = x(a+\Delta u, b) \approx x(a,b) + x_u \Delta u$
$a_3 = x(a, b+\Delta v) \approx x(a,b) + x_v \Delta v$
$a_4 = x(a+\Delta u, b+\Delta v) \approx x(a,b) + x_u \Delta u + x_v \Delta v$

Here, $x_u = x_u(a,b)$ and $x_v = x_v(a,b)$. Similar approximations hold for b_1, b_2, b_3, b_4.

Now, Area of the image of the rectangle $A_1A_2A_3A_4$ is approximately equal to the area of the parallelogram $B_1B_2B_3B_4$ in xy-plane, which is twice the area of the triangle $B_1B_2B_4$ and is

$$|(a_4 - a_1)(b_4 - b_2) - (a_4 - a_2)(b_4 - b_1)| = \left|\det \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}\right| (a,b)\Delta u \Delta v.$$

This determinant is called the **Jacobian** of the map $(u,v) \mapsto (x,y)$; and is denoted by $J(x(u,v), y(u,v))$.

The Jacobian is also written as $J(x(u,v), y(u,v)) = \dfrac{\partial(x,y)}{\partial(u,v)}$.

We write this as Area of image of a rectangle with one corner at (a,b) and sides of length Δu and Δv is approximately $|J(x(u,v), y(u,v)|\Delta u \Delta v$, where the Jacobian $J(\cdot,\cdot)$ is evaluated at (a,b).

In deriving this approximation, we have assumed that x_u, x_v, y_u, y_v are continuous.

Assume that $x = x(u,v)$ and $y = y(u,v)$ have continuous partial derivatives with respect to u and v. Assume also that a region D in the uv-plane is in one-one correspondence with a region R in the xy-plane by the map $(u,v) \mapsto (x,y)$. Let $f(x,y)$ be a real valued continuous function on the region R. Then we have the map $\tilde{f}(u,v) = f(x(u,v), y(u,v))$.

To see how the integrals of f over R and integral of $\tilde{f}$ over D are related, divide D in the uv-plane into smaller rectangles. Now, the images of the smaller rectangles are related by

$$\text{Area of } R = |J(x(u,v), y(u,v))| \text{Area of } D.$$

By forming the Riemann sum and taking the limit, we obtain:

$$\iint_R f(x,y)dA = \iint_D \tilde{f}(u,v)\, |J(x(u,v), y(u,v)|\, dA.$$

For example, in the case of polar coordinates, we have

$$x = x(r,\theta) = r\cos\theta, y = y(r,\theta) = r\sin\theta.$$

Thus, the Jacobian is

$$J(x(r,\theta), y(r,\theta)) = x_r y_\theta - x_\theta y_r = \cos\theta(r\cos\theta) - (-r\sin\theta)\sin\theta = r.$$

Therefore, the double integral in polar coordinates for a function $f(x, y)$ takes the form

$$\iint_R f(x,y)dA = \iint_D f(r\cos\theta, r\sin\theta)\, r\, dA.$$

as we had seen earlier.

For $x = x(u, v, w),\ y = y(u, v, w),\ z = z(u, v, w)$, we write the Jacobian as

$$J(x(u,v,w), y(u,v,w), z(u,v,w)) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{bmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{bmatrix}.$$

If R is the region in $\mathbb{R}^3$ on which f has been defined and D is the region in the uvw-space so that the functions x, y, z map D onto R in a one-one manner, then

$$\iiint_R f(x,y,z)dV = \iiint_D f(x(u,v,w), y(u,v,w), z(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\, dv\, dw.$$

In the case of cylindrical coordinates, $x = r\cos\theta,\ y = r\sin\theta,\ z = z$. The Jacobian is

$$J(x(r,\theta,z), y(r,\theta,z), z(r,\theta,z)) = \left| \det \begin{bmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{bmatrix} \right| = r.$$

$$\iiint_R f(x,y,z)dV = \iiint_D f(r\cos\theta, r\sin\theta, z)\, r\, dr\, d\theta\, dz.$$

For the *spherical coordinates,* we see that

$$x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi.$$

The triple integral looks like

$$\iiint_R f(x,y,z)dV = \iiint_D f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta.$$

We had already derived these results independently.

These formulas help us in evaluating double and triple integrals in x, y, z as integrals in u, v, w by choosing a transformation $(u, v, w) \mapsto (x, y, z)$ suitably.

Example 2.32. Evaluate the double integral $\iint_R (y - x)dA$, where R is the region bounded by the lines $y - x = 1,\ y - x = -3,\ 3y + x = 7,\ 3y + x = 15.$

Take $u = y - x,\ v = 3y + x$. That is, $x = \frac{1}{4}(v - 3u), y = \frac{1}{4}(u + v)$. Then

$$D = \{(u, v) : -3 \le u \le 1,\ 7 \le v \le 15\}.$$

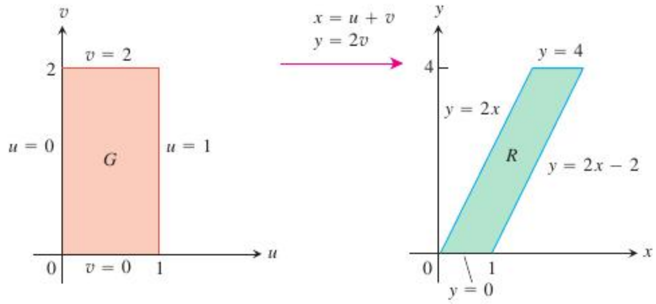
The Jacobian is

$$J = |x_u y_v - x_v y_u| = |(-3/4)(1/4) - (1/4)(1/4)| = -1/4.$$

Therefore,

$$\iint_R (y-x)dA = \iint_D u\,|J|\,dA = \iint_D u\,\frac{1}{4}\,dA = \int_{-3}^{1}\int_{7}^{15} \frac{1}{4}u\,dv\,du = \int_{-3}^{1} \frac{1}{4}(15-7)u\,du = -8.$$

Example 2.33. Evaluate $\displaystyle\int_0^4 \int_{y/2}^{1+y/2} \frac{2x-y}{2}\, dx\, dy$ by using the transformation $u = x - y/2,\ v = y/2.$



The domains R in the xy-plane and G in the uv-plane are

$$R = \{(x, y) : 0 \le y \le 4, y/2 \le x \le 1 + y/2\},\ \ G = \{(u, v) : 0 \le u \le 1, 0 \le v \le 2\}.$$

And $f(x, y) = \dfrac{2x - y}{2} = u.$

Notice that $x = u + v,\ y = 2v$. Thus, $|J| = |x_u y_v - x_v y_u| = |(1)(2) - (0)(1)| = 2.$

$$\int_0^4 \int_{y/2}^{1+y/2} \frac{2x-y}{2} dxdy = \int_0^1 \int_0^2 u\, du\, dv = \int_0^1 \frac{2^2 - 0^2}{2}\, dv = 2.$$

Caution: The change of variables formula turns an xy-integral into a uv-integral. But the map that changes the variables goes from uv-domain onto xy-domain. This map must be one-one on the interior of the uv-domain. Sometimes it is easier to get such a map from xy-domain to uv-domain. Then we will be tackling with the inverse of such an easy map. Here the fact that

the Jacobian of the inverse map is the inverse of the Jacobian of the original map

helps us. This may be expressed as

$$\frac{\partial(x, y)}{\partial(u, v)} = \left(\frac{\partial(u, v)}{\partial(x, y)}\right)^{-1}.$$

Similarly, triple integrals undergo change of variables by using the inverse of the Jacobian.

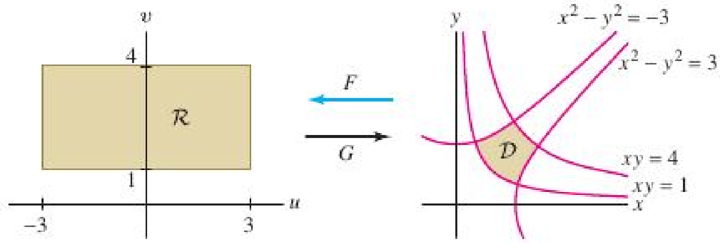
Example 2.34. Integrate $f(x, y) = xy(x^2 + y^2)$ over the domain

$$R : \ \ -3 \le x^2 - y^2 \le 3,\ 1 \le xy \le 4.$$

There is a simple map that goes in the *wrong direction*: $u = x^2 - y^2,\ v = xy$. Then the image of R, which we denote as D in the uv-plane is the rectangle

$$D : \ \ -3 \le u \le 3,\ 1 \le v \le 4.$$

We have $F : D \to R$ defined by $F(x, y) = (u, v) = (x^2 - y^2, xy)$. And its inverse is $G = F^{-1}$, where $G : R \to D$.



We need not compute the map G. Instead, we go for the Jacobian.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix} = 2(x^2+y^2).$$

Therefore, $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{2(x^2+y^2)}$. Then

$$\iint_R xy(x^2+y^2)\,dA = \iint_D \left[xy(x^2+y^2) \left| \frac{1}{2(x^2+y^2)} \right| \right] dA.$$

Notice that the integral on the right side is in the uv-plane and the bracketed term inside $[\,\cdot\,]$ is a function of (u,v). Since the bracketed term simplifies to $xy/2$ which is equal to $v/2$, we have the integral as

$$\iint_D \frac{v}{2}\,dA = \frac{1}{2}\int_{-3}^{3}\int_{1}^{4} v\,dv\,du = \frac{1}{2}\int_{-3}^{3} \frac{4^2-1^2}{2}\,du = \frac{15}{4}\big[3-(-3)\big] = \frac{45}{2}.$$

2.8 Review Problems

Problem 2.1: Find the area of the region bounded by the curves $y = x$ and $y = 2 - x^2$.

The points of intersection of the curves satisfy $y = x$ and $x = 2 - x^2$. The last equation is same as $(x+2)(x-1) = 0$. Thus the points of intersection are $(-2,-2)$ and $(1,1)$. Hence the area is

$$\Big| \int_{-2}^{1}\int_{x}^{2-x^2} dydx \Big| = \Big| \int_{-2}^{1} (2-x^2-x)dx \Big| = \Big| \Big[2x - \frac{x^3}{3} - \frac{x^2}{2}\Big]_{-2}^{1} \Big| = \frac{9}{2}.$$

Since the significant portion of the curve $y = 2 - x^2$ lies above the portion of the line $y = x$, there is no need to take the absolute value. The calculation also confirms this.

Problem 2.2: Evaluate $I = \iint_D (4 - x^2 - y^2)\,dA$ if D is the region bounded by the straight lines $x = 0,\ x = 1,\ y = 0$ and $y = 3/2$.

$$I = \int_0^{3/2}\int_0^1 (4-x^2-y^2)dxdy = \int_0^{3/2} \big[4x - x^3/3 - y^2x\big]_0^1 dy = \int_0^{3/2} \Big(\frac{11}{3} - y^2\Big)\,dy = \frac{35}{8}.$$

Problem 2.3: Evaluate the double integral of $f(x,y) = 1 + x + y$ over the region bounded by the lines $y = -x,\ y = 2$ and the parabola $x = \sqrt{y}$.

Draw the region. The integral is equal to

$$\int_0^2 \int_{-y}^{\sqrt{y}} (1+x+y)dx\,dy = \int_0^2 \big(\sqrt{y} + \frac{y}{2} + \sqrt{y}\,y - \big(-y + \frac{y^2}{2} - y^2\big)\,dy$$

$$= \int_0^2 \big(\sqrt{y} + \frac{3y}{2} + y\sqrt{y} + \frac{y^2}{2}\big)dy = \Big[\frac{2}{3}y^{3/2} + \frac{5}{2}y^{5/2} + \frac{3}{4}y^2 + \frac{1}{6}y^3\Big]_0^2 = \frac{1}{3}(13 + 44\sqrt{2}).$$

Problem 2.4: Change the order of integration in $\int_0^1 \int_x^{\sqrt{x}} f(x,y)dydx$.

The domain D of integration is bounded by the straight line $y = x$ and the parabola $y = \sqrt{x}$. Every straight line parallel to x-axis cuts the boundary of D in no more than two points, and it remains in between y^2 to y. Also, y lies between 0 and 1. Hence

$$\int_0^1 \int_x^{\sqrt{x}} f(x,y)dydx = \int_0^1 \int_{y^2}^{y} f(x,y)dx\,dy.$$

Problem 2.5: Evaluate $\displaystyle\iint_D e^{y/x}\,dA$, where D is a triangle bounded by the straight lines $y = x$, $y = 0$, and $x = 1$.

In D, the variable x remains in between 0 and 1, and y lies between 0 and x. Hence

$$\iint_D e^{y/x}\,dA = \int_0^1 \int_0^x e^{y/x} dy\,dx = \int_0^1 x(e-1)\,dx = \frac{e-1}{2}.$$

Problem 2.6: Find $I = \iint_D e^{x+y}\,dA$, where D is the annular region bounded by two squares of sides 2 and 4 each having center at $(0,0)$.

Draw the picture. D is not a simply connected domain. Divide D into four simply connected domains by drawing lines $x = -1$ and $x = 1$. Let D_1 be the rectangle to the left of the inner square; D_2 be the square on top of the inner square; D_3 be the square below the inner square; and D_4 be the rectangle to the right of the inner square; so that D is the disjoint union of D_1, D_2, D_3, D_4. Then

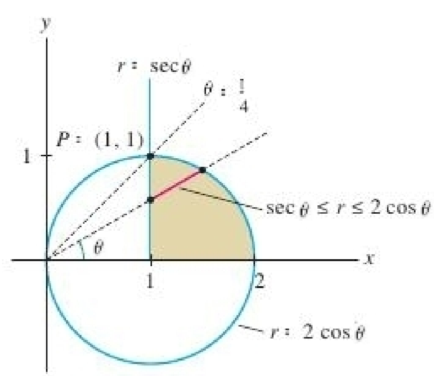
$$I = \iint_{D_1} e^{x+y}\,dA + \iint_{D_2} e^{x+y}\,dA + \iint_{D_3} e^{x+y}\,dA + \iint_{D_4} e^{x+y}\,dA.$$

Converting each integral to an iterated integral, we have

$$I = \int_{-2}^{-1} \int_{-2}^{2} e^{x+y} dydx + \int_{-1}^{1} \int_{1}^{2} e^{x+y} dydx$$

$$+ \int_{-1}^{1} \int_{-2}^{-1} e^{x+y} dydx + \int_{1}^{2} \int_{-2}^{2} e^{x+y} dydx = e^4 - e^2 - e^{-2} + e^{-4}.$$

Problem 2.7: Evaluate $\iint_D (x^2+y^2)^{-2}\,dA$, where D is the shaded region in the figure below:



The integrand in polar coordinates is $f(r, \theta) = r^{-4}$. The region D is given by $0 \le \theta \le \pi/4$, $\sec\theta \le r \le 2\cos\theta$. Thus

$$\iint_D (x^2+y^2)^{-2}\, dA = \int_0^{\pi/4}\int_{\sec\theta}^{2\cos\theta} r^{-4}\, r\, dr\, d\theta = \frac{1}{8}\int_0^{\pi/4}(4\cos^2\theta - \sec^2\theta)\, d\theta = \frac{\pi}{16}.$$

Problem 2.8: Calculate the volume of the solid bounded by the planes $x = 0,\ y = 0,\ z = 0$, and $x + y + z = 1$.

The volume $V = \iint_D (1 - x - y) dA$, where D is the base of the solid on the xy-plane. We see that D is the triangular region bounded by the straight lines $x = 0,\ y = 0,\ x + y = 1$. Thus,

$$V = \int_0^1\int_0^{1-x}(1-x-y)dydx = \int_0^1 \frac{1}{2}(1-x)^2 dx = \frac{1}{6}.$$

Problem 2.9: Compute the volume V of the solid bounded by the spherical surface $x^2 + y^2 + z^2 = 4a^2$ and the cylinder $x^2 + y^2 = 2ay$, where $a > 0$.

The domain of integration is the base of the cylinder. This is the circle $x^2 + y^2 - 2ay = 0$, whose centre is $(0, a)$ and radius a. We calculate $V/4$, the volume of the portion of the solid in the first octant. Now, the domain of integration D is the semicircular disk whose boundaries are given by

$$x = g_1(y) = 0,\ x = g_2(y) = \sqrt{2ay - y^2},\ y = 0,\ y = 2a.$$

The integrand is $z = f(x, y) = \sqrt{4a^2 - x^2 - y^2}$. Then

$$\frac{V}{4} = \int_0^{2a}\int_0^{\sqrt{2ay-y^2}} \sqrt{4a^2 - x^2 - y^2}\, dx\, dy.$$

To evaluate this, use polar coordinates: $x = r\cos\theta,\ y = r\sin\theta$. For the limits of integration, use $x^2 + y^2 = r^2,\ y = r\sin\theta$ to get:

$$x^2 + y^2 - 2ay = 0 \Rightarrow r^2 - 2ar\sin\theta = 0 \Rightarrow r = 2a\sin\theta.$$

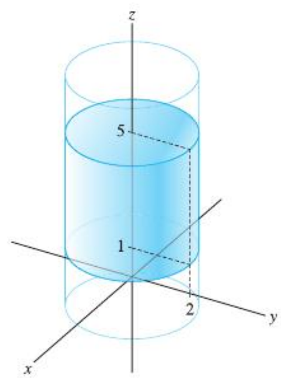
That is, in polar coordinates, the boundaries of D are given by

$$r = g_1(\theta) = 0,\ r = g_2(\theta) = 2a\sin\theta,\ 0 \le \theta \le \pi/2.$$

The integrand is $f(r, \theta) = \sqrt{4a^2 - r^2}$. Hence,

$$\begin{aligned} V &= 4\int_0^{\pi/2}\int_0^{2a\sin\theta} \sqrt{4a^2 - r^2}\, r\, dr\, d\theta \\ &= \frac{-4}{3}\int_0^{\pi/2}\left[(4a^2 - 4a^2\sin^2\theta)^{3/2} - (4a^2)^{3/2}\right] d\theta = \frac{16}{9}a^3(3\pi - 4). \end{aligned}$$

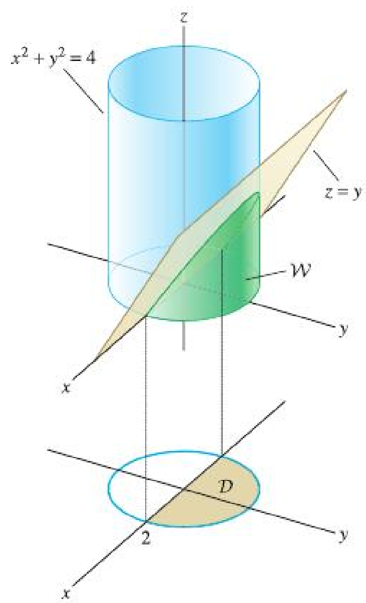
Problem 2.10: Integrate $f(x, y, z) = z\sqrt{x^2 + y^2}$ over the solid cylinder $x^2+y^2 \le 4$ for $1 \le z \le 5$.



The domain of integration D in cylindrical coordinates is given by $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 2$, $1 \leq z \leq 5$. The integrand is zr. Thus

$$\iiint_D z\sqrt{x^2+y^2}\,dV = \int_0^{2\pi}\int_0^2\int_1^5 (zr)\,r\,dz\,dr\,d\theta = 64\pi.$$

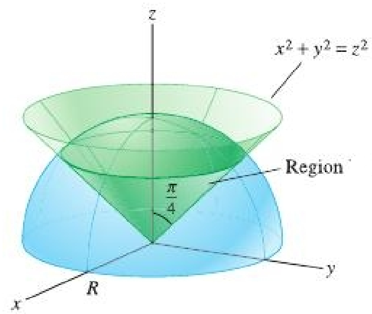
Problem 2.11: Integrate $f(x,y,z) = z$ over the part of the solid cylinder $x^2+y^2 \leq 4$ for $0 \leq z \leq y$.



The domain W has the projection D on the xy-plane as the semicircle depicted in the figure. The z-coordinate varies from 0 to y and $y = r\sin\theta$. Thus W is given by $0 \leq \theta \leq \pi$, $0 \leq r \leq 2$, $0 \leq z \leq r\sin\theta$. In cylindrical coordinates,

$$\iiint_W z\,dV = \int_0^{\pi}\int_0^2\int_0^{r\sin\theta} z\,r\,d\theta\,dr\,dz = \int_0^{\pi}\int_0^2 \frac{1}{2}\big(r\sin\theta\big)^2\,r\,d\theta\,dr = \pi.$$

Problem 2.12: Compute $\iiint_D z\,dV$, where D is the solid lying above the cone $x^2+y^2 = z^2$ and below the unit sphere.



The upper branch of the cone, which is relevant to D, has the equation $\phi = \pi/4$ in spherical coordinates. The sphere has the equation $\rho = 1$. Thus D is given by

$$D: \quad 0 \le \theta \le 2\pi, \ 0 \le \phi \le \pi/4, \ 0 \le \rho \le 1.$$

Since $z = \rho\cos\phi$, the required integral is

$$\begin{aligned}\iiint_D z\,dV &= \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 (\rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\ &= 2\pi\int_0^{\pi/4}\int_0^1 \rho^3\cos\phi\sin\phi\,d\rho\,d\phi = \frac{\pi}{2}\int_0^{\pi/4}\cos\phi\sin\phi\,d\phi = \frac{\pi}{8}.\end{aligned}$$

Problem 2.13: Evaluate $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$.

$$\begin{aligned}I^2 &= \lim_{a\to\infty}\Big(\int_{-a}^{a} e^{-x^2}\,dx\Big)^2 = \lim_{a\to\infty}\Big[\Big(\int_{-a}^{a} e^{-x^2}\,dx\Big)\Big(\int_{-a}^{a} e^{-y^2}\,dy\Big)\Big] \\ &= \lim_{a\to\infty}\Big[\int_{-a}^{a}\int_{-a}^{a} e^{-x^2-y^2}\,dx\,dy\Big] = \lim_{a\to\infty}\iint_R e^{-x^2-y^2}\,dA\end{aligned}$$

where R is the square $[-a, a]\times[-a, a]$ for $a > 0$.

Let $D = B(0, a)$ and $S = B(0, \sqrt{2}\,a)$, the balls centred at 0 and with radii a and $\sqrt{2}\,a$, respectively. Then $D \subseteq R \subseteq S$. Since $e^{-x^2-y^2} > 0$ for all $(x, y) \in \mathbb{R}^2$, we have

$$\iint_D e^{-x^2-y^2}\,dA \le \iint_R e^{-x^2-y^2}\,dA \le \iint_S e^{-x^2-y^2}\,dA.$$

Now,

$$\iint_D e^{-x^2-y^2}\,dA = \int_0^{2\pi}\int_0^a e^{-r^2}\,r\,dr\,d\theta = -\frac{1}{2}\int_0^{2\pi}(e^{-a^2}-1)\,d\theta = \pi(1-e^{-a^2}).$$

Similarly, $\displaystyle\iint_S e^{-x^2-y^2}\,dA = \pi(1-e^{-2a^2})$. We see that

$$\lim_{a\to\infty}\iint_D e^{-x^2-y^2}\,dA = \pi, \quad \lim_{a\to\infty}\iint_S e^{-x^2-y^2}\,dA = \pi.$$

Therefore, by sandwich theorem, we have

$$I^2 = \lim_{a\to\infty}\iint_R e^{-x^2-y^2}\,dA = \pi \Rightarrow I = \sqrt{\pi}.$$

Problem 2.14: Compute the volume of the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1.$

Projection of this solid on the xy-plane is the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. Therefore, the required volume is

$$V=\int_{-a}^{a}\int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}}\int_{-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}dz\,dy\,dx=2c\int_{-a}^{a}\int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}}\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dy\,dx.$$

Substitute $y=b(1-x^2/a^2)^{1/2}\sin t$. Then $dy=b(1-x^2/a^2)\cos t\,dt$ and
$-\pi/2\le t\le\pi/2$. Therefore,

$$\begin{aligned}V&=2c\int_{-a}^{a}\int_{-\pi/2}^{\pi/2}\Big[\big(1-\frac{x^2}{a^2}\big)-\big(1-\frac{x^2}{a^2}\big)\sin^2 t\Big]^{1/2}b\big(1-\frac{x^2}{a^2}\big)\cos t\,dt\,dx\\&=\frac{bc\pi}{a^2}\int_{-a}^{a}(a^2-x^2)\,dx=\frac{4\pi abc}{3}.\end{aligned}$$

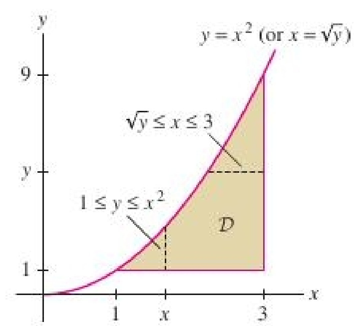
Problem 2.15: Evaluate $\displaystyle\int_0^\infty\frac{e^{-ax}-e^{-bx}}{x}\,dx$ for $a>0,b>0$.

$$\begin{aligned}\int_0^\infty\frac{e^{-ax}-e^{-bx}}{x}\,dx&=\int_0^\infty\int_a^b e^{-yx}dy\,dx\\&=\int_a^b\int_0^\infty e^{-yx}dx\,dy=\int_a^b\frac{1}{y}\,dy=\ln\frac{b}{a}.\end{aligned}$$

Notice the change in order of integration above.

Problem 2.16: Evaluate $\displaystyle\int_1^9\int_{\sqrt{y}}^3 xe^y\,dx\,dy.$

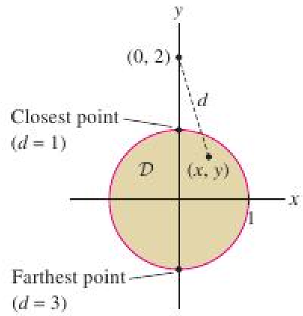
The domain of integration is given by $1\le y\le 9,\ \sqrt{y}\le x\le 3.$



The same is expressed as $1\le x\le 3,\ 1\le y\le x^2$. Changing the order of integration, we have

$$\int_1^9\int_{\sqrt{y}}^3 xe^y\,dx\,dy=\int_1^3\int_1^{x^2}xe^y\,dx\,dy=\int_1^3(xe^{x^2}-e\,x)dx=\frac{1}{2}(e^9-9e).$$

Problem 2.17: Show that $\dfrac{\pi}{3}\le\displaystyle\iint_D\frac{dA}{\sqrt{x^2+(y-2)^2}}\le\pi$, where D is the unit disc.



The quantity $f(x,y) = \sqrt{x^2+(y-2)^2}$ is the distance of any point (x,y) from $(0,2)$.
For $(x,y) \in D$, maximum of $f(x,y)$ is thus 3 and minimum is 1. Therefore,

$$\frac{1}{3} \le \frac{1}{\sqrt{x^2+(y-2)^2}} \le 1.$$

Integrating over D, we have

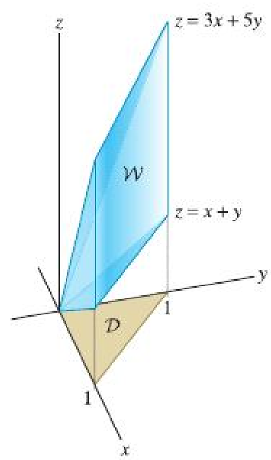
$$\iint_D \frac{1}{3}\, dA \le \iint_D \frac{1}{\sqrt{x^2+(y-2)^2}}\, dA \le \iint_D 1\, dA.$$

Since $\iint_D dA =$ area of D, we obtain

$$\frac{\pi}{3} \le \iint_D \frac{dA}{\sqrt{x^2+(y-2)^2}} \le \pi.$$

Problem 2.18: Evaluate $\iiint_W z\, dV$, where W is the solid bounded by the planes $x=0$, $y=0$, $x+y=1$, $z=x+y$, and $z=3x+5y$ in the first octant.

W lies over the triangle D in the xy-plane defined by $0 \le x \le 1,\ 0 \le y \le 1-x$. Hence



$$\begin{aligned}
\iiint_D z\, dV &= \int_0^1 \int_0^{1-x} \int_{x+y}^{3x+5y} z\, dz\, dy\, dx \\
&= \int_0^1 \int_0^{1-x} (4x^2+14xy+12y^2)\, dy\, dx = \int_0^1 (4-5x+2x^2-x^3)\, dx = \frac{23}{12}.
\end{aligned}$$

Fun Problem: The n-dimensional cube with side a has volume a^n. What is the volume of an n-dimensional ball?

Denote by $V_n(r)$ the volume of the n-dimensional ball with radius r. Also, write $A_n = V_n(1)$. For $n = 1$, we have the interval $[-1, 1]$, whose volume we take as its length, that is, $A_1 = 2,\ V_1 = 2\pi$. For $n = 2$, we have the unit disk, whose volume is its area; that is, $A_2 = \pi,\ V_2 = \pi r^2$. For $n = 3$, we know that $A_3 = 4\pi/3$ and $V_3(r) = 4\pi r^3/3$.

Exercise 1: Show by induction that $V_n(r) = A_n\, r^n$.

Suppose $V_{n-1}(r) = A_{n-1}r^{n-1}$. The slice of the n-dimensional ball $x_1^2 + \cdots x_{n-1}^2 + x_n^2 = r^n$ at the height $x_n = c$, has the equation

$$x_1^2 + \cdots x_{n-1}^2 + c^2 = r^2.$$

This slice has the radius $\sqrt{r^2 - c^2}$. Thus

$$V_n(r) = \int_{-r}^{r} V_{n-1}\sqrt{r^2 - x_n^2}\, dx_n = A_{n-1}\int_{-r}^{r} (\sqrt{r^2 - x_n^2})^{n-1}\, dx_n.$$

Substitute $x_n = r\sin\theta$. So, $dx_n = r\cos\theta$ and $-\pi/2 \le \theta \le \pi/2$. Then

$$V_n(r) = A_{n-1}r^n \int_{-\pi/2}^{\pi/2} \cos^n\theta\, d\theta = A_{n-1}C_n r^n,$$

where $C_n = \int_{-\pi/2}^{\pi/2} \cos^n\theta\, d\theta$. This says that $A_n = A_{n-1}C_n$.

Exercise 2: Prove that $C_3 = 4/5, C_4 = 3\pi/8$ and $C_n = \frac{n-1}{n}C_{n-2}$.

Exercise 3: Prove that $A_{2m} = \dfrac{\pi^m}{m!}$ and $A_{2m+1} = \dfrac{2^{m+1}\pi^m}{1\cdot 3\cdots(2m+1)}$.

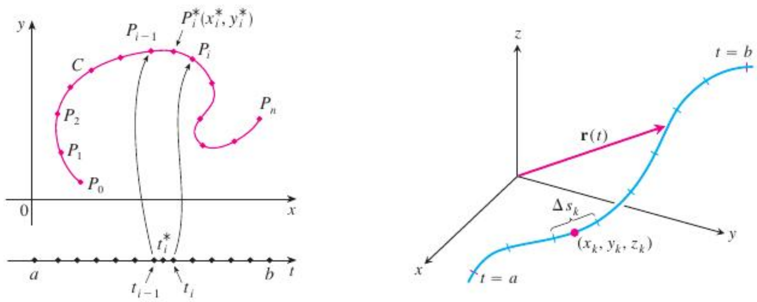
This sequence of numbers have a curious property: A_n increases up to $n = 5$ and then it decreases to 0 as $n \to \infty$.

Chapter 3

Vector Integrals

3.1 Line Integral

Line integrals are single integrals which are obtained by integrating a function over a curve instead of integrating over an interval.



Let $f(x, y, z)$ be a real valued function with domain D. Let C be a curve that lies in D given in parametric form as

$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}, \quad a \le t \le b.$$

The values of f on the curve C are given by the composite function $f(x(t), y(t), z(t))$. We want to integrate this composite function on the curve C.

Partition C into n sub-arcs. Choose a point (x_k, y_k, z_k) on the kth subarc. Suppose the kth subarc has length Δs_k. Form the Riemann sum

$$S_n = \sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta s_k.$$

When n approaches ∞, the length s_k approaches 0. In such a case, if $\lim_{n\to\infty} S_n$ exists, then this limit is called the **line integral of f over the curve** C.

$$\int_C f(x, y, z)ds = \lim_{n\to\infty} S_n.$$

In practice, the line integral is computed by parameterizing the curve C.

Theorem 3.1. *Let* $C: \quad x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$ *be a parametrization of the curve* C *lying in a domain* $D \subseteq \mathbb{R}^3$. *If* $f : D \to \mathbb{R}$ *is continuous and the component functions* $x(t), y(t), z(t)$ *are differentiable, then the line integral of* f *over* C *exists and is given by*

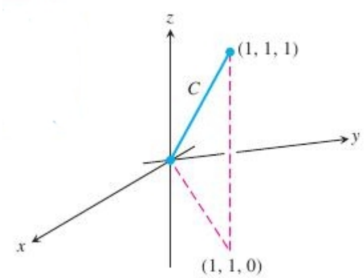
$$\int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt.$$

We also write $ds = |\vec{r}\,'(t)|dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt.$

Example 3.1. Integrate $f(x,y,z) = x - 3y^2 + z$ over the line segment from $(0,0,0)$ to $(1,1,1)$.

Parametrize the curve $C: \ \vec{r}(t) = t\,\hat{i} + t\,\hat{j} + t\,\hat{k}, \quad 0 \le t \le 1.$

Then $x(t) = y(t) = z(t) = t$. So, $|\vec{r}\,'(t)| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$.



$$\int_C f ds = \int_0^1 \left[x(t) - 3y^2(t) + z(t)\right]\sqrt{3}dt = \int_0^1 [t - 3t^2 + t]\sqrt{3}dt = \sqrt{3}\left[t^2 - t^3\right]_0^1 = 0.$$

Example 3.2. Evaluate $\int_C (2 + x^2y)ds$, where C is the upper half of the unit circle in the xy-plane.

Here, $f = f(x,y)$ is a function of two variables.
Parametrize the curve. $C: \ x(t) = \cos t, \ y(t) = \sin t, \quad 0 \le t \le \pi$. Then

$$\int_C (2 + x^2y)ds = \int_0^\pi (2 + \cos^2 t\, \sin t)\sqrt{(x'(t))^2 + (y'(t))^2}dt = 2\pi + \frac{2}{3}.$$

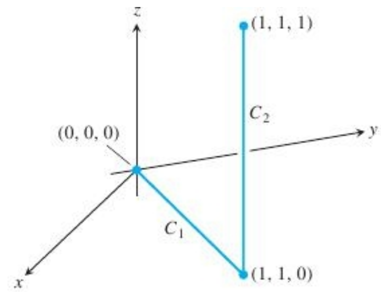
If C is a piecewise smooth curve, i.e., it is a join of finite number of smooth curves, written as $C = C_1 \cup \cdots \cup C_m$, then we define

$$\int_C f(x,y,z)ds = \int_{C_1} f(x,y,z)ds + \cdots + \int_{C_m} f(x,y,z)ds.$$

Example 3.3. Let C be the curve consisting of line segments joining $(0,0,0)$ to $(1,1,0)$ and $(1,1,0)$ to $(1,1,1)$. Evaluate $\int_C (x - 3y^2 + z)ds$.

C is the join of C_1 and C_2, whose parametrization are given by

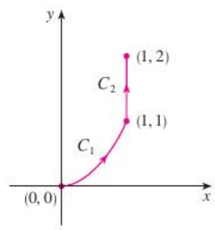
$$C_1: \ \vec{r}(t) = t\,\hat{i} + t\,\hat{j}, \ 0 \le t \le 1; \quad C_2: \vec{r}(t) = \hat{i} + \hat{j} + t\,\hat{k}, \ 0 \le t \le 1.$$



Then On $C_1,\ |\vec{r}\,'(t)| = \sqrt{2}$ and on $C_2,\ |\vec{r}\,'(t)| = 1$. Now,

$$\begin{aligned}
\int_C (x - 3y^2 + z)ds &= \int_{C_1} (x - 3y^2 + z)ds + \int_{C_2} (x - 3y^2 + z)ds \\
&= \int_0^1 f(t, t, 0)\sqrt{2}dt + \int_0^1 f(1, 1, t)\, 1\, dt \\
&= \int_0^1 (t - 3t^2 + 0)\sqrt{2}dt + \int_0^1 (1 - 3 + t)dt = \frac{-3 - \sqrt{2}}{2}.
\end{aligned}$$

Example 3.4. Evaluate $\displaystyle\int_C 2x ds$, where C is the arc of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$ followed by the line segment joining $(1, 1)$ to $(1, 2)$.



Parametrize: $C = C_1 \cup C_2$, where

$$C_1 : x = x,\ y = x^2, \quad 0 \le x \le 1; \qquad C_2 : x = 1,\ y = y, \quad 1 \le y \le 2.$$

Choosing $x = t$ for C_1 and $y = t$ for C_2, we have

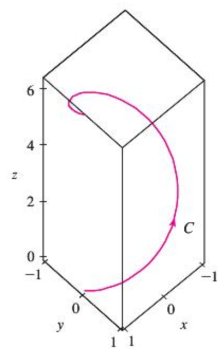
$$C_1 : x = t,\ y = t^2, \quad 0 \le t \le 1; \quad C_2 : x = 1,\ y = t, \quad 1 \le t \le 2.$$

On C_1, $dx = 1\, dt,\ dy = 2t\, dt,\ ds = \sqrt{1 + 4t^2}\, dt$. Similarly, on C_2, $ds = dt$. Then

$$\begin{aligned}
\int_C 2x ds &= \int_{C_1} 2x ds + \int_{C_2} 2x ds = \int_0^1 2t\sqrt{1 + 4t^2}\, dt + \int_1^2 2(1)dt \\
&= \frac{(1 + 4t^2)^{3/2}}{6}\Big|_0^1 + 2 = \frac{5\sqrt{5} - 1}{6} + 2.
\end{aligned}$$

Example 3.5. Evaluate $\displaystyle\int_C y \sin z ds$, where C is the circular helix given by

$$x(t) = \cos t,\ y(t) = \sin t,\ z(t) = t,\ \ 0 \le t \le 2\pi.$$



$$\int_C y \sin z ds = \int_0^{2\pi} \sin t \, t\sqrt{\sin^2 t + \cos^2 t + 1} dt = \sqrt{2}\pi.$$

If the curve C happens to be a line segment on the x-axis, then $ds = dx$. In that case, the line integral over the curve becomes

$$\int_C f(x,y,z)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta x_k.$$

As earlier, if $f(x,y,z)$ has continuous partial derivatives and $\vec{r}(t)$ is smooth, and C has parametrization as $x = x(t),\ y = y(t),\ z = z(t),\ \ a \le t \le b$, then

$$\int_C f(x,y,z)dx = \int_a^b f(x(t),y(t),z(t))x'(t)dt.$$

Similarly, if the curve C is a segment on the y or z-axis, then the line integrals are, respectively

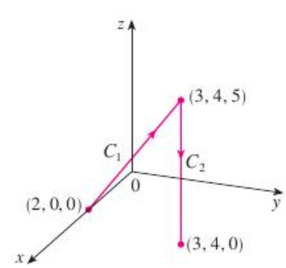
$$\int_C f(x,y,z)dy = \int_a^b f(x(t),y(t),z(t))y'(t)dt, \quad \int_C f(x,y,z)dz = \int_a^b f(x(t),y(t),z(t))z'(t)dt.$$

These line integrals are called as the **line integrals of f over C with respect to** x, y, z respectively.

Example 3.6. Evaluate $\int_C ydx + zdy + xdz$, where C is the curve joining the line segments from $(2,0,0)$ to $(3,4,5)$ to $(3,4,0)$.

Parameterize: $C = C_1 \cup C_2$, where

$$C_1 : \ x = 2+t,\ y = 4t,\ z = 5t,\ \ 0 \le t \le 1; \quad C_2 : \ x = 3,\ y = 4,\ z = 5-5t,\ \ 0 \le t \le 1.$$



Then $\int_C ydx + zdy + xdz = \int_{C_1} ydx + zdy + xdz + \int_{C_2} ydx + xdz + zdx$

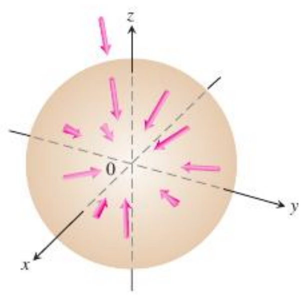
$$= \int_0^1 (4t)dt + (5t)4dt + (2+t)5dt \ + \ \int_0^1 3(-5)dt = 49/2 - 15 = 9.5.$$

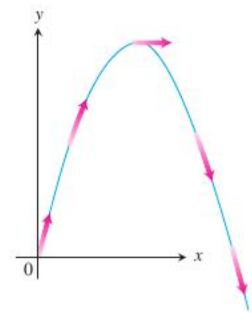
3.2 Line Integral of Vector Fields

We want to generalize line integrals to vector fields.

A **vector field** is a function defined on a domain D in the plane or space that assigns a vector to each point in D. If D is a domain in space, a vector field on D may be written as

$$\vec{F}(x, y, z) = M(x, y, z)\,\hat{\imath} + N(x, y, z)\,\hat{\jmath} + P(x, y, z)\,\hat{k}.$$





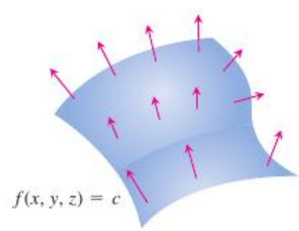
Vectors in a gravitational field point toward the center of mass that gives the source of the field. The velocity vectors on a projectile's motion make a vector field along the trajectory.

Let $f(x, y, z)$ be a function from a domain in $\mathbb{R}^3$ to $\mathbb{R}$. If f_x, y_y, f_z exist, then the **gradient field** of $f(x, y, z)$ is the field of gradient vectors

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial x}\,\hat{\imath} + \frac{\partial f}{\partial y}\,\hat{\jmath} + \frac{\partial f}{\partial z}\,\hat{k}.$$

The gradient field of the surface $f(x, y, z) = c$ may be drawn as follows:

> At each point on the surface, we have a vector, the gradient vector, which is normal to the surface. And we draw it there itself to show it.



For example, the gradient field of $f(x, y, z) = xyz$ is

$$\text{grad } f = yz\,\hat{\imath} + zx\,\hat{\jmath} + xy\,\hat{k}.$$

Notice that $f(x, y, z)$ has a continuous gradient iff f_x, f_y, f_z are continuous on the domain of definition of f.

A vector field $\vec{F}$ is called **conservative** if there exists a scalar function f such that $\vec{F} = \text{grad } f$. In such a case, the scalar function f is called the **potential** of the vector field $\vec{F}$.

For example, consider the gravitational force field $\vec{F} = -\dfrac{mMG}{|r|^3}\vec{r}$. It is also written in the form:

$$\vec{F}(x, y, z) = -\frac{mMG}{(x^2 + y^2 + z^2)^{3/2}}[x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}]$$

Here, $\vec{F}$ is a conservative field. Reason?

Define $f(x,y,z) = \dfrac{mMG}{(x^2+y^2+z^2)^{1/2}}$. Then

$$\text{grad } f = \frac{\partial f}{\partial x}\,\hat{i} + \frac{\partial f}{\partial y}\,\hat{j} + \frac{\partial f}{\partial z}\,\hat{k} = \vec{F}.$$

Physically, the law of conservation of energy holds in every conservative field.

Let $\vec{F}(x,y,z)$ be a continuous vector filed defined over a curve C given by

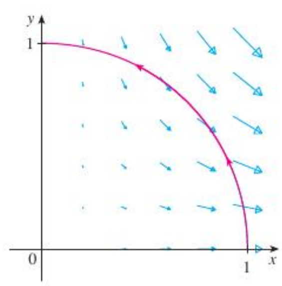
$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k} \text{ for } a \le t \le b.$$

The **line integral of $\vec{F}$ along** C, also called the **work done by moving a particle on C under the force field $\vec{F}$** is

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt = \int_C \vec{F}\cdot\vec{T}\,ds,$$

where $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$ is the unit tangent vector at a point on C.

Example 3.7. Evaluate the line integral of the vector field $\vec{F}(x,y,z) = x^2\,\hat{i} - xy\,\hat{j}$ along the first quarter unit circle in the first quadrant.



The curve C is given by $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j},\ 0 \le t \le \pi/2$. Then

$$\vec{F}(\vec{r}(t)) = \cos^2 t\,\hat{i} - \cos t \sin t\,\hat{j}, \quad d\vec{r} = -\sin t\,\hat{i} + \cos t\,\hat{j}.$$

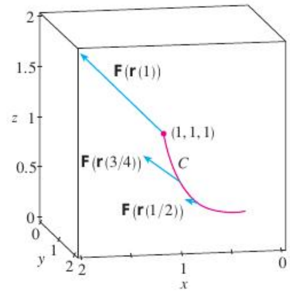
The work done is

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{\pi/2} \vec{F}(\vec{r})\cdot\vec{r}\,'\,dt = \frac{-2}{3}.$$

Let the vector filed be $\vec{F}(x,y,z) = M(x,y,z)\,\hat{i} + N(x,y,z)\,\hat{j} + P(x,y,z)\,\hat{k}$.
Let C be the curve given by $\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$ for $a \le t \le b$. Then

$$\begin{aligned}
\int_C \vec{F}\cdot d\vec{r} &= \int_a^b \vec{F}(\vec{r}(t))\vec{r}\,'(t)\,dt \\
&= \int_a^b [M(x(t),y(t),z(t))\,x'(t) + N\,y'(t) + P\,z'(t)]\,dt \\
&= \int_a^b Mdx + Ndy + Pdz.
\end{aligned}$$

Example 3.8. Evaluate $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F} = xy\,\hat{i} + yz\,\hat{j} + zx\,\hat{k}$ and C is the twisted cube given by $x = t,\ y = t^2,\ z = t^3,\ \ 0 \le t \le 1$.



$\int_C M dx = \int_0^1 t\,t^2\,1\,dt = 1/4, \quad \int_C N dy = t^2\,t^3\,2t\,dt = 2/7$, and
$\int_C P dz = t^3\,t\,3t^2\,dt = 3/7$. So,

$$\int_C \vec{F}\cdot d\vec{r} = 1/4 + 2/7 + 3/7 = 27/28.$$

Also,

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 [xyx' + yzy' + zxz']dt = \int_0^1 [t^3 + 2t^6 + 3t^6]dt = 27/28.$$

3.3 Conservative Fields

Recall: $\int_a^b f'(t)dt = f(b) - f(a)$ for a function $f(t)$. In case of line integrals, the gradient acts as a sort of derivative.

Theorem 3.2. *Let C be a smooth curve given by $\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$ for $a \le t \le b$. Suppose C joins points (x_1, y_1, z_1) to (x_2, y_2, z_2). That is,*

$$\vec{r}(a) = x_1\,\hat{i} + y_1\,\hat{j} + z_1\,\hat{k} \quad \textit{and} \quad \vec{r}(b) = x_2\,\hat{i} + y_2\,\hat{j} + z_2\,\hat{k}.$$

Let $f(x, y, z)$ be a function whose gradient vector is continuous on a domain containing C. Then

$$\int_C \nabla f\cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) = f(x_2, y_2, z_2) - f(x_1, y_1, z_1)$$

Proof:

$$\begin{aligned}
\int_C \nabla f\cdot d\vec{r} &= \int_a^b \nabla f(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt \\
&= \int_a^b \Big[\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\Big]\,dt \\
&= \int_a^b \frac{d}{dt} f(\vec{r}(t))\,dt = f(\vec{r}(t))\Big|_a^b = f(\vec{r}(b)) - f(\vec{r}(a)).
\end{aligned}$$

□

Theorem 3.2 is sometimes called as the *Fundamental theorem for line integrals*. It says that if $\vec{F}$ is a conservative vector field with potential f, then the line integral over any smooth curve joining points A to B can be evaluated from the potential by:

$$\int_C \vec{F}\cdot d\vec{r} = f(B) - f(A).$$

In such a case, the line integral is independent of path of C; it only depends on the initial point and the end point of C.

We say that a line integral $\int_C \vec{F} \cdot d\vec{r}$ is **independent of path** iff for any curve C' that is lying in the domain of $\vec{F}$, and having the same initial and end points as that of C, we have

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C'} \vec{F} \cdot d\vec{r}.$$

Thus, if $\vec{F}$ is conservative, then the line integral $\int_C \vec{F} \cdot d\vec{r}$ is path independent. Then the following result is obvious:

Theorem 3.3. *Let $\vec{F}$ be a continuous vector field defined on a domain D. Let C be any smooth curve lying in D. The line integral $\int_C \vec{F} \cdot d\vec{r}$ is path independent iff $\int_{C'} \vec{F} \cdot d\vec{r} = 0$ for every closed curve C' lying in D.*

Remark: A **closed curve** is a curve having the same initial and end points. "Smooth curve" may be replaced by "Piecewise smooth curve" everywhere. When C is a closed curve, the line integral over C is written as $\oint_C \vec{F} \cdot d\vec{r}$.

Theorem 3.4. *Let $\vec{F}$ be a continuous vector field defined on an open connected region D. If $\int_C \vec{F} \cdot d\vec{r}$ is path independent for each smooth curve C lying in D, then $\vec{F}$ is conservative.*

Hints for the Proof: Suppose D is in the plane. Fix any point (a, b) in D. Let C be a curve from (a, b) to (x, y). Define

$$f(x, y) := \int_C \vec{F} \cdot d\vec{r} = \int_{(a,b)}^{(x,y)} \vec{F} \cdot d\vec{r},$$

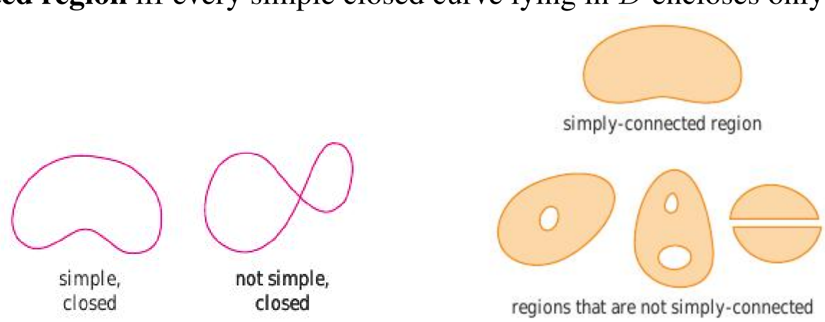
due to path independence. Next show that $\vec{F} = \text{grad } f$. $\square$

If $\vec{F}(x, y) = M(x, y)\,\hat{\imath} + N(x, y)\,\hat{\jmath}$ is conservative, then we have a scalar function $f(x, y)$ such that $f_x = M,\ f_y = N$. Then using Clairaut's theorem, we have $f_{xy} = M_y = f_{yx} = N_x$. That is, if $\vec{F} = M\,\hat{\imath} + N\,\hat{\jmath}$ is conservative, then $M_y = N_x$. Similar result holds in three dimensions.

Theorem 3.5. *Let $\vec{F}(x, y, z) = M(x, y, z)\,\hat{\imath} + N(x, y, z)\,\hat{\jmath} + P(x, y, z)\,\hat{k}$, where the gradients of the component functions M, N, P are continuous on a domain D. If $\vec{F}$ is conservative, then we have $M_y = N_x,\ N_z = P_y,\ P_x = M_z$ on D.*

The converse of Theorem 3.5 holds if the domain of $\vec{F}$ is a simply connected domain.

A **simple curve** is a curve which does not intersect itself. A connected region D is said to be a **simply connected region** iff every simple closed curve lying in D encloses only points from D.



Theorem 3.6. *Let $\vec{F} = M\,\hat{\imath} + N\,\hat{\jmath} + P\,\hat{k}$ be a vector field on a simply connected region D, where gradients of M, N, P are continuous. If $M_y = N_x$, $N_z = P_y$, and $P_x = M_z$ hold on D, then $\vec{F}$ is conservative.*

Proof of this can be done here, but it follows from Green's theorem in the plane and from Stokes' theorem in space, which we will do later.

These equations help in determining the potential function of a conservative field.

Example 3.9. Find the line integral of the field $\vec{F} = yz\,\hat{\imath} + zx\,\hat{\jmath} + xy\,\hat{k}$ along any smooth curve joining the points $A(-1, 3, 9)$ to $B(1, 6, -4)$.

Notice that $\vec{F}$ is conservative since $\vec{F} = \text{grad}\,(xyz) = \nabla f$, where $f = xyz$. Let C be any such curve. Then

$$\int_C \vec{F} \cdot d\vec{r} = \int_A^B \nabla f \cdot d\vec{r} = f(B) - f(A) = 3.$$

Example 3.10. Are the following vector fields conservative?

(a) $\vec{F}(x, y) = (x - y)\,\hat{\imath} + (x - 2)\,\hat{\jmath}$

(b) $\vec{F}(x, y) = (3 + 2xy)\,\hat{\imath} + (x^2 - 3y^2)\,\hat{\jmath}$.

(c) $\vec{F}(x, y, z) = (2x - 3)\,\hat{\imath} + z\,\hat{\jmath} + \cos z\,\hat{k}$.

(a) $\vec{F} = M\,\hat{\imath} + N\,\hat{\jmath}$, where $M = x - y,\ N = x - 2$. $M_y = -1,\ N_x = 1$. Since $M_y \neq N_x$, the vector field F is not conservative.

(b) Here, $M = 3 + 2xy,\ N = x^2 - 3y^2$. $M_y = 2x = N_x$. The vector filed is defined on $\mathbb{R}^2$, which is a simply connected region. The partial derivatives of M and N are continuous. Therefore, $\vec{F}$ is a conservative field.

(c) $\vec{F} = M\,\hat{\imath} + N\,\hat{\jmath} + P\,\hat{k}$, where $M = 2x - 3,\ N = z,\ P = \cos z$.
$M_y = 0,\ N_x = 0,\ N_z = 1, P_y = 0,\ P_x = 0,\ M_z = 0$.
Since $N_z \neq P_y$, the field $\vec{F}$ is not conservative.

Example 3.11. Find a potential for the vector field $\vec{F} = (3 + 2xy)\,\hat{\imath} + (x^2 - 3y^2)\,\hat{\jmath}$. Then evaluate $\int_C \vec{F} \cdot d\vec{r}$, where C is given by $\vec{r}(t) = e^t \sin t\,\hat{\imath} + e^t \cos t\,\hat{\jmath},\ 0 \le t \le \pi$.

We know that $\vec{F}$ is conservative. To determine the scalar function $f(x, y, z)$ such that $\vec{F} = \text{grad}\ f$, we have

$$f_x = 3 + 2xy, \quad f_y = x^2 - 3y^2.$$

Integrate the first one with respect to x and integrate the second with respect to y to obtain:

$$f(x, y) = 3x + x^2 y + g(y), \quad f(x, y) = x^2 y - y^3 + h(x).$$

Taking $g(y) = y^3 + \text{const.}$ and $h(x) = 3x + \text{const.}$, we have

$$f(x, y) = 3x + x^2 y - y^3 + k \quad \text{for any constant } k.$$

Next, $\int_C \vec{F} \cdot d\vec{r} = f(x(\pi), y(\pi)) - f(x(0), y(0)) = e^{3\pi} + 1$.

Example 3.12. Find a potential for the vector field $\vec{F} = y^2\,\hat{\imath} + (2xy + e^{3z})\,\hat{\jmath} + 3ye^{3z}\,\hat{k}$.

Denote the potential by $f(x, y, z)$. Then

$$f_x = y^2,\ f_y = 2xy + e^{3z},\ f_z = 3ye^{3z}.$$

Integrate with respect to suitable variables:

$$f = xy^2 + g(y, z),\ f = xy^2 + ye^{3z} + h(x, z),\ f = ye^{3z} + \phi(x, y).$$

Taking $g(x, z) = ye^{3z},\ \phi(x, y) = xy^2,\ h(x, z) = k$, a constant, we get one such f.

Sometimes matching may not be obvious. So, differentiate the first:

$$f_y = 2xy + g_y(y, z) = 2xy + e^{3z}.$$

Thus, $g_y(y, z) = e^{3z}$. Integrate: $g(y, z) = ye^{3z} + \psi(z)$. Then

$$f = xy^2 + ye^{3z} + \psi(z).$$

This gives $f_z = 3e^{3z} + \psi'(z) = 3y^{3z}$. Thus, $\psi(z) = k$, a const. Therefore,

$$f(x, y, z) = xy^2 + ye^{3z} + k.$$

Example 3.13. Show that the vector field $\vec{F} = (e^x \cos y + yz)\,\hat{\imath} + (xz - e^x \sin y)\,\hat{\jmath} + (xy + z)\,\hat{k}$ is conservative by finding a potential for it.

Let the potential be $f(x, y, z)$. Then

$$f_x = e^x \cos y + yz,\ f_y = xz - e^x \sin y,\ f_z = xy + z.$$

Integrate the first w.r.t. x to get

$$f = e^x \cos y + xyz + g(y, z).$$

Differentiate w.r.t. y to get

$$f_y = -e^x \sin y + xz + g_y(y, z) = xz - e^x \sin y \Rightarrow g_y(y, z) = 0.$$

Thus $g(y, z) = h(z)$. And then $f = e^x \cos y + xyz + h(z)$. Differentiate w.r.t. z to obtain

$$f_z = xy + h'(z) = xy + z \Rightarrow h'(z) = z \Rightarrow h(z) = z^2/2 + k.$$

Then $f(x, y, z) = e^x \cos y + xyz + z^2/2 + k$.

If M, N, P are functions of x, y, z, on a domain D in space, then the expression

$$M(x, y, z)dx + N(x, y, z)dy + P(x, y, z)dz$$

Is called a **differential form**. The differential form is called **exact** iff there exists a function $f(x, y, z)$ such that

$$M(x, y, z) = \frac{\partial f}{\partial x},\ N(x, y, z) = \frac{\partial f}{\partial y},\ P(x, y, z) = \frac{\partial f}{\partial z}.$$

Notice that if the differential form is exact, then

$$M(x,y,z)dx + N(x,y,z)dy + P(x,y,z)dz = df,$$

which is an exact differential. In that case, if C is any curve joining points A to B in the domain D, then

$$\int_A^B [Mdx + Ndy + Pdz] = \int_C \nabla f \cdot d\vec{r} = \int_A^B df = f(B) - f(A).$$

Therefore, the differential form is exact iff $\vec{F} = M\,\hat{i} + N\,\hat{j} + P\,\hat{k}$ is conservative. Then the scalar function $f(x,y,z)$ is the potential of the field $\vec{F}$.

Example 3.14. Show that the differential form $ydx + xdy + 4dz$ is exact. Then evaluate the integral $\int_C (ydx + xdy + 4dz)$ over the line segment C joining the points $(1,1,1)$ to $(2,3,-1)$.

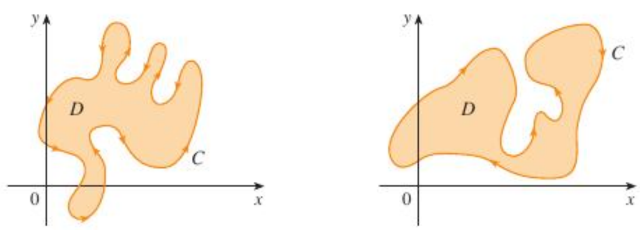
$M = y,\ N = x,\ P = 4$. Then $M_y = 1 = N_x,\ N_z = 0 = P_y,\ P_x = 0 = M_z$.

Therefore, the differential form is exact.
Also, notice that $ydx + xdy + 4dz = d(xy + 4z + k)$. Hence it is exact.
In case, f is not obvious, we can determine it as earlier by differentiating and integrating etc. Next,

$$\int_C (ydx + xdy + 4dz) = \int_{(1,1,1)}^{(2,3,-1)} d(xy + 4z + k) = (xy + 4z + k)\Big|_{(1,1,1)}^{(2,3,-1)} = -3.$$

3.4 Green's Theorem

Let C be a simple closed curve in the plane. The **positive orientation** of C refers to a single counter-clockwise traversal of C. If C is given by $\vec{r}(t),\ a \le t \le b$, then its positive orientation refers to a traversal of C keeping the region D bounded by the curve to the left.



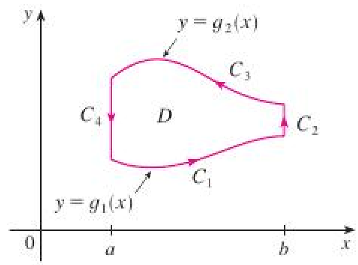
(a) Positive orientation (b) Negative orientation

Theorem 3.7. **(Green's Theorem)** *Let C be a positively oriented simple closed piecewise smooth curve in the plane. Let D be the region bounded by C. (That is, $C = \partial D$.) If $M(x,y)$ and $N(x,y)$ have continuous partial derivatives on an open region containing D, then*

1. $\displaystyle\oint_C (Mdx + Ndy) = \iint_D \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big) dA.$

2. $\displaystyle\oint_C (Mdy - Ndx) = \iint_D \Big(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\Big) dA.$

Green's theorem helps in evaluating an integral of the type $\int_a^b \vec{F} \cdot d\vec{r}$ in a non-conservative vector field $\vec{F}$. It gives a relationship between a line integral around a simple closed curve C and the double integral over the plane region D bounded by this closed curve.



Proof: We only prove for a special kind of regions to give an idea of how it is proved.

Consider the region $D = \{(x, y) : a \le x \le b,\ f(x) \le y \le g(x)\}$. Assume that f, g are continuous functions. Then

$$\iint_D \frac{\partial M}{\partial y} dA = \int_a^b \int_{f(x)}^{g(x)} M_y dy dx = \int_a^b [M(x, g(x)) - M(x, f(x))] dx.$$

Now we compute $\int_C M dx$ by breaking C into four parts C_1, C_2, C_3 and C_4.
The curve C_1 is given by $x = x,\ y = f(x),\ a \le x \le b$. Thus

$$\int_{C_1} M dx = \int_a^b M(x, f(x)) dx.$$

On C_2 and also on C_4, the variable x is a single point. So,

$$\int_{C_2} M dx = \int_{C_4} M dx = 0.$$

As x increases, C_3 is traversed backward. That is, $-C_3$ is given by $x = x$, $y = g(x)$, $a \le x \le b$. So,

$$\int_{C_3} M dx = -\int_{C_3} M dx = -\int_a^b M(x, g(x)) dx.$$

Therefore, $\displaystyle\iint_D \frac{\partial M}{\partial y} dA = -\int_C M dx$. Similarly, express D using the variable of integration as y. Then we have $\displaystyle\iint_D \frac{\partial N}{\partial x} dA = \int_C N dy$. Next, add the two results obtained to get

$$\int_C (M dx + N dy) = \iint_D \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \Big) dA.$$

The second form follows similarly. □

Example 3.15. Verify Green's theorem for the field $\vec{F} = (x - y)\,\hat{\imath} + x\,\hat{\jmath}$, where C is the unit circle oriented positively.

Here, we have $C :\ \vec{r}(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath},\ 0 \le t \le 2\pi$. The region D is the unit disk.

$$M = \cos t - \sin t,\ N = \cos t, \quad dx = -\sin t\, dt,\ dy = \cos t\, dt.$$
$$M_x = 1,\ M_y = -1,\ N_x = 1,\ N_y = 0.$$

Now,
$$\oint_C (Mdy - Ndx) = \int_0^{2\pi} [(\cos t - \sin t)\cos t - \cos t(-\sin t)]dt = \pi.$$
$$\iint_D (M_x + N_y)dA = \iint_D (1+0)dA = \text{Area of } D = \pi.$$

Similarly,
$$\oint_C (Mdx + Ndy) = \int_0^{2\pi} [(\cos t - \sin t)(-\sin t) + \cos^2 t]dt = 2\pi.$$
$$\iint_D (N_x - M_y)dA = \iint_D (1-(-1))dA = 2 \times \text{Area of } D = 2\pi.$$

Example 3.16. Evaluate the integral $I = \oint_C xy\, dy + y^2\, dx$, where C is the square cut from the first quadrant by the lines $x = 1$ and $y = 1$, with positive orientation.

Take $M(x,y) = xy,\ N(x,y) = y^2,\ D$ as the region bounded by C. Then
$$I = \oint_C (Mdy - Ndx) = \iint_D (M_x + N_y)dA = \int_0^1 \int_0^1 (y + 2y)dxdy = 3/2.$$

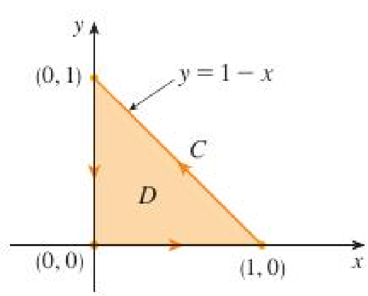
Also, taking $M = -y^2,\ N = xy$, we have
$$I = \oint_C (Mdx + Ndy) = \iint_D (N_x - M_y)dA = \int_0^1 \int_0^1 (y + 2y)dxdy = 3/2.$$

Example 3.17. Evaluate the integral $I = \oint_C (3y - e^{\sin x})dx + (7x + \sqrt{1+y^4})dy$,

where C is the positively oriented circle $x^2 + y^2 = 9$.

Take D as the disk $x^2 + y^2 \le 9$. Then by Green's theorem,
$$I = \iint_D \big[(7x + \sqrt{1+y^4})_x - (3y - e^{\sin x})_y\big]dA = \iint_D (7-3)dA = 36\pi.$$

Example 3.18. Evaluate $I = \oint_C x^4\, dx + xy\, dy$, where C is the triangle with vertices at $(0,0),\ (0,1)$ and $(1,0)$; its orientation being from $(0,0)$ to $(1,0)$ to $(0,1)$ to $(0,0)$.



The triangle is positively oriented. Let D be the region bounded by the triangle.
Take $M = x^4,\ N = xy$. Then
$$I = \iint_D [(xy)_x - (x^4)_y]dA = \int_0^1 \int_0^{1-x} y\, dy\, dx = \frac{1}{2}\int_0^1 (1 - x^2)dx = \frac{1}{6}.$$

Example 3.19. Evaluate $\int_C (xdy - y^2dx)$, where C is the positively oriented square bounded by the lines $x = \pm 1$ and $y = \pm 1$.

Let $\vec{F}$ be the vector field $x\,\hat{\imath} + y^2\,\hat{\jmath}$. Here, $M = x,\ N = y^2$, and D is the region bounded by C. By Green's theorem,

$$\oint_C (Mdy - Ndx) = \iint_D (M_x + N_y)dA = \int_{-1}^{1}\int_{-1}^{1}(1-2y)dxdy = 4.$$

Two important Observations

1. Suppose $M(x,y)$ and $N(x,y)$ are zero on a simple closed curve C.
   If D is the region bounded by C, then

$$\iint_D (N_x - M_y)dA = \oint_C (Mdx + Ndy) = 0, \quad \iint_D (M_x + N_y)dA = \oint_C (Mdy - Ndx) = 0.$$

2. Let D be the region bounded by a simple closed curve C.
   Suppose $N_x - M_y = 1$. Then Area of D = $\iint_D (N_x - M_y)dA = \oint_C (Mdx + Ndy)$ gives

$$\text{Area of } D = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C (x\,dy - y\,dx).$$

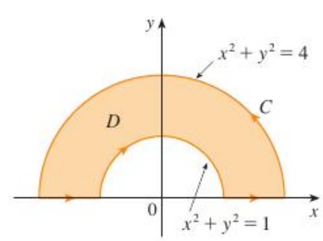
For example, the constraint is satisfied when

$$M = 0,\ N = x; \quad \text{Or,} \quad M = -y,\ N = 0; \quad \text{Or,} \quad M = -y/2,\ N = x/2.$$

As an application, to compute the area enclosed by the ellipse $C : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, we parameterize C as $x = a\cos t,\ y = b\sin t,\ 0 \le t \le 2\pi$. And then the area is

$$\frac{1}{2}\oint_C (x\,dy - y\,dx) = \frac{1}{2}\int_0^{2\pi}[(a\cos t\, b\cos t) - (b\sin t\,(-b\sin t))]dt = \frac{1}{2}\int_0^{2\pi} ab\,dt = \pi\, ab.$$

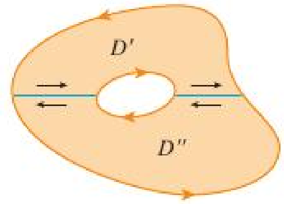
Example 3.20. Evaluate $\oint_C (y^2\,dx + xy\,dy)$, where C is the boundary of the semi- annular region between the semicircles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ in the upper half plane.



Write, in polar coordinates, $D = \{(r,\theta) : 1 \le r \le 2,\ 0 \le \theta \le \pi\}$. Then

$$\begin{aligned}\oint_C (y^2\,dx + xy\,dy) &= \iint_D \left[\frac{\partial}{\partial x}(3xy) - \frac{\partial}{\partial y}(y^2)\right] dA = \iint_D y\,dA \\ &= \int_1^2\int_0^{\pi} r\sin\theta r\,dr\,d\theta = \int_1^2 r^2 dr \int_0^{\pi}\sin\theta\,d\theta = \frac{14}{3}.\end{aligned}$$

In fact, Green's theorem can be applied to domains having holes, provided the domain can be divided into simply connected regions.



The boundary C of the region D consists of two simple closed curves C_1 (Outer) and C_2 (inner). Assume that these boundary curves are oriented so that the region D is always on the left as the curve C is traversed.

Thus the positive direction is counterclockwise for the outer curve C_1 but clockwise for the inner curve C_2. Divide D into two regions D' and D'' as shown in the figure. Green's theorem on D' and D'' gives

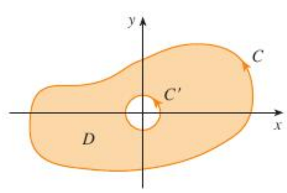
$$\begin{aligned}\iint_D (N_x - M_y)dA &= \iint_{D'} (N_x - M_y)dA + \iint_{D''} (N_x - M_y)dA \\ &= \int_{\partial D'} (Mdx + Ndy) + \int_{\partial D''} (Mdx + Ndy) = \int_C (Mdx + Ndy).\end{aligned}$$

This is the general version of Green's Theorem.

Example 3.21. Show that if C is any positively oriented simple closed path that encloses the origin, then

$$\oint_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2} = 2\pi,$$

No idea how to show it for every such curve. So, take a positively oriented circle C', of radius a, around origin that lies entirely in the region bounded by C. Let D be the annular region bounded by C and C'. Take $\vec{F}(x,y) = (-y\,\hat{\imath} + x\,\hat{\jmath})/(x^2+y^2)$.



Then the positively oriented boundary of D is $\partial D = C \cup (-C')$. Green's theorem on D gives

$$\oint_C (Mdx + Ndy) + \oint_{-C'} (Mdx + Ndy) = \iint_D (N_x - M_y)dA = 0$$

Reason? Here, $\vec{F} = M\,\hat{\imath} + N\,\hat{\jmath}$ gives $N_x = M_y = (y^2 - x^2)/(x_2 + y^2)^2$. Then

$$\oint_C (Mdx + Ndy) = \oint_{C'} (Mdx + Ndy).$$

But C' is parameterized by $x(t) = \cos t,\ y(t) = \sin t,\ 0 \le t \le 2\pi$. So,

$$\int_{C'} (Mdx + Ndy) = \int_0^{2\pi} \vec{F}(a\cos t\,\hat{\imath} + a\sin t\,\hat{\jmath}) \cdot (a\cos t\,\hat{\imath} + a\sin t\,\hat{\jmath})'dt = 2\pi.$$

Generalize this example by taking the constraint $N_x = M_y$ on the vector field.

3.5 Curl and Divergence of a vector field

If $\overrightarrow{F} = M\,\hat{\imath} + N\,\hat{\jmath} + P\,\hat{k}$ is a vector field in $\mathbb{R}^3$, where the partial derivatives of the component functions exist, then $\operatorname{curl}\overrightarrow{F}$ is a vector field given by

$$\operatorname{curl}\overrightarrow{F} = \Big(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\Big)\,\hat{\imath} + \Big(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\Big)\,\hat{\jmath} + \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big)\,\hat{k}.$$

Writing in operator notation, recall that $\operatorname{grad} = \nabla = \hat{\imath}\dfrac{\partial}{\partial x} + \hat{\jmath}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}$.

Then $\operatorname{curl}\overrightarrow{F} = \nabla \times \overrightarrow{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$.

For example, if $\overrightarrow{F} = zx\,\hat{\imath} + xyz\,\hat{\jmath} - y^2\,\hat{k}$, then $\operatorname{curl}\overrightarrow{F} = -y(2+x)\,\hat{\imath} + x\,\hat{\jmath} + yz\,\hat{k}$.

Theorem 3.8. *Let $\overrightarrow{F}$ be a vector field defined over a simply connected region D whose component functions have continuous second order partial derivatives. Then $\overrightarrow{F}$ is conservative iff* $\operatorname{curl}\overrightarrow{F} = 0$.

Proof of $\Rightarrow$: If $\overrightarrow{F}$ is conservative, then $\overrightarrow{F} = \nabla f$ for some f, where f is some scalar function defined on D. Now,

$$\operatorname{curl}\nabla f = \nabla \times (\nabla f) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} = (f_{yz} - f_{zy})\,\hat{\imath} + (f_{zx} - f_{xz})\,\hat{\jmath} + (f_{xy} - f_{yx})\,\hat{k} = 0.$$

The converse follows from Stokes' theorem, which we will discuss later. □

Remember: The curl of gradient of any scalar function is zero:

$$\operatorname{curl}\operatorname{grad} f = 0.$$

Example 3.22. Is the vector field $\overrightarrow{F} = zx\,\hat{\imath} + xyz\,\hat{\jmath} - y^2\,\hat{k}$ conservative?

Here, $\operatorname{curl}\overrightarrow{F} = -y(2+x)\,\hat{\imath} + x\,\hat{\jmath} + yz\,\hat{k} \neq 0$. So, $\overrightarrow{F}$ is not conservative.

Example 3.23. Is the vector field $\overrightarrow{F} = y^2z^3\,\hat{\imath} + 2xyz^3\,\hat{\jmath} + 3xy^2z^2\,\hat{k}$ conservative?

Here, $\overrightarrow{F}$ is defined on $\mathbb{R}^2$ and

$$\operatorname{curl}\overrightarrow{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix} \quad = \quad \begin{array}{c} (6xy^2z^2 - 6xy^2z^2)\,\hat{\imath} \\ -(3y^2z^2 - 3y^2z^2)\,\hat{\jmath} \\ +(2yz^3 - 2yz^3)\,\hat{k} \end{array} \quad = \quad 0.$$

Hence $\overrightarrow{F}$ is conservative. In fact, $\overrightarrow{F} = \operatorname{grad} f$, where $f(x,y,z) = xy^2z^3$.

The name game: $\operatorname{curl}\overrightarrow{F}$ measures how quickly a tiny peddle (at a point) in some fluid in a vector field moves around itself. If $\operatorname{curl}\overrightarrow{F} = 0$, then there is no rotation of such a tiny peddle.

If $\vec{F} = M\,\hat{i} + N\,\hat{j} + P\,\hat{k}$ is a vector field defined on a domain, where its component functions have first order partial derivatives, then

$$\text{div}\,\vec{F} = \nabla \cdot \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}.$$

The **divergence** is also called **flux** or **flux density**.

For example, if $\vec{F} = zx\,\hat{i} + xyz\,\hat{j} - y^2\,\hat{k}$, then $\text{div}\,\vec{F} = z + xz$.

The divergence of the vector field $\vec{F} = (x^2 - y)\,\hat{i} + (xy - y^2)\,\hat{j}$ is

$$\frac{\partial(x^2 - y)}{\partial x} + \frac{\partial(xy - y^2)}{\partial y} = 3x - 2y.$$

Intuitively, $\text{div}\,\vec{F}$ measures the tendency of the fluid to diverge from the point (a, b). When the gas (fluid) is expanding, divergence is positive; and when it is compressing, the divergence is negative. The fluid is said to be **incompressible** iff $\text{div}\,\vec{F} = 0$.

Theorem 3.9. *Let $\vec{F} = M\,\hat{i} + N\,\hat{j} + P\,\hat{k}$ be a vector field defined on a simply connected domain $D \subseteq \mathbb{R}^3$, where M, N, P have continuous second order partial derivatives. Then* $\text{div curl}\,\vec{F} = 0$.

Proof: $\text{div curl}\,\vec{F} = \nabla \cdot (\nabla \times \vec{F}) = \frac{\partial}{\partial x}\Big(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\Big) + \frac{\partial}{\partial y}\Big(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\Big) + \frac{\partial}{\partial z}\Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big)$

This is equal to zero, due to Clairaut's Theorem. □

Example 3.24. Does there exist a vector field G such that $\vec{F} = zx\,\hat{i} + xyz\,\hat{j} - y^2\,\hat{k} = \text{curl}\,G$?

$\text{div}\,\vec{F} = z + xz \neq 0$. Hence there is no such G.

Divergence of $\text{grad}\,f$ is the **Laplacian** of a scalar function f since

$$\text{div grad}\,f = \nabla \cdot (\nabla \vec{F}) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} := \nabla^2 f.$$

The operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called the Laplacian.

Green's Theorem - Vector form - 1

Let D be a simply connected region whose boundary is the simple closed curve C.
Let $\vec{F} = M\,\hat{i} + N\,\hat{j}$ be a vector field defined on D.
Let C be parameterized by $\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}$. Then

$$\vec{F} \cdot \vec{T}(t)dt = \vec{F} \cdot \vec{r}\,'(t)dt = \vec{F} \cdot d\vec{r} = Mdx + Ndy.$$

The line integral of $\vec{F}$ over C is

$$\oint \vec{F} \cdot \vec{T}(t)dt = \oint_C \vec{F} \cdot d\vec{r} = \oint_C (M\,dx + N\,dy).$$

Consider $\vec{F}$ as a vector field on $\mathbb{R}^3$ with $P = 0$. Then

$$\text{curl}\,\vec{F} = (N_x - M_y)\,\hat{k} \Rightarrow \text{curl}\,\vec{F} \cdot \hat{k} = N_x - M_y.$$

Thus Green's theorem takes the form

$$\oint_C \vec{F} \cdot \vec{T}(t)\, dt = \oint_C \vec{F} \cdot d\vec{r} = \iint_D (\text{curl } \vec{F} \cdot \hat{k})\, dA.$$

Recall: $\vec{T}$ is the unit tangent vector and $\hat{n}$ is the unit normal vector.

Green's Theorem - Vector form - 2

Let C be given by $\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}$. Then

$$\vec{T} = \frac{x'(t)}{|\vec{r}\,'(t)|}\,\hat{i} + \frac{y'(t)}{|\vec{r}\,'(t)|}, \quad \hat{n}(t) = \frac{y'(t)}{|\vec{r}\,'(t)|}\,\hat{i} - \frac{x'(t)}{|\vec{r}\,'(t)|}.$$

Consequently, $\vec{F} \cdot \hat{n} = [M(x(t), y(t))y'(t) - N(x(t), y(t))x'(t)]/|\vec{r}\,'(t)|$.

Now, $\oint_C \vec{F} \cdot \hat{n}\, ds = \int_a^b \vec{F} \cdot \hat{n}|\vec{r}\,'(t)|\, dt = \oint_C (M dy - N dx)$.

Also, $\iint_D \text{div } \vec{F}\, dA = \iint_D (M_x + N_y) dA$.

Hence Green's theorem takes the form

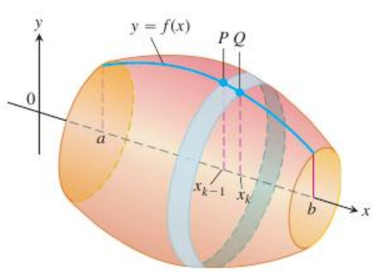
$$\oint_C \vec{F} \cdot \hat{n}\, ds = \iint_D \text{div } \vec{F}\, dA.$$

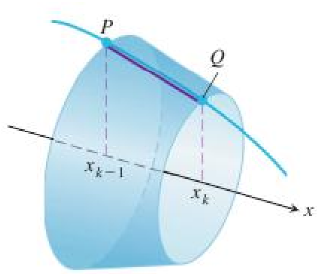
The first form is called the **tangent-form** and the second form is called the **normal-form** of Green's theorem.

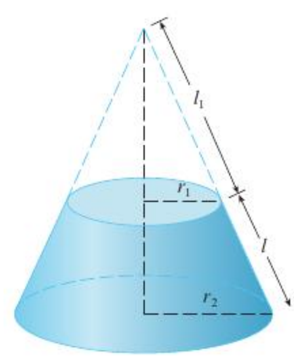
3.6 Surface Area of solid of Revolution

Suppose a smooth curve is given by $y = f(x)$, where $f(x) \geq 0$. Its arc when $a \leq x \leq b$ is revolved about the x-axis to generate a solid. How do we compute the area of the surface of this solid?

We follow a strategy similar to computing the volume of revolution. Partition $[a, b]$ into n subintervals $[x_{k-1}, x_k]$. When each Δx_k is small, the surface area corresponding to this subinterval is approximately same as the area on the frustum of a right circular cone.







If a right circular cone has base radius R and slant height ℓ, then its surface area is given by $\pi R\ell$. Now, for the frustum, we subtract the smaller cone surface area from the larger. Look at the figure. The area of the frustum is

$$A = \pi r_2(\ell_1 + \ell) - \pi r_1 \ell_1 = \pi[(r_2 - r_1)\ell_1 + r_2 \ell].$$

Using similarity of triangles, we have $\dfrac{\ell_1}{r_1} = \dfrac{\ell_1 + \ell}{r_2}$.

This gives $r_2\ell_1 = r_1\ell_1 + r_1\ell \Rightarrow (r_2 - r_1)\ell_1 = r_1\ell$. Therefore,

$$A = \pi(r_1\ell + r_2\ell) = 2\pi\, r\, \ell, \quad \text{where } \ r = \frac{r_1 + r_2}{2}.$$

To use this this formula on the frustum obtained on the subinterval $[x_{k-1}, x_k]$, we notice that the slant height ℓ is approximated by $\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$, where $\Delta x_k = x_k - x_{k-1}$ and $\Delta y_k = f(x_k) - f(x_{k-1})$. Next, the average radius $r = \frac{r_1+r_2}{2}$ is $\frac{f(x_{k-1})+f(x_k)}{2}$. Thus the area of the frustum is

$$A_k = 2\pi \frac{f(x_{k-1}) + f(x_k)}{2} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}.$$

Due to MVT, we have $c_k \in [x_{k-1}, x_k]$ such that

$$\Delta y_k = f(x_k) - f(x_{k-1}) = f'(c_k)(x_k - x_{k-1}) = f'(c_k)\Delta x_k.$$

So, $\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sqrt{1 + (f'(c_k))^2}\, \Delta x_k$. The surface of revolution is approximated by

$$\sum_{k=1}^{n} A_k = 2\pi\, \frac{f(x_{k-1}) + f(x_k)}{2}\, \sqrt{1 + (f'(c_k))^2}\, \Delta x_k.$$

Its limit as $n \to \infty$ is the Riemann sum of an integral, which is the required area:

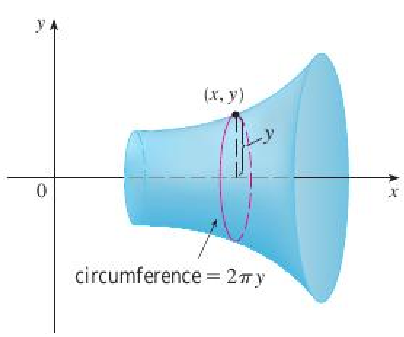
$$S = \int_a^b 2\pi\, f(x)\, \sqrt{1 + (f'(x))^2}\, dx = \int_a^b 2\pi y \sqrt{1 + (f'(x))^2} dx.$$

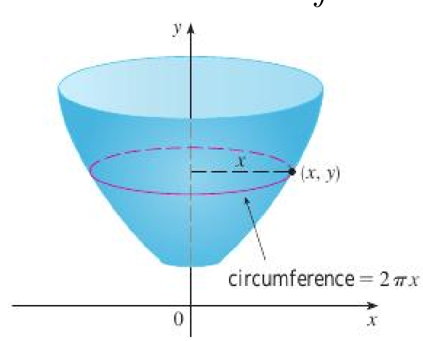
If the arc is given by $x = g(y),\ c \le y \le d$, then the surface area of revolution is given by

$$S = \int_c^d 2\pi\, g(y)\, \sqrt{1 + (g'(y))^2}\, dy = \int_c^d 2\pi x \sqrt{1 + (g'(y))^2}\, dy.$$

Notice that with relevant limits of integration, if the revolution is about x-axis, then $S = \displaystyle\int 2\pi y ds$.

If the revolution is about y-axis, then the surface area of revolution is $S = \displaystyle\int 2\pi x ds$.





For parameterized curves, suppose the smooth curve is given by $x = x(t),\ y = y(t)$ for $a \le t \le b$. If the curve is traversed exactly once while t increases from a to b, then the surface area of the solid generated by revolving the curve about the coordinate axes are as follows:

1. Revolution about the x-axis $(y \geq 0)$: $S = \displaystyle\int_a^b 2\pi\, y(t)\, \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$.

2. Revolution about the y-axis $(x \geq 0)$: $S = \displaystyle\int_a^b 2\pi\, x(t)\, \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$.

Example 3.25. Find the surface area of the solid obtained by revolving about x-axis, the arc of the curve $y = 2\sqrt{x}$, $1 \leq x \leq 2$.

Since $y = 2\sqrt{x},\ y' = 1/\sqrt{x},\ \sqrt{1+(y')^2} = \sqrt{1+1/x}$. Then

$$S = \int_1^2 2\pi y\Big(1+[y']^2\Big)^{1/2} dx = \int_1^2 2\pi\, 2\sqrt{x}\,\sqrt{1+\frac{1}{x}}\, dx = \frac{8\pi}{3}(3\sqrt{3} - 2\sqrt{2}).$$

Example 3.26. The arc of the parabola $y = x^2,\ 1 \leq x \leq 2$ is revolved about the y-axis. Find the surface area of revolution.

The curve can be parameterized by $\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath},\ 1 \leq t \leq 2$, where $x(t) = t$ and $y(t) = t^2$. Then $x'(t) = 1$ and $y'(t) = 2t$. The surface area is

$$\begin{aligned} S &= \int_1^2 2\pi\, x(t)\, \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = 2\pi \int_1^2 t\,\sqrt{1+4t^2}\, dt \\ &= \frac{\pi}{4}\int_1^2 \sqrt{1+4t^2}\, d(1+4t^2) = \frac{\pi}{6}\Big[\big(1+4t^2\big)^{3/2}\Big]_1^2 = \frac{\pi}{6}(17\sqrt{17} - 5\sqrt{5}) \end{aligned}$$

Example 3.27.

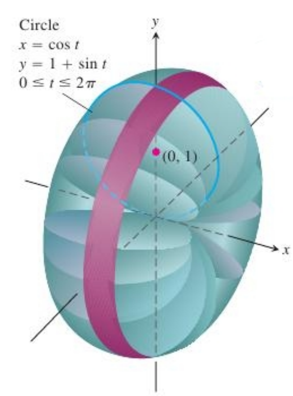
The circle of radius 1 centered at $(0, 1)$ is revolved about the x-axis. Find the surface area of the solid so generated.

The circle can be parameterized as

$$x = \cos t,\ y = 1 + \sin t,\ 0 \leq t \leq 2\pi.$$

Then $(x'(t))^2 + (y'(t))^2 = 1$. Thus the area is

$$S = \int_0^{2\pi} 2\pi\,(1+\sin t)\, dt = 4\pi^2.$$



3.7 Surface area

As we know, a smooth surface can be given by a function such as $z = f(x, y)$. More generally, a smooth surface is given parametrically by $x = x(u, v),\ y = y(u, v),\ z = z(u, v)$, where (u, v) varies over a given parameter domain. Normally, we say that the point (u, v) varies over a domain in uv-plane. The parametric equation is also written in vector form as

$$\vec{r} = x(u, v)\hat{\imath} + y(u, v)\hat{\jmath} + z(u, v)\hat{k}.$$

Some examples:

The cone $z=\sqrt{x^2+y^2},\ 0\le z\le 1$ can be parametrized by

$$x=r\cos\theta,\ y=r\sin\theta,\ z=r,\ \text{where } 0\le r\le 1 \text{ and } 0\le\theta\le 2\pi.$$

Then its vector form is

$$\vec{r}(r,\theta)=r\cos\theta\,\hat{i}+r\sin\theta\,\hat{j}+r\,\hat{k}.$$

The sphere $x^2+y^2+z^2=a^2$ can be parametrized by

$$x=a\cos\theta\sin\phi,\ y=a\sin\theta\sin\phi,\ z=a\cos\phi \text{ for } 0\le\theta\le 2\pi,\ 0\le\phi\le\pi.$$

In vector form the parametrization is

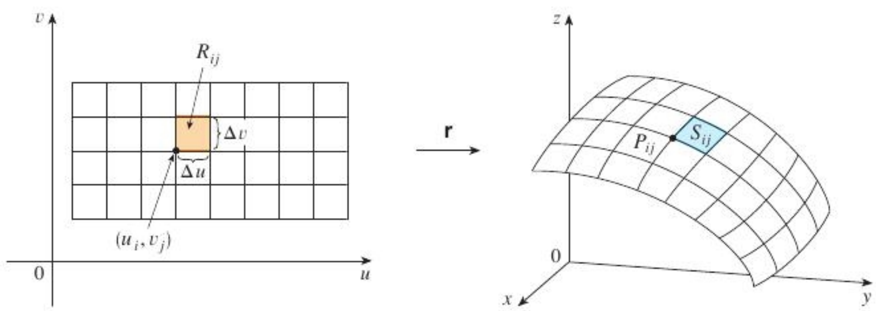
$$\vec{r}(\theta,\phi)=a\cos\theta\sin\phi\,\hat{i}+a\sin\theta\sin\phi\,\hat{j}+a\cos\phi\,\hat{k}.$$

The cylinder $x^2+y^2=a^2,\ 0\le z\le 5$ can be parametrized by

$$\vec{r}(\theta,z)=a\cos\theta\,\hat{i}+a\sin\theta\,\hat{j}+z\,\hat{k},\ \text{for } 0\le\theta\le 2\pi.$$

Let S be a smooth surface given parametrically by $x=x(u,v),\ y=y(u,v),\ z=z(u,v)$, where (u,v) ranges over a parameter domain D in the uv-plane. Suppose that S is covered exactly once as (u,v) varies over D. For simplicity, assume that D is a rectangle. We write S in vector form:

$$\vec{r}=x(u,v)\hat{i}+y(u,v)\hat{j}+z(u,v)\hat{k}.$$



Divide D into smaller rectangles R_{ij} with the lower left corner point as $P_{ij}=(u_i,v_j)$. For simplicity, let the partition be uniform with u-lengths as Δu and v-lengths as Δv. The part S_{ij} of S that corresponds to R_{ij} has the corner P_{ij} with position vector $\vec{r}(u_i,v_j)$. The tangent vectors to S at P_{ij} are given by

$$\vec{r}^{\,*}_u := \vec{r}_u(u_i,v_j)=x_u(u_i,v_j)\hat{i}+y_u(u_i,v_j)\hat{j}+z_u\hat{k}(u_i,v_j)$$

$$\vec{r}^{\,*}_v := \vec{r}_v(u_i,v_j)=x_v(u_i,v_j)\hat{i}+y_v(u_i,v_j)\hat{j}+z_v\hat{k}(u_i,v_j)$$

The tangent plane to S is the plane that contains the two tangent vectors $\vec{r}_u(u_i,v_j)$ and $\vec{r}_v(u_i,v_j)$. The normal to S at P_{ij} is the vector $\vec{r}_u(u_i,v_j)\times\vec{r}_v(u_i,v_j)$. Notice that since S is assumed to be smooth, the normal vector is non-zero.

The part S_{ij} is a curved parallelogram on S whose sides can be approximated by the vectors $\vec{r}^{\,*}_u\Delta u$ and $\vec{r}^{\,*}_v\Delta v$. Then the area of S_{ij} can be approximated by

$$\text{Area of } S_{ij}\simeq|\vec{r}^{\,*}_u\times\vec{r}^{\,*}_v|\Delta u\Delta v.$$

Then an approximation to the area of S is obtained by summing over both indices i and j:

$$\text{Area of } S \simeq \sum_j \sum_i |\vec{r}_u^{\,*} \times \vec{r}_v^{\,*}| \Delta u \Delta v.$$

We thus define the surface area by taking the limit of the above approximated quantity. It is as follows:

Let S be a smooth surface given parametrically by

$$\vec{r} = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k},$$

where $(u,v) \in D$, a domain in the uv-plane. Suppose that S is covered exactly once as (u,v) varies over D. Then the **surface area** of S is given by

$$\text{Area of } S = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA$$

where $\vec{r}_u = x_u\hat{i} + y_u\hat{j} + z_u\hat{k}$ and $\vec{r}_v = x_v\hat{i} + y_v\hat{j} + z_v\hat{k}$.

In case, the surface S is given by the graph of a function such as $z = f(x,y)$, where $(x,y) \in D$, then we take the parameters as $u = x,\ v = y$ and $z = z(u,v) = f(x,y)$. That is, S is given by

$$\vec{r} = u\hat{i} + v\hat{j} + z\hat{k}.$$

We see that

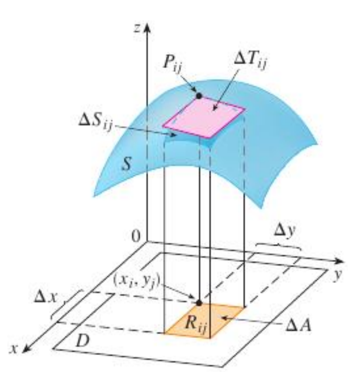
$$\vec{r}_u = \hat{i} + z_u\hat{k} = \hat{i} + f_x\hat{k}, \quad \vec{r}_v = \hat{j} + z_v\hat{k} = \hat{j} + f_y\hat{k}.$$

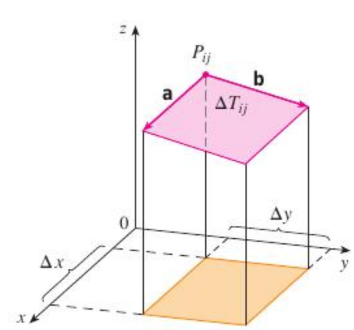
$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = -f_x\hat{i} - f_y\hat{j} + \hat{k}.$$

Therefore,

$$\text{Area of } S = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA = \iint_D \sqrt{f_x^2 + f_y^2 + 1} \, dA.$$

This formula can also be derived from the first principle as we had done for the parametric form. For this, suppose that S is given by the equation $z = f(x,y)$ for $(x,y) \in D$. Divide D into smaller rectangles R_{ij} with area $\Delta(R_{ij}) = \Delta x \Delta y$. For the corner (x_i, y_j) in R_{ij}, closest to the origin, let P_{ij} be the point $(x_i, y_j, f(x_i, y_j))$ on the surface. The tangent plane to S at P_{ij} is an approximation to S near P_{ij}.





The area T_{ij} of the portion of the tangent plane that lies above R_{ij} approximates the area of S_{ij}, the portion of S that is directly above R_{ij}. Therefore, we define the **area of the surface** S as

$$\Delta(S) = \lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} T_{ij}.$$

Let $\vec{a}$ and $\vec{b}$ be the vectors that start at P_{ij} and lie along the sides of the parallelogram whose area is T_{ij}. Then $T_{ij} = |\vec{a} \times \vec{b}|$. However, $f_x(x_i, y_j)$ and $f_y(x_i, y_j)$ are the slopes of the tangent lines through P_{ij} in the directions of $\vec{a}$ and $\vec{b}$, respectively. Therefore,

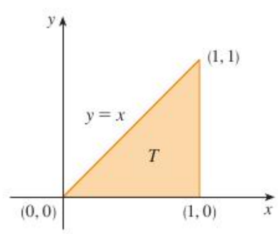
$$\vec{a} = \Delta x\,\hat{\imath} + f_x(x_i, y_j)\Delta x\,\hat{k}, \quad \vec{b} = \Delta y\,\hat{\jmath} + f_y(x_i, y_j)\Delta y\,\hat{k}.$$

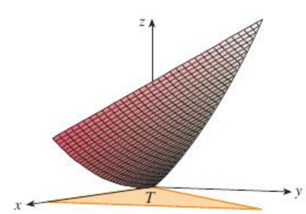
$$\begin{aligned} T_{ij} &= |\vec{a} \times \vec{b}| = |-f_x(x_i, y_j)\,\hat{\imath} - f_y(x_i, y_j)\,\hat{\jmath} + k|\,\Delta(R_{ij}) \\ &= \sqrt{f_x^2(x_i, y_j) + f_y^2(x_i, y_j) + 1}\,\Delta(R_{ij}). \end{aligned}$$

Summing over these T_{ij} and taking the limit, we obtain:

$$\text{Area of } S = \iint_D \sqrt{f_x^2 + f_y^2 + 1}\,dA.$$

Example 3.28. Find the surface area of the part of the surface $z = x^2 + 2y$ that lies above the triangular region in the xy-plane with vertices $(0, 0)$, $(1, 0)$ and $(1, 1)$.





$$T = \{(x, y) : 0 \le x \le 1,\ 0 \le y \le x\}, \quad f(x, y) = x^2 + 2y.$$

The required surface area is

$$\iint_T \sqrt{(2x)^2 + 2^2 + 1}\,dA = \int_0^1 \int_0^x \sqrt{4x^2 + 5}\,dydx = \frac{1}{12}(27 - 5\sqrt{5}).$$

Surface Area - a generalized form

Recall that for a surface S which is given by $f(x, y) = z$, the surface area is $\displaystyle\iint_D \sqrt{f_x^2 + f_y^2 + 1}dA$. Here, D is the rectangle on the xy-plane obtained by projecting S onto the plane.

Look at this surface as $f(x, y) - z = 0$. Then $\nabla f = f_x\,\hat{\imath} + f_y\,\hat{\jmath} - 1\,\hat{k}$. If $\vec{p}$ is the unit normal to the projected rectangle, then $\vec{p} = \hat{k}$. Then

$$\frac{|\nabla f|}{|\nabla f \cdot \vec{p}|} = \frac{\sqrt{f_x^2 + f_y^2 + 1}}{1^2},$$

which is the integrand in the surface area formula.

Warning: $\nabla f \cdot \vec{p}$ must not be ZERO.

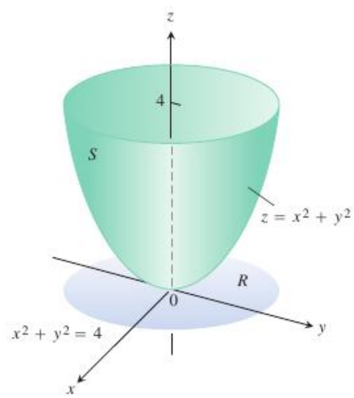
A derivation similar to the surface area formula gives the following:

Let the surface S be given by $f(x, y, z) = c$. Let R be a closed bounded region which is obtained by projecting the surface to a plane whose unit normal is $\vec{p}$. Suppose that ∇f is continuous on R and $\nabla f \cdot \vec{p} \neq 0$ on R. Then

$$\text{The surface area of } S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \vec{p}|} dA.$$

Of course, whenever possible, we project onto the coordinate planes.

Example 3.29. Find the area of the surface cut from the bottom of the paraboloid $x^2 + y^2 = z$ by the plane $z = 4$.

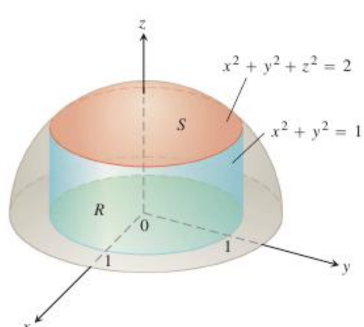


Surface S is given by $f(x, y, z) = x^2 + y^2 - z = 0$. Project it onto xy-plane to get the region R as $x^2 + y^2 \leq 4$. Then $\nabla f = 2x\,\hat{i} + 2y\,\hat{j} - \hat{k}$. $|\nabla f| = \sqrt{1 + 4x^2 + 4y^2}$.
$\vec{p} = \hat{k}$. $|\nabla f \cdot \vec{p}| = 1$.

R is given by $x = r\cos\theta,\ y = r\sin\theta,\ 0 \leq \theta \leq 2\pi, 0 \leq r \leq 2$. So, the surface area is

$$\iint_R \sqrt{1 + 4x^2 + 4y^2}\, dA = \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2}\, r\, dr d\theta = \frac{\pi}{6}(17\sqrt{17} - 1).$$

Example 3.30. Find the surface area of the cap cut from the hemisphere $x^2 + y^2 + z^2 = 2,\ z \geq 0$ by the cylinder $x^2 + y^2 = 1$.



The surface projected on xy-plane gives R as the disk $x^2+y^2 \le 1$. The surface is $f(x,y,z)=2$, where $f(x,y,z)=x^2+y^2+z^2$. Then

$$\nabla f = 2x\,\hat{i} + 2y\,\hat{j} + 2z\,\hat{k}, \quad |\nabla f| = 2\sqrt{x^2+y^2+z^2} = 2\sqrt{2}.$$

$\vec{p}=k$. $|\nabla f \cdot \vec{p}| = |2z| = 2z$. Thus the surface area is

$$\Delta = \iint_R \frac{2\sqrt{2}}{2z}\,dA = \sqrt{2}\iint_R z^{-1}\,dA = \sqrt{2}\iint_R (2-x^2-y^2)^{-1}\,dA.$$

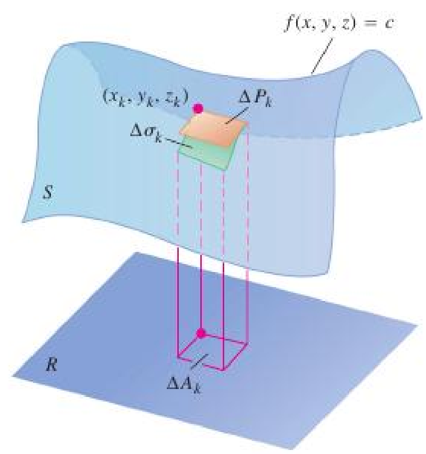
R is given by $x=r\cos\theta, y=r\sin\theta,\ 0\le\theta\le 2\pi,\ 0\le r\le 1$. So,

$$\Delta = \sqrt{2}\int_0^{2\pi}\int_0^1 \frac{r\,dr\,d\theta}{\sqrt{2-r^2}} = 2\pi(2-\sqrt{2}).$$

3.8 Integrating over a surface

Suppose over a surface $f(x,y,z)=c$, we have distribution of charge. The charge density, that is, the charge per unit area, may be given by a real valued function $g(x,y,z)$ defined on the surface. Then we may calculate the total charge on the surface as an integral.

So, we consider a real valued function $g(x,y,z)$ defined over a surface S given by $f(x,y,z)=c$; and our task is to compute the integral of g, where the area elements are taken over the surface. We look at the region R in the xy-plane on which this surface is defined by $f(x,y,z)=c$. Divide the region R into smaller rectangles ΔA_k. Consider the corresponding surface areas $\Delta\sigma_k$.



Let ΔP_k denote the projection of $\Delta\sigma_k$ onto the tangent plane at (x_k,y_k,z_k). Then

$$\Delta\sigma_k \simeq \Delta P_k.$$

If $\vec{p}$ is the unit normal to the region R, and if $\vec{u}_k$ and $\vec{v}_k$ are the vectors that lie along the edges of the patch ΔP_k, then

$$\Delta P_k = |\vec{u}_k \times \vec{v}_k|.$$

Write $\gamma_k =$ the angle between $\vec{u_k}\times\vec{v_k}$ and $\vec{p}$. Since $\vec{p}$ is a unit vector, we have

$$\Delta A_k = |\vec{u}_k \times \vec{v}_k \cdot \vec{p}| = |\vec{u}_k\times\vec{v}_k|\,|\vec{p}|\,|\cos(\gamma_k)| = \Delta P_k\,|\cos\gamma_k|.$$

Also, we have $|\nabla f \cdot \vec{p}| = |\nabla f|\,|\vec{p}|\,|\cos\gamma| = |\nabla f|\,|\cos\gamma|$. Therefore,

$$\Delta\sigma_k \approx \Delta P_k = \frac{A_k}{|\cos\gamma_k|} = \Big(\frac{|\nabla f|}{|\nabla f \cdot \vec{p}|}\Big)_k \Delta A_k.$$

Assuming that g is nearly constant on the smaller surface fragments σ_k, we form the sum

$$\sum_k g(x_k, y_k, z_k)\Delta\sigma_k \approx g(x_k, y_k, z_k)\Big(\frac{|\nabla f|}{|\nabla f \cdot \vec{p}|}\Big)_k \Delta A_k.$$

If this sum converges to a limit as the number of partitions, k approaches ∞, then we define that limit as the integral of g over the surface S. We thus define the surface integral as follows.

Let S be a surface S given by $f(x, y, z) = c$. Let the projection of S onto a plane with unit normal $\vec{p}$ be the region R. Let $g(x, y, z)$ be defined over S. Then the **surface integral of** g **over** S is

$$\iint_S g\,d\sigma = \iint_R g(x, y, z)\frac{|\nabla f|}{|\nabla f \cdot \vec{p}|}\,dA.$$

Also, we write the surface differential as

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \vec{p}|}\,dA.$$

Warning: $|\nabla f \cdot \vec{p}|$ must not be ZERO.

If the surface S can be represented as a union of non-overlapping smooth surfaces $S_1, \ldots, S_n$, then

$$\iint_S g\,d\sigma = \iint_{S_1} g\,d\sigma + \cdots + \iint_{S_n} g\,d\sigma.$$

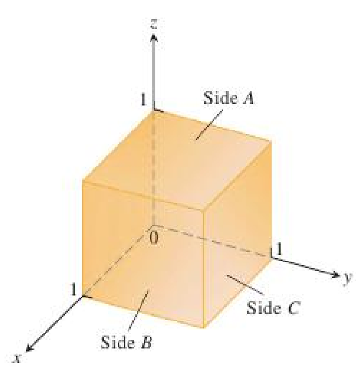
If $g(x, y, z) = g_1(x, y, z) + \cdots + g_m(x, y, z)$ over the surface S, then

$$\iint_S g\,d\sigma = \iint_S g_1\,d\sigma + \cdots + \iint_S g_m\,d\sigma.$$

Similarly, if $g(x, y, z) = k\,h(x, y, z)$ holds for a constant k, over S, then

$$\iint_S g(x, y, z)\,d\sigma = \iint_S k\,h(x, y, z)\,d\sigma.$$

Example 3.31. Integrate $g(x, y, z) = xyz$ over the surface of the cube cut from the first octant by the planes $x = 1,\ y = 1$, and $z = 1$.



We integrate g over the six surfaces and add the results. As $g = xyz$ is zero on the coordinate planes, we need integrals on sides A, B and C.

Side A is the surface defined on the region $R_A : 0 \le x \le 1,\ 0 \le y \le 1$ on the xy-plane. For this surface and the region,

$$\vec{p} = \hat{k},\ \nabla f = \hat{k},\ |\nabla f| = 1,\ |\nabla f \cdot \vec{p}| = |\hat{k} \cdot \hat{k}| = 1,\ g(x,y,z) = xyz\big|_{z=1} = xy.$$

Therefore,

$$\iint_A g(x,y,z)\,d\sigma = \iint_{R_1} xy\frac{|\nabla f|}{|\nabla f \cdot \vec{p}|}dxdy = \int_0^1 \int_0^1 xydxdy = \int_0^1 \frac{y}{2} = \frac{1}{4}.$$

Similarly,

$$\iint_B g(x,y,z)\,d\sigma = \frac{1}{4} = \iint_C g(x,y,z)\,d\sigma.$$

Thus, $\displaystyle\iint_S g\,d\sigma = \frac{3}{4}.$

Example 3.32. Evaluate the surface integral of $g(x,y,z) = x^2$ over the unit sphere.

S can be divided into the upper hemisphere and the lower hemisphere. Let S be the upper hemisphere $f(x,y,z) := x^2 + y^2 + z^2 = 1,\ z \ge 0$. Its projection on the xy-plane is the region

$$R:\ x = r\cos\theta, y = r\sin\theta,\ 0 \le r \le 1, 0 \le \theta \le 2\pi.$$

Here,

$$\vec{p} = \hat{k},\ |\nabla f| = 2\sqrt{x^2+y^2+z^2} = 2,$$
$$|\nabla f \cdot \vec{p}| = 2|z| = 2\sqrt{1-(x^2+y^2)} = 2\sqrt{1-r^2}.$$

Hence

$$\begin{aligned}\iint_S x^2\,d\sigma &= \iint_R x^2\frac{|\nabla f|}{|\nabla f \cdot \vec{p}|}\,dA = \iint_R \frac{x^2}{\sqrt{1-r^2}}dA \\ &= \int_0^{2\pi}\int_0^1 \frac{r^2\cos^2\theta}{\sqrt{1-r^2}}\,r\,dr\,d\theta \int_0^{2\pi} \cos^2\theta\,d\theta \int_0^1 \frac{r^3}{\sqrt{1-r^2}}dr = \frac{2\pi}{3}.\end{aligned}$$

Since the integral of x^2 on the upper hemisphere is equal to that on the lower hemisphere, the required integral is $2 \times \dfrac{2\pi}{3} = \dfrac{4\pi}{3}$.

Recall that when $\vec{p} = \hat{k}$, that is, when the region R is obtained by projecting the surface S onto the xy-plane, $\dfrac{|\nabla f|}{|\nabla f \cdot \vec{p}|} = \sqrt{1+z_x^2+z_y^2}$. Now, if the surface $f(x,y,z) = c$ can be written explicitly by $z = h(x,y)$, then the surface integral takes the form

$$\iint_S g(x,y,z)\,d\sigma = \iint_R g(x,y,h(x,y))\,\sqrt{1+h_x^2+h_y^2}\,dx\,dy.$$

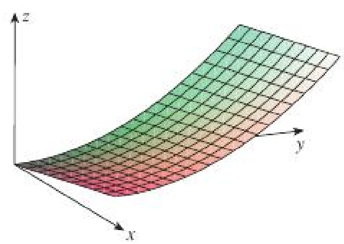
Similarly, if the surface can be written as $y = h(x, z)$ and R is obtained by projecting S onto the xz-plane, then

$$\iint_S g(x, y, z)\, d\sigma = \iint_R g(x, h(x, z), z)\, \sqrt{1 + h_x^2 + h_z^2}\, dx\, dz.$$

If the surface can be written as $x = h(y, z)$ and R is obtained by projecting S onto the yz-plane, then

$$\iint_S g(x, y, z)\, d\sigma = \iint_R g(h(y, z), y, z)\, \sqrt{1 + h_y^2 + h_z^2}\, dy\, dz.$$

Example 3.33. Evaluate $\iint_S y\, d\sigma$, where S is the surface $z = x + y^2,\ \ 0 \le x \le 1,\ 0 \le y \le 2$.



Projecting the surface onto xy-plane, we obtain the region R as the rectangle

$$R:\ \ 0 \le x \le 1,\ 0 \le y \le 2.$$

Here, the surface is given by $z = h(x, y) = x + y^2$. So,

$$\iint_S y\, d\sigma = \iint_R y\, \sqrt{1 + 1 + (2y)^2}\, dA = \int_0^1 \int_0^2 \sqrt{2}y\sqrt{(1 + 2y^2)}\, dy\, dx = \frac{13\sqrt{2}}{3}.$$

Suppose the surface S is given in a parameterized form:

$$\vec{r}(u, v) = x(u, v)\,\hat{\imath} + y(u, v)\,\hat{\jmath} + z(u, v)\,\hat{k},$$

where (u, v) ranges over the region D in the uv-plane. Here, a change of variable happens. The Jacobian is simply $\vec{r}_u \times \vec{r}_v$. Then

$$d\sigma = |\vec{r}_u \times \vec{r}_v|\, dA,$$

where $\vec{r}_u = x_u\,\hat{\imath} + y_u\,\hat{\jmath} + z_u\,\hat{k}$ and $\vec{r}_v = x_v\,\hat{\imath} + y_v\,\hat{\jmath} + z_v\,\hat{k}$. Then

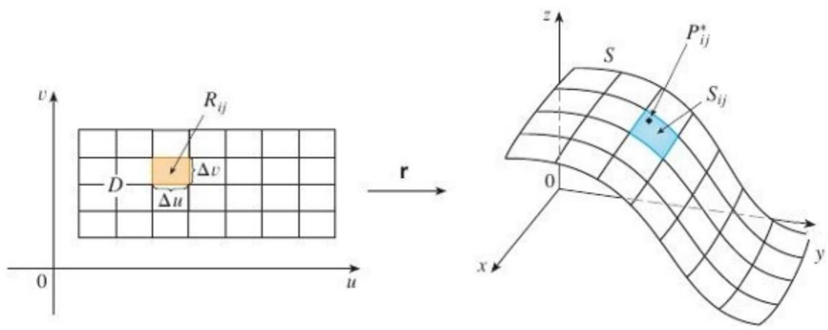
$$\iint_S f(x, y, z)\, d\sigma = \iint_D f(\vec{r}(u, v))|\vec{r}_u \times \vec{r}_v|\, dA.$$

Also this formula can directly be derived as we had done for computing surface area when a surface is given parametrically. It is as follows.

Suppose the smooth surface S has the parametric equation in vector form as

$$\vec{r} = x(u, v)\hat{\imath} + y(u, v)\hat{\jmath} + z(u, v)\hat{k}.$$

Assume that the parameter domain D is a rectangle. Divide D into smaller rectangles R_{ij} by taking grid lengths Δu and Δv.



Then the surface S is divided into corresponding patches S_{ij}. We evaluate f at a point P_{ij} in S_{ij} and form the Riemann sum $\sum_i \sum_j f(P_{ij})\Delta S_{ij}$, where ΔS_{ij} is the area of the patch S_{ij}. Taking limit as the number of sub-rectangles approach ∞, we obtain the surface integral of f over S as

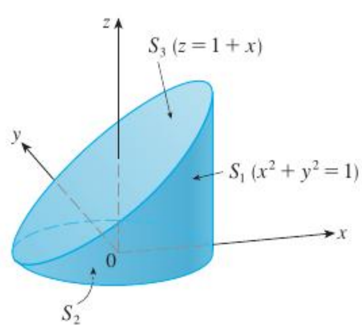
$$\iint_S f(x,y,z)\,d\sigma = \lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{m} f(P_{ij})\Delta S_{ij}.$$

However, $\Delta S_{ij} = |\vec{r}_u(P_{ij}) \times \vec{r}_v(P_{ij})|\Delta u \Delta v$. Therefore, the surface integral is given by

$$\iint_S f(x,y,z)\,d\sigma = \iint_D f(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,dA.$$

Observe that the surface area of S is simply $\iint_S 1\,d\sigma$ as it should be. The relation between a surface integral and surface area is much the same as that between a line integral and the arc length of a curve.

Example 3.34. Evaluate $\iint_S z\,d\sigma$, where S is the surface whose sides S_1 are given by the cylinder $x^2+y^2=1$, bottom S_2 is the disk $x^2+y^2 \le 1,\ z=0$, and whose top S_3 is part of the plane $z=1+x$ that lies above S_2.



S_1 is given by $\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$ with $x=\cos\theta,\ y=\sin\theta,\ z=z$, where D is given by $0\le\theta\le 2\pi$ and $0 \le z \le 1+x = 1+\cos\theta$. Then

$$|\vec{r}_\theta \times \vec{r}_z| = |\cos\theta\,\hat{i} + \sin\theta\,\hat{j}| = 1;$$

$$\iint_{S_1} z\,d\sigma = \iint_D z\,|\vec{r}_\theta \times \vec{r}_z|\,dA = \int_0^{2\pi}\int_0^{1+\cos\theta} z\,dz\,d\theta = \int_0^{2\pi}\frac{(1+\cos\theta)^2}{2}\,d\theta = \frac{3\pi}{2}.$$

S_2 lies in the plane $z=0$. Hence $\iint_{S_2} z\,d\sigma = 0$.
S_3 lies above the unit disk and lies in the plane $z = 1+x$.
Here, $u=x, v=y$ and $\vec{r} = x\hat{i} + y\hat{j} + z(x,y)\hat{j}$. Then

$$|\vec{r}_u \times \vec{r}_v| = |(\hat{i} + z_x\hat{k}) \times (\hat{j} + z_y\hat{k})| = \sqrt{z_x^2 + z_y^2 + 1}.$$

So,

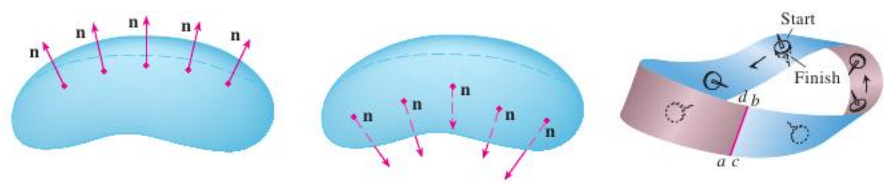
$$\begin{aligned}\iint_{S_3} z\,d\sigma &= \iint_D (1+x)\sqrt{1+z_x^2+z_y^2}\,dA\\ &= \int_0^{2\pi}\int_0^1 (1+r\cos\theta)\sqrt{1+1+0}\,r\,dr\,d\theta = \sqrt{2}\pi.\end{aligned}$$

Hence,

$$\iint_S z\,d\sigma = \iint_{S_1} z\,d\sigma + \iint_{S_2} z\,d\sigma + \iint_{S_3} z\,d\sigma = \frac{3\pi}{2} + \sqrt{2}\pi.$$

3.9 Surface Integral of a Vector Field

A smooth surface is called **orientable** iff it is possible to define a vector field of unit normal vectors $\hat{n}$ to the surface which varies continuously with position. Once such normal vectors are chosen, the surface is considered an **oriented** surface.



If the surface S is given by $z = f(x, y)$, then we take its orientation by considering the unit normal vectors $\hat{n} = \dfrac{-f_x\,\hat{i} - f_y\,\hat{j} + \hat{k}}{\sqrt{1+f_x^2+f_y^2}}$.

If S is a part of a level surface $g(x, y, z) = c$, then we may take $\hat{n} = \dfrac{\nabla g}{|\nabla g|}$.

If S is given parametrically as $\vec{r}(u, v) = x(u, v)\,\hat{i} + y(u, v)\,\hat{j} + z(u, v)\,\hat{k}$, then $\hat{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$.

Sometimes we may take negative sign if it is preferred. Conventionally, the outward direction is taken as the positive direction. Note that the outward direction of a normal makes sense when the surface is oriented.

Let $\vec{F}$ be a continuous vector field defined over an oriented surface S with unit normal $\hat{n}$. The **surface integral of $\vec{F}$ over** S, also called, the **flux of $\vec{F}$ across** S is

$$\iint_S \vec{F}\cdot\hat{n}\,d\sigma.$$

The flux is the integral of the scalar component of $\vec{F}$ along the unit normal to the surface. Thus in a flow, the flux is the net rate at which the fluid is crossing the surface S in the chosen positive direction.

If S is part of a level surface $g(x, y, z) = c$, which is defined over the domain D, then $d\sigma = \dfrac{|\nabla g|}{|\nabla g.\vec{p}\,|}\,dA$.

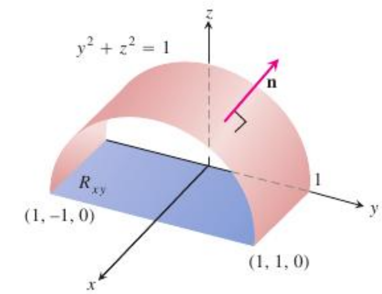
So, the flux across S is

$$\iint_S \vec{F}\cdot\hat{n}\,d\sigma = \iint_S \vec{F}\cdot\frac{\pm\nabla g}{|\nabla g|}d\sigma = \iint_D \vec{F}\cdot\frac{\pm\nabla g}{|\nabla g\cdot\vec{p}\,|}dA.$$

If S is parametrized by $\vec{r}(u,v)$, where D is the domain in uv-plane, then $d\sigma = |\vec{r}_u \times \vec{r}_v|\, dA$. So, flux across S is

$$\iint_S \vec{F} \cdot \hat{n}\, d\sigma = \iint_S \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} d\sigma = \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, dA.$$

Example 3.35. Find the flux of $\vec{F} = yz\,\hat{j} + z^2\,\hat{k}$ outward through the surface S which is cut from the cylinder $y^2 + z^2 = 1, z \geq 0$ by the planes $x = 0$ and $x = 1$.



S is given by $g(x,y,z) := y^2 + z^2 - 1 = 0$, defined over the rectangle $R = R_{xy}$ as in the figure.

The outward unit normal is $\hat{n} = +\dfrac{\nabla g}{|\nabla g|} = y\,\hat{j} + z\,\hat{k}$.

Here, $\vec{p} = \hat{k}$. So, $d\sigma = \dfrac{|\nabla g|}{|\nabla g \cdot \hat{k}|} dA = \dfrac{\sqrt{y^2+z^2}}{z} = \dfrac{1}{z}\, dA$.

$\vec{F} \cdot \hat{n}$ on S is $y^2 z + z^3 = z(y^2 + z^2) = z$. Therefore, outward flux through S is

$$\iint_S \vec{F} \cdot \hat{n}\, d\sigma = \iint_R z\,\frac{1}{z}\, dA = \iint_R dA = \text{Area of } R = 2.$$

Example 3.36. Find the flux of the vector field $\vec{F} = z\,\hat{\imath} + y\,\hat{\jmath} + x\,\hat{k}$ across the unit sphere.

If no direction of the normal vector is given and the surface is a closed surface, we take $\hat{n}$ in the positive direction, which is directed outward.

Using the spherical coordinates, the unit sphere S is parametrized by

$$\vec{r}(\phi,\theta) = \sin\phi\cos\theta\,\hat{\imath} + \sin\phi\sin\theta\,\hat{\jmath} + \cos\phi\,\hat{k},$$

where $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$ give the domain D. Then

$$\vec{F}(\vec{r}(\phi,\theta)) = \cos\phi\,\hat{\imath} + \sin\phi\sin\theta\,\hat{\jmath} + \sin\phi\cos\theta\,\hat{k}.$$

$$\vec{r}_\phi \times \vec{r}_\theta = \sin^2\phi\cos\theta\,\hat{\imath} + \sin^2\phi\sin\theta\,\hat{\jmath} + \sin\phi\cos\theta\,\hat{k}.$$

Consequently,

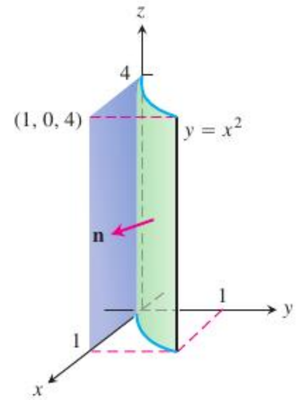
$$\begin{aligned}
\iint_S \vec{F} \cdot \vec{n}\, d\sigma &= \iint_D \vec{F} \cdot (\vec{r}_\phi \times \vec{r}_\theta)\, dA \\
&= \int_0^{2\pi} \int_0^{\pi} (2\sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta)\, d\phi\, d\theta = \frac{4\pi}{3}.
\end{aligned}$$

Example 3.37. Find the surface integral of the vector field

$$\vec{F} = yz\,\hat{i} + x\,\hat{j} - z^2\,\hat{k}$$

over the portion of the parabolic cylinder given by

$$y = x^2,\ 0 \le x \le 1,\ 0 \le z \le 4.$$



We assume the positive direction of the normal $\hat{n}$. On the surface, we have $x = x,\ y = x^2,\ z = z$ giving the parametrization as $\vec{r}(x,z) = x\,\hat{i} + x^2\,\hat{j} + z\,\hat{k}$ where D is given by $0 \le x \le 1,\ 0 \le z \le 4$.

On the surface $\vec{F} = x^2 z\,\hat{i} + x\,\hat{j} - z^2\,\hat{k}$. So,

$$\begin{aligned}
\iint_S \vec{F} \cdot \hat{n}\, d\sigma &= \iint_D \vec{F} \cdot (\vec{r}_x \times \vec{r}_y)\, dA \\
&= \iint_D (x^2 z\,\hat{i} + x\,\hat{j} - z^2\,\hat{k}) \cdot (2x\,\hat{i} - \hat{j}) \\
&= \int_0^4 \int_0^1 (2x^3 z - x)\, dx\, dz = \int_0^4 \frac{z-1}{2}\, dz = 2.
\end{aligned}$$

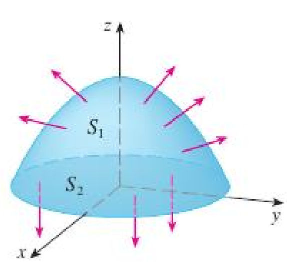
If S is given by $z = f(x,y)$, then think of x, y as the parameters u and v. We have $\vec{F} = M(x,y)\,\hat{i} + N(x,y)\,\hat{j} + P(x,y)\,\hat{k}$ and $\vec{r} = x\,\hat{i} + y\,\hat{j} + f(x,y)\,\hat{k}$.

Then $\vec{r}_x \times \vec{r}_y = (\hat{i} + f_x\,\hat{k}) \times (\hat{j} + f_y\,\hat{k}) = -f_x\,\hat{i} - f_y\,\hat{j} + \hat{k}$.

Therefore, the flux is

$$\iint_S \vec{F} \cdot \hat{n}\, d\sigma = \iint_D \vec{F} \cdot (\vec{r}_x \times \vec{r}_y)\, dA = \iint_D (-M f_x - N f_y + P)\, dA.$$

Example 3.38. Evaluate $\iint_S \vec{F} \cdot \hat{n}\, d\sigma$, where $\vec{F} = y\,\hat{i} + x\,\hat{j} + z\,\hat{k}$ and S is the boundary of the solid enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.



The surface S has two parts: the top portion S_1 and the base S_2. Since S is a closed surface, we consider its outward unit normal $\hat{n}$. Projections of both S_1 and S_2 on xy-plane are D, the unit disk.

By the simplified formula for the flux, we have

$$\begin{aligned}\iint_{S_1} \vec{F}\cdot\hat{n}\,d\sigma &= \iint_D (-Mf_x - Nf_y + P)dA\\ &= \iint_D [-y(-2x) - x(-2y) + 1 - x^2 - y^2]dA\\ &= \int_0^{2\pi}\int_0^1 (1 + 4r^2\cos\theta\sin\theta - r^2)\,r\,dr\,d\theta\\ &= \int_0^{2\pi} (\frac{1}{4} + \cos\theta\sin\theta)\,d\theta = \frac{\pi}{2}.\end{aligned}$$

The disk S_2 has positive direction, when $\hat{n} = -\hat{k}$. Thus

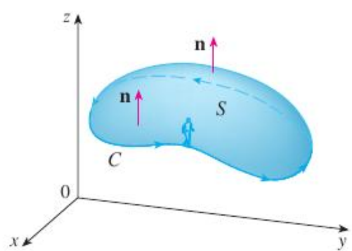
$$\iint_{S_2} \vec{F}\cdot\hat{n}\,d\sigma = \iint_{S_2} (-\vec{F}\cdot\hat{k})\,d\sigma = \iint_D (-z)dA = 0$$

since on $D = S_2$, $z = 0$. Then

$$\iint_S \vec{F}\cdot\hat{n}\,d\sigma = \iint_{S_1} \vec{F}\cdot\hat{n}\,d\sigma + \iint_{S_2} \vec{F}\cdot\hat{n}\,d\sigma = \frac{\pi}{2}.$$

3.10 Stokes' Theorem

Consider an oriented surface with a unit normal vector $\hat{n}$. Call the boundary curve of S as C. The orientation of S induces a *positive orientation on C.*



If you walk in the positive direction of C keeping your head pointing towards $\hat{n}$, then S will be to your left.

Recall that Green's theorem relates a double integral in the plane to a line integral over its boundary. We will have a generalization of this to 3 dimensions. Write the boundary curve of a given smooth surface as ∂S. The boundary is assumed to be a closed curve, positively oriented unless specified otherwise.

Theorem 3.10. (Stokes' Theorem) *Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve ∂S with positive orientation.*
Let $F = M\,\hat{i} + N\,\hat{j} + P\,\hat{k}$ be a vector field with M, N, P having continuous partial derivatives on an open region in space that contains S. Then

$$\oint_{\partial S} \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot\hat{n}\,d\sigma.$$

In particular, if S is a bounded region D in the xy-plane, $\partial S = C$, the smooth boundary of D, then $\hat{n} = \hat{k}$ and $d\sigma = dA$. We obtain

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D \text{curl}\, \vec{F} \cdot \hat{k}\, dA = \iint_D (N_x - M_y)\, dx\, dy.$$

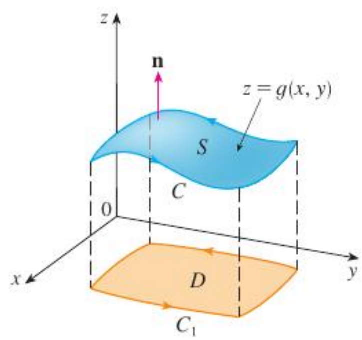
as Green's theorem states. In fact, we can use Green's theorem to prove Stokes' theorem in case S is the graph of a smooth function $z = f(x, y)$ with a smooth boundary, and the vector field $\vec{F}$ is smooth.

Proof: Let $\vec{F} = M\,\hat{\imath} + N\,\hat{\jmath} + P\,\hat{k}$. We see that

$$\oint_{\partial S} \vec{F} \cdot d\vec{r} = \oint_{\partial S} M\, dx + N\, dy + P\, dz.$$

And

$$\begin{aligned} \iint_S \text{curl}\, \vec{F} \cdot \hat{n}\, d\sigma &= \iint_S \text{curl}\, (M\,\hat{\imath}) \cdot \hat{n}\, d\sigma \\ &+ \iint_S \text{curl}\, (N\,\hat{\jmath}) \cdot \hat{n}\, d\sigma + \iint_S \text{curl}\, (P\,\hat{k}) \cdot \hat{n}\, d\sigma. \end{aligned}$$



We show that the M-, N- and P- components in both are equal.

Suppose S is given by $z = f(x, y)$ for $(x, y) \in D$. Orient ∂D positively, i.e., counter-clock-wise. Choose a parameterization for this. Suppose ∂D is given by

$$\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} \quad \text{for } a \le t \le b.$$

Then ∂S has the parameterization as

$$\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + f(x(t), y(t))\,\hat{\jmath} \quad \text{for } a \le t \le b.$$

Thus

$$\oint_{\partial S} M(x, y, z)\, dx = \int_a^b M(x(t), y(t), f(x(t), y(t))\frac{dx}{dt}\, dt.$$

Or that

$$\oint_{\partial S} M(x, y, z)\, dx = \int_{\partial D} M(x, y, z)\, dx.$$

Next, we apply Green's theorem on the integral on the right to obtain:

$$\oint_{\partial S} M(x, y, z)\, dx = -\iint_D M_y(x, y, f(x, y))\, dA.$$

Apply Chain rule on the right side integrand to obtain

$$\oint_{\partial S} M_y(x,y,z)\,dx = -\iint_D \Big[M_y(x,y,f(x,y)) + M_z(x,y,f(x,y))f_y\Big]\,dA.$$

We now compute $\iint_S \text{curl }(M\,\hat{i})d\sigma$. For this, notice that S has the parameterization:

$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + f(x,y)\,\hat{k}.$$

So, $\hat{n} = \dfrac{-f_x\,\hat{i} - f_y\,\hat{j} + \hat{k}}{c}$, where $c = |-f_x\,\hat{i} - f_y\,\hat{j} + \hat{k}|$. Then

$$\text{curl }(M\,\hat{i})\cdot\,\hat{n} = (0\,\hat{i} + M_z\,\hat{j} - M_y\,\hat{k})\cdot\hat{n} = [-M_z f_y - M_y]/c.$$

$$\iint_S \text{curl }(M\,\hat{i})\cdot\hat{n}\,d\sigma = -\iint_D \frac{1}{c}\big[M_y(x,y,f(x,y))dy + M_z(x,y,f(x,y))\big]\,(c\,dA),$$

since $c = |\nabla(z - f(x,y))|/|\nabla(z-f(x,y))\ \cdot\hat{k}|$. Therefore,

$$\iint_S \text{curl }(M\,\hat{i})\cdot\,\hat{n} = \oint_{\partial S} M(x,y,z)\,dx.$$

Similarly, other components become respectively equal. □

Example 3.39. Consider S as the hemisphere $x^2+y^2+z^2=9,\ z\ge 0$. Let $\vec{F}(\vec{r}) = y\,\hat{i} - x\,\hat{j}$.

The bounding curve for S in the xy-plane is ∂S given by $x^2+y^2=9,\ z=0$.

Parameterization of ∂S is $\vec{r}(\theta) = 3\cos\theta\,\hat{i} + 3\sin\theta\,\hat{j}$ for $0\le\theta\le 2\pi$. Then

$$\begin{aligned}\oint_{\partial S}\vec{F}\cdot d\vec{r} &= \int_0^{2\pi}[(3\sin\theta)\,\hat{i} - (3\cos\theta)\,\hat{j}]\cdot[(-3\sin\theta)\,\hat{i} + (3\cos\theta)\,\hat{j}]\,d\theta\\ &= \int_0^{2\pi}[-9\sin^2\theta - 9\cos^2\theta]\,d\theta = -18\pi.\end{aligned}$$

This is the line integral in Stokes' theorem. For the surface integral, we have

$$\text{curl }\vec{F} = (P_y - N_z)\,\hat{i} + (M_z - P_x)\,\hat{j} + (N_x - M_y)\,\hat{k} = -2\,\hat{k}.$$

Since on the surface $g := x^2+y^2+z^2-9$, we have

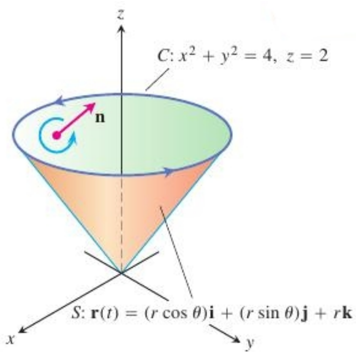
$$\hat{n} = \frac{\text{grad } g}{|\text{grad } g|} = \frac{1}{3}(x\,\hat{i} + y\,\hat{j} + z\,\hat{k}).$$

$$\vec{p} = \hat{k},\quad d\sigma = \frac{|\text{grad } g|}{|\text{grad } g\cdot\ \vec{p}|}\,dA = \frac{2\times 3}{2z}dA = \frac{3}{z}dA,$$

where dA is the differential in the projected area $D:\ x^2+y^2\le 9$. Then

$$\iint_S \text{curl }\vec{F}\cdot\hat{n}\,d\sigma = \iint_S \frac{-2z}{3}\,d\sigma = \iint_D \frac{-2z}{3}\,\frac{3}{z}\,dA = \iint_D(-2)\,dA = -18\pi.$$

Example 3.40. Evaluate $\oint_C ((x^2-y)\,\hat{i}+4z\,\hat{j}+x^2\,\hat{k})\cdot d\vec{r}$, where C is the intersection of the plane $z=2$ and the cone $z=\sqrt{x^2+y^2}$.



Parameterize the cone as

$$\vec{r}(r,\theta)=r\cos\theta\,\hat{i}+r\sin\theta\,\hat{j}+r\,\hat{k}$$

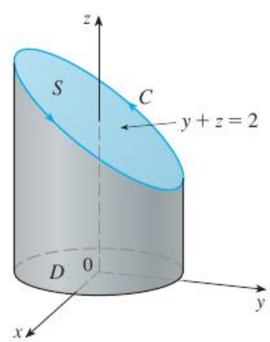
for $0\le r\le 2$, $0\le\theta\le 2\pi$. Then

$$\begin{aligned}
\hat{n} &= \frac{\vec{r}_r\times\vec{r}_\theta}{|\vec{r}_r\times\vec{r}_\theta|}=\frac{1}{\sqrt{2}}(-\cos\theta\,\hat{i}-\sin\theta\,\hat{j}+\,\hat{k}).\\
\text{curl}\,\vec{F} &= (P_y-N_z)\,\hat{i}+(M_z-P_x)\,\hat{j}+(N_x-M_y)\,\hat{k}=-4\,\hat{i}-2r\cos\theta\,\hat{j}+\,\hat{k}.\\
\text{curl}\,\vec{F}\cdot\hat{n} &= \frac{1}{\sqrt{2}}(4\cos\theta+r\sin(2\theta)+1)\\
d\sigma &= r\sqrt{2}\,dr\,d\theta.
\end{aligned}$$

By Stokes' theorem,

$$\oint_C \vec{F}\cdot d\vec{r}=\iint_S \text{curl}\,\vec{F}\cdot\,\hat{n}\,d\sigma=\int_0^{2\pi}\int_0^2(4\cos\theta+r\sin(2\theta)+1)r\,dr\,d\theta=4\pi.$$

Example 3.41. Evaluate $\oint_C(-y^2\,\hat{i}+x\,\hat{j}+z^2\,\hat{k})\cdot d\vec{r}$, where C is the curve of intersection of the plane $y+z=2$ and the cylinder $x^2+y^2=1$, oriented counter-clock-wise when looked from above.



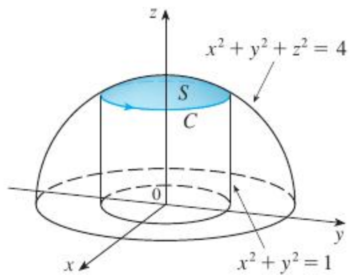
$\vec{F}=M\,\hat{i}+N\,\hat{j}+P\,\hat{k}$, where $M=-y^2,\ N=x,\ P=z^2$.

$$\text{curl}\,\vec{F}=(P_y-N_z)\,\hat{i}+(M_z-P_x)\,\hat{j}+(N_x-M_y)\,\hat{k}=(1+2y)\,\hat{k}.$$

Here, there are many surfaces with boundary C. We choose a convenient one: the surface S on the plane $y+z=2$ with boundary as C. Its projection on the xy-plane is the disc $D:\ x^2+y^2\le 1$. Then $\vec{p}=\hat{k}$. With $g(x,y)=y+z-2$, we have $\hat{n}=(\text{grad}\ g)/|\text{grad}\ g|=(\hat{j}+\hat{k})/\sqrt{2}$, grad $g\cdot\vec{p}=1$, and $d\sigma=\sqrt{2}\,dA$. Stokes' theorem gives

$$\begin{aligned}
\oint_C\vec{F}\cdot d\vec{r} &= \iint_S \text{curl}\,\vec{F}\cdot\,\hat{n}\,d\sigma=\iint_D\frac{1+2y}{\sqrt{2}}\sqrt{2}\,dA\\
&= \int_0^{2\pi}\int_0^1(1+2r\sin\theta)\,r\,dr\,d\theta=\int_0^{2\pi}\Big(\frac{1}{2}+\frac{2}{3}\sin\theta\Big)\,d\theta=\pi.
\end{aligned}$$

Example 3.42. Compute $\iint_S \operatorname{curl} \vec{F} \cdot \hat{n}\, d\sigma$, where $\vec{F} = xz\,\hat{\imath} + yz\,\hat{\jmath} + xy\,\hat{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy-plane.



The boundary curve C is obtained by solving the two equations to get $z^2 = 3$. Since $z > 0$, we have the curve C as $x^2 + y^2 = 1,\ z = \sqrt{3}$. In vector parametric form,

$$C: \quad \vec{r}(\theta) = \cos\theta\ \hat{\imath} + \sin\theta\,\hat{\jmath} + \sqrt{3}\ \hat{k} \ \text{ for } 0 \le \theta \le 2\pi.$$

Then

$$\vec{F}(\vec{r}(\theta)) = \sqrt{3}\cos\theta\,\hat{\imath} + \sqrt{3}\sin\theta\,\hat{\jmath} + \cos\theta\sin\theta\,\hat{k}.$$

By Stokes' theorem,

$$\begin{aligned}\iint_S \operatorname{curl} \vec{F} \cdot \hat{n}\, d\sigma &= \oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F} \cdot \vec{r}\,'(\theta)\, d\theta \\ &= \int_0^{2\pi} (-\sqrt{3}\cos\theta\sin\theta + \sqrt{3}\sin\theta\cos\theta)\, d\theta = 0.\end{aligned}$$

Stokes' theorem can be generalized to piecewise smooth surfaces like union of sides of a polyhedra. Here, we take the integral over the sides as the sum of integrals over each individual side.

Similarly, Stokes' theorem can be generalized to surfaces with holes. The line integrals are to be taken over all the curves which form the boundaries of the holes.

The surface integral over S of the normal component of $\operatorname{curl} \vec{F}$ is equal to the sum of the line integrals around all the boundary curves of the tangential component of $\vec{F}$. Here, the curves are traced in the direction induced by the orientation of S.

Recall that a conservative field is one which can be expressed as a gradient of another scalar field. In such a case, $\operatorname{curl} \vec{F} = 0$. Then from Stokes' theorem, it follows that $\oint_C \vec{F} \cdot d\vec{r} = 0$.

Theorem 3.11. *If* $\operatorname{curl} \vec{F} = 0$ *at each point of an open simply connected region D in space, then on any piecewise smooth closed path C lying in D,* $\oint_C \vec{F} \cdot d\vec{r} = 0$.

3.11 Gauss' Divergence Theorem

We have seen how to relate an integral of a function over a region with the integral of possibly some other related function over the boundary of the region.

For definite integrals on intervals: $\int_a^b f'(t)\, dt = f(b) - f(a)$.

For a path from a point P to a point Q in $\mathbb{R}^3$, $\int_C \operatorname{grad} f \cdot ds = f(Q) - f(P)$.

For a domain D in $\mathbb{R}^2$, $\iint_D (N_y - M_x)\, dA = \int_{\partial D} \vec{F} \cdot d\vec{r}$.

For a surface S in $\mathbb{R}^3$, $\iint_S \operatorname{curl} \vec{F} \cdot \hat{n}\, d\sigma = \int_C \vec{F} \cdot d\vec{r}$.

It suggests a generalization to three dimensions; and we use the divergence of a vector field for this purpose.

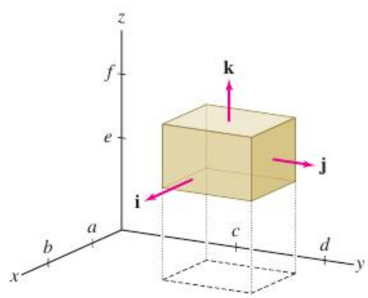
Recall that div $\vec{F} = \text{grad} \cdot \vec{F} = \nabla \cdot \vec{F}$. That is, the **divergence** of a vector field $\vec{F} = M(x,y,z)\,\hat{i} + N(x,y,z)\,\hat{j} + P(x,y,z)\,\hat{k}$ is the scalar function div $\vec{F} = M_x + N_y + P_z$.

Our generalization is $\iiint_D \operatorname{div} \vec{F}\, dV = \iint_S \vec{F} \cdot \hat{n}\, d\sigma$.

Theorem 3.12. (Gauss' Divergence Theorem) *Let S be a piecewise smooth simple closed bounded surface that encloses a solid region D in $\mathbb{R}^3$. Suppose S has been oriented positively by its outward normals. Let $\vec{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains D. Then*

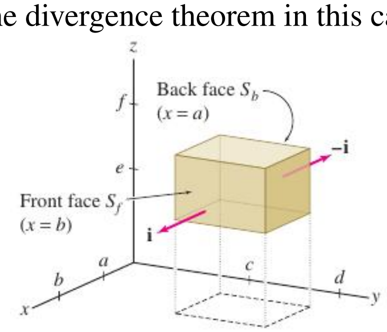
$$\iint_S \vec{F} \cdot \hat{n}\, d\sigma = \iiint_D \operatorname{div} \vec{F}\, dV.$$

Proof: We prove this in the special case that D is a box in $\mathbb{R}^3$ given by $D = [a,b] \times [c,d] \times [e,f]$. Let $\vec{F} = M\,\hat{i} + N\,\hat{j} + P\,\hat{k}$. Then



$$\iiint_D \operatorname{div} \vec{F}\, dV = \iiint_D \operatorname{div} M\, dV + \iiint_D \operatorname{div} N\, dV + \iiint_D \operatorname{div} \vec{F}\, dV.$$
$$\iint_S \vec{F} \cdot \hat{n}\, d\sigma = \iint_S M \cdot \hat{n}\, d\sigma + \iint_S N \cdot \hat{n}\, d\sigma + \iint_S P \cdot \hat{n}\, d\sigma.$$

We prove that the respective components are equal. We thus consider only the $\hat{i}$-component. That is, we take $\vec{F} = M\,\hat{i}$ and prove the divergence theorem in this case.



So, let $\vec{F} = M\,\hat{i}$. The solid has six faces. The surface integral over S is the sum of integrals over these faces. A simplification occurs. $\vec{F} = M\,\hat{i}$ we have $\vec{F} \cdot \hat{j} = F \cdot \hat{k} = 0$. That is, $\vec{F}$ is orthogonal to the normals of the top, bottom, and the two side faces.

Writing the remaining faces as S_f and S_b, we have

$$\iint_S \vec{F}\cdot\hat{n}\,d\sigma = \iint_{S_f} \vec{F}\cdot\hat{n}\,d\sigma + \iint_{S_b} \vec{F}\cdot\hat{n}\,d\sigma.$$

Parameterization of these faces give

$$S_f: \quad \vec{r} = b\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}, \quad S_b: \quad \vec{r} = a\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}.$$

for $c \le y \le d,\ e \le z \le f$. The outward normal to S_f is $\hat{\imath}$, and to S_b is $-\hat{\imath}$. Then

$$\begin{aligned}
\iint_S \vec{F}\cdot\hat{n}\,d\sigma &= \int_e^f\int_c^d M(b,y,z)\,dydz - \int_e^f\int_c^d M(a,y,z)\,dydz \\
&= \int_e^f\int_c^d [M(b,y,z) - M(a,y,z)]\,dydz \\
&= \int_e^f\int_c^d\int_a^b M_x(x,y,z)\,dxdydz \\
&= \iiint_D \operatorname{div}\vec{F}\,dV,
\end{aligned}$$

since $\vec{F} = M\,\hat{\imath} \Rightarrow \operatorname{div}\vec{F} = \operatorname{div} M = M_x$. $\square$

Example 3.43. Consider the field $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$ over the sphere $S:\ x^2+y^2+z^2 = a^2$.

The outer unit normal to S computed from $\operatorname{grad} f$, with $f = x^2+y^2+z^2-a^2$, is

$$\hat{n} = \frac{2(x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k})}{\sqrt{4(x^2+y^2+z^2)}} = \frac{1}{a}(x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}).$$

Hence on the given surface,

$$\vec{F}\cdot\hat{n}\,d\sigma = \frac{1}{a}(x^2+y^2+z^2)\,d\sigma = a\,d\sigma.$$

Therefore,

$$\iint_S \vec{F}\cdot\,\hat{n}\,d\sigma = \iint_S a\,d\sigma = a \times \text{Area of } S = 4\pi a^3.$$

Now, for the triple integral,

$$\operatorname{div}\vec{F} = M_x + N_y + P_z = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3.$$

Therefore, with D as the ball bounded by S,

$$\iiint_D \operatorname{div}\vec{F}\,dV = \iiint_D 3\,dV = 3 \times \text{Volume of } D = 4\pi a^3.$$

Example 3.44. Find the outward flux of the vector field $xy\,\hat{\imath} + yz\,\hat{\jmath} + zx\,\hat{k}$ through the surface cut from the first octant by the planes $x = 1,\ y = 1$ and $z = 1$.

The solid D is a cube having six faces. Call the surface of the cube as S. Instead of computing the surface integral, we use Divergence theorem.

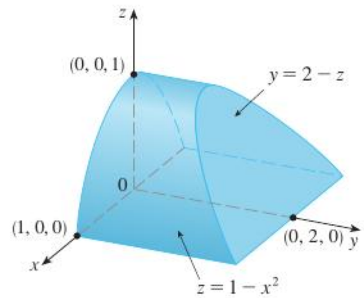
With $\vec{F} = xy\,\hat{i} + yz\,\hat{j} + zx\,\hat{k}$, we have

$$\text{div } \vec{F} = \frac{\partial xy}{\partial x} + \frac{\partial yz}{\partial y} + \frac{\partial zx}{\partial z} = y + z + x.$$

Therefore the required flux is

$$\iint_S \vec{F} \cdot \hat{n}\, d\sigma = \iiint_D \text{div } \vec{F}\, dV = \int_0^1 \int_0^1 \int_0^1 (y+z+x)\, dxdydz = \frac{3}{2}.$$

Example 3.45. Evaluate $\iint_S \vec{F} \cdot \hat{n}\, d\sigma$, where $\vec{F} = xy\,\hat{i} + y^2 + e^{xz^2}\,\hat{j} + \sin(xy)\,\hat{k}$ and S is the surface of the solid D bounded by the parabolic cylinder $z = 1 - x^2$, and the planes $y = 0$, $z = 0$, and $y + z = 2$.



S has four sides. Instead of computing the surface integrals, we use Divergence theorem. We have

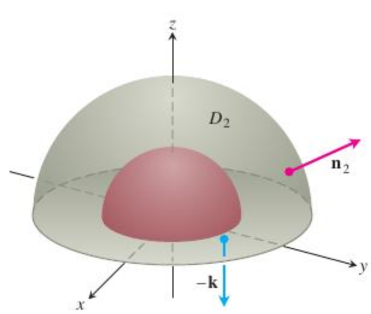
$$\text{div } \vec{F} = (xy)_x + (y^2 + e^{xz^2})_y + (\sin(xy))_z = 3y.$$

And D is given by $-1 \le x \le 1,\ 0 \le z \le 1 - x^2,\ 0 \le y \le 2 - z.$

Therefore,

$$\begin{aligned}
\iint_S \vec{F} \cdot \hat{n}\, d\sigma &= \iiint_D \text{div } \vec{F}\, dV = \iiint_D 3y\, dV \\
&= \int_{-1}^1 \int_0^{1-x^2} \int_0^{2-z} 3y\, dy\, dz\, dx = \int_{-1}^1 \int_0^{1-x^2} \frac{(2-z)^2}{2}\, dz\, dx \\
&= -\frac{1}{2} \int_{-1}^1 [(x^2+1)^3 - 8]\, dx = \frac{184}{35}.
\end{aligned}$$

Example 3.46. Find the outward flux of the vector field $\vec{F}$ across the boundary of the solid D where $\vec{F} = \dfrac{x\,\hat{i} + y\,\hat{j} + z\,\hat{k}}{(x^2+y^2+z^2)^{3/2}}$ and $D:\ 0 < a^2 \le x^2 + y^2 + z^2 \le b^2$.



Write $\rho = \sqrt{x^2+y^2+z^2}$. Then $\dfrac{d\rho}{dx} = \dfrac{x}{\rho}$. With $\vec{F} = M\,\hat{\imath} + N\,\hat{\jmath} + P\,\hat{k}$, we have

$$M_x = \frac{\partial(x\rho^{-3})}{\partial x} = \rho^{-3} - 3x\rho^{-4}\frac{\partial \rho}{\partial x} = \frac{1}{\rho^3} - \frac{3x^2}{\rho^5}.$$

Similarly, $N_y = \dfrac{1}{\rho^3} - \dfrac{3y^2}{\rho^5}$ and $P_z = \dfrac{1}{\rho^3} - \dfrac{3z^2}{\rho^5}$.

Then $\operatorname{div} \vec{F} = \dfrac{3}{\rho^3} - \dfrac{3x^2+3y^2+3z^2}{\rho^5} = 0$.

Thus the required flux is $\iiint_D \operatorname{div} \vec{F}\, dV = 0$.

In fact, flux through the inner surface and flux through the outer surface are in opposite directions. Are their magnitudes equal?

Example 3.47. Consider the vector field $\vec{F} = \dfrac{1}{a^3}(x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k})$ on the sphere S of radius a centered at the origin. Show that the flux through S is a constant.

We compute the flux directly. Let S be the sphere $x^2+y^2+z^2 = a^2$ for any $a > 0$. The gradient computed from $f = x^2+y^2+z^2-a^2$ gives the outward unit normal to S as

$$\hat{n} = \frac{2x\,\hat{\imath} + 2y\,\hat{\jmath} + 2z\,\hat{k}}{\sqrt{4x^2+4y^2+4z^2}} = \frac{x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}}{a}.$$

Therefore, on the sphere S with $\vec{F} = (x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k})/(x^2+y^2+z^2)^{3/2}$,

$$\vec{F}\cdot\hat{n} = \frac{x^2+y^2+z^2}{a^4} = \frac{1}{a^2}.$$

Then

$$\iint_S \vec{F}\cdot\hat{n}\, d\sigma = \iint_S \frac{1}{a^2}\, d\sigma = \frac{1}{a^2} \times \text{Area of } S = 4\pi.$$

3.12 Review Problems

Problem 3.1: Compute the line integral of the vector function $x^3\,\hat{\imath} + 3zy^2\,\hat{\jmath} - x^2y\,\hat{k}$ along the straight line segment L from the point $(3,2,1)$ to $(0,0,0)$.

The parametric equation of the line segment joining these points is

$$x = -3t,\ y = -2t,\ z = -t \text{ for } -1 \le t \le 0.$$

The derivatives of these with respect to t are

$$x_t = -3,\ y_t = -2,\ z_t = -1.$$

Then the required line integral is

$$\int_L x^3\,dx + 3zy^2\,dy - x^2y\,dz = \int_{-1}^{0} [(-3t)^3(-3) + 3(-t)(-2t)^2(-2) - (-3t)^2(-2t)(-1)]\,dt = \frac{-87}{4}.$$

Problem 3.2: Let C be the portion of the curve $y = x^3$ from $(1,1)$ to $(2,8)$. Compute

$$\int_C (6x^2y\,dx + 10xy^2\,dy).$$

C is parametrized as $x = t,\ y = t^3,\ 1 \le t \le 2$. Then $x_t = 1,\ y_t = 3t^2$. The line integral is

$$\int_C (6x^2y\,dx + 10xy^2\,dy) = \int_1^2 (6t^5 \cdot 1 + 10t^7 \cdot 3t^2)\,dt = 3132.$$

Problem 3.3: Evaluate $\int_C (-y\,\hat{i} - xy\,\hat{j}) \cdot d\vec{r}$, where C is the circular arc joining $(1,0)$ to $(0,1)$ of a circle centered at the origin.

Prameterize C by $\vec{r}(\theta) = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$, for $0 \le \theta \le \pi/2$. Thus $x(\theta) = \cos\theta$, $y(\theta) = \sin\theta$. Then

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_0^{\pi/2} \vec{F}(\vec{r}(\theta)) \cdot \vec{r}\,'(\theta)\,d\theta \\ &= \int_0^{\pi/2} (-\sin\theta\,\hat{i} - \cos\theta\sin\theta\,\hat{j}) \cdot (-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})\,d\theta \\ &= \int_0^{\pi/2} (\sin^2\theta - \cos^2\theta\sin\theta)\,d\theta = \frac{\pi}{4} - \frac{1}{3}.\end{aligned}$$

Problem 3.4: Let $\vec{F} = 5z\,\hat{i} + xy\,\hat{j} + x^2z\,\hat{k}$. Is $\int_C \vec{F} \cdot d\vec{r}$ the same if C is a curve joining $(0,0,0)$ to $(1,1,1)$, given by
(a) $\vec{r}(t) = t\,\hat{i} + t\,\hat{j} + t\,\hat{k}$ for $0 \le t \le 1$; (b) $\vec{r}(t) = t\,\hat{i} + t\,\hat{j} + t^2\,\hat{k}$ for $0 \le t \le 1$?

(a) $\vec{F}(\vec{r}(t)) = 5t\,\hat{i} + t^2\,\hat{j} + t^3\,\hat{k}$. $d\vec{r}(t) = \hat{i} + \hat{j} + \hat{k}$. Thus

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (5t + t^2 + t^3)dt = \frac{37}{12}.$$

(b) $\vec{F}(\vec{r}(t)) = 5t\,\hat{i} + t^2\,\hat{j} + t^3\,\hat{k}$. $d\vec{r}(t) = \hat{i} + \hat{j} + 2t\,\hat{k}$. Thus

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (5t^2 + t^2 + 2t^5)dt = \frac{28}{12}.$$

As we see the line integral is not path-independent.

Problem 3.5: Let D be a simply connected domain containing a smooth curve C from $(0,0,0)$ to $(1,1,1)$. Evaluate $\int_C (2xdx + 2ydy + 4zdz)$.

$\vec{F} = 2x\,\hat{i} + 2y\,\hat{j} + 4z\,\hat{k} = \text{grad } f$, where $f = x^2 + y^2 + 2z^2$. Therefore, the line integral is independent of path C. Hence its value is $f(1,1,1) - f(0,0,0) = 4$.

Problem 3.6: Evaluate $\iint_S (7x\,\hat{i} - z\,\hat{k}) \cdot \hat{n}\,d\sigma$ over the surface $S:\ x^2 + y^2 + z^2 = 4$.

$\text{div } \vec{F} = \text{div } (7x\,\hat{i} - z\,\hat{k}) = 7 - 1 = 6$. So, the integral $= 6 \times$ volume of $S = 64\pi$.

Problem 3.7: Evaluate $I = \int_C (3x^2\,dx + 2yz\,dy + y^2\,dz)$, where C is a smooth curve joining $(0,1,2)$ to $(1,-1,7)$ by showing that $\vec{F}$ has a potential.

In order that $\vec{F} = \operatorname{grad} f$, we should have

$$f_x = M = 3x^2, \quad f_y = N = 2yz, \quad f_z = P = y^2.$$

To obtain such a possible f, we use integration and differentiation:

$$f = x^3 + g(y,z), \quad f_y = g_y = 2yz, \quad g = y^2 z + h(z),$$

$$f_z = y^2 + h'(z) = y^2, \quad h'(z) = 0, \quad h(z) = 0, \text{ say.}$$

Then $f = x^3 + y^2 z$. We verify that $\vec{F} = \operatorname{grad} f$. Therefore, $I = \vec{F}(1,-1,7) - f(0,1,2) = 6$.

Problem 3.8: Determine whether $I = \int_C (2xyz^2\,dx + (x^2z^2 + z\cos(yz))\,dy + (2x^2yz + y\cos(yz)\,dz)$ is independent of path. Evaluate I, where C is the line segment joining $(0,0,1)$ to $(1,\pi/4,2)$.

Here, $M = 2xyz^2,\ N = x^2z^2 + z\cos(yz),\ P = 2x^2yz + y\cos(yz)$. Then

$$M_y = 2xz^2 = N_x,\ \ N_z = 2x^2z + \cos(yz) - yz\sin(yz) = P_y,\ \ P_x = 4xyz = M_z.$$

Hence the line integral is independent of path. We find f such that $\vec{F} = \operatorname{grad} f$. Now,

$$f = \int N dy = x^2z^2y + \sin(yz) + g(x,z),\ f_x = 2xz^2y + g_x = M = 2xyz^2.$$

$$g_x = 0,\ g = h(z),\ f_z = 2x^2yz + y\cos(yz) + h'(z) = P = 2x^2yz + y\cos(yz),\ h'(z) = 0.$$

Taking $h(z) = 0$, we get $f(x,y,z) = x^2yz^2 + \sin(yz)$ as a possible potential. Then

$$I = f(1,\pi/4,2) - f(0,0,1) = \pi + 1.$$

Problem 3.9: Use Green's theorem to compute the area of the region
(a) bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$.
(b) bounded by the cardioid $r = a(1-\cos\theta)$ for $0 \le \theta \le 2\pi$.

(a) Recall: Green's theorem gave Area of $D = \frac{1}{2}\oint_{\partial D}(x\,dy - y\,dx)$. The ellipse $x^2/a^2 + y^2/b^2 = 1$ has the parameterization $x(t) = a\cos t,\ y = b\sin t$ for $0 \le t \le 2\pi$. Then its area is

$$\frac{1}{2}\int_0^{2\pi}(xy' - yx')\,dt = \frac{1}{2}\int_0^{2\pi}(ab\cos^2 t - (-ab\sin^2 t))\,dt = \pi ab.$$

(b) In polar form, $x = r\cos\theta,\ y = r\sin\theta$. Then $dx = \cos\theta\,dr - \sin\theta\,d\theta$ and $dy = \sin\theta\,dr + r\cos\theta\,d\theta$. Consequently the area is equal to

$$\frac{1}{2}\oint_{\partial D}(x\,dy - y\,dx) = \frac{1}{2}\oint_{\partial D} r^2\,d\theta = \frac{a^2}{2}\int_0^{2\pi}(1-\cos\theta)^2\,d\theta = \frac{3\pi}{2}a^2.$$

Problem 3.10: Compute the flux of the water through the parabolic cylinder $S:\ y = x^2,\ 0 \le x \le 2,\ 0 \le z \le 3$ if the velocity vector $\vec{F} = 3z^2\,\hat{\imath} + 6\,\hat{\jmath} + 6zx\,\hat{k}$, speed being measured in m/sec.

Write $x = u,\ z = v$. We have $y = x^2 = u^2$. The surface is

$$S:\ \ \vec{r} = u\,\hat{\imath} - u^2\,\hat{\jmath} + v\,\hat{k},\ \text{ for } 0 \le u \le 2,\ 0 \le v \le 3.$$

Then

$$\vec{n} = \vec{r}_u \times \vec{r}_v = (\hat{i} + 2\hat{j}) \times \hat{k} = 2u\,\hat{i} - \hat{j}.$$

On S,

$$\vec{F}(\vec{r}(u,v)) = 3v^2\,\hat{i} + 6\,\hat{j} + 6uv\,\hat{k}.$$

Hence $\vec{F}(\vec{r}(u,v)) \cdot \vec{n} = 6uv^2 - 6$. Consequently the flux is

$$\iint_S \vec{F} \cdot \vec{n}\, d\sigma = \int_0^3 \int_0^2 (6uv^2 - 6)\, du\, dv = \int_0^3 (12v^2 - 12)\, dv = 72\,\mathrm{m}^3/\mathrm{sec}.$$

Problem 3.11: Find the area of the portion of the surface of the cylinder $x^2 + y^2 = a^2$ which is cut out by the cylinder $x^2 + z^2 = a^2$.

One-eighth of the required surface area is in the first octant. This portion of the surface has the equation $y = \sqrt{a^2 - x^2}$. This gives

$$\frac{\partial y}{\partial x} = -\frac{x}{\sqrt{a^2 - x^2}}, \ \frac{\partial y}{\partial z} = 0 \ \Rightarrow \ \sqrt{1 + y_x^2 + y_z^2} = \sqrt{1 + \frac{x^2}{a^2 - x^2}} = \frac{a}{\sqrt{a^2 - x^2}}.$$

The domain of integration is a quarter of a disk given by

$$x^2 + x^2 \le a^2 \le a^2, \quad x \ge 0, \ \ z \ge 0.$$

Therefore, the required area is

$$8 \times \int_0^a \Big[\int_0^{\sqrt{a^2-x^2}} \frac{a}{\sqrt{a^2 - x^2}}\, dz\Big]\, dx = 8a \int_0^a dx = 8a^2.$$

Problem 3.12: A torus is generated by rotating a circle C about a straight line L in space so that C does not intersect or touch L. If L is the z-axis and C has radius b and its centre has distance a $(> b)$ from L, then compute the surface area of the torus.

The surface S of the torus is represented by

$$\vec{r}(u,v) = (a + b\cos v)\cos u\,\hat{i} + (a + b\cos v)\,\sin u\,\hat{j} + b\sin v\,\hat{k}.$$

Here, v is the angle in describing the circle and u is the angle of rotation. Thus $0 \le u, v \le 2\pi$. Projection onto the uv-plane shows that

$$\begin{aligned}
\vec{r}(u) &= -(a + b\cos v)\sin u\,\hat{i} + (a + b\cos v)\cos u\,\hat{j} \\
\vec{r}(v) &= -b\sin v\cos u\,\hat{i} - b\sin v\sin u\,\hat{j} + b\cos v\,\hat{k} \\
\vec{r}(u) \times \vec{r}(v) &= b(a + b\cos v)(\cos u\cos v\,\hat{i} + \sin u\cos v\,\hat{j} + \sin v\,\hat{k})
\end{aligned}$$

Hence $|\vec{r}(u) \times \vec{r}(v)| = b(a + b\cos v)$ and the area is

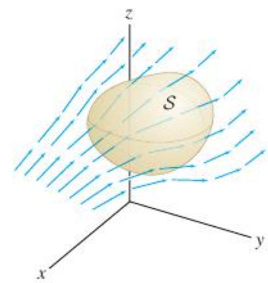
$$\iint_C |\vec{r}(u) \times \vec{r}(v)|\, du\, dv = \int_0^{2\pi} \int_0^{2\pi} b(a + b\cos v)\, du\, dv = 4\pi^2 ab.$$

Problem 3.13: Let S be the closed surface consisting of the cylinder $x^2 + y^2 = a^2$, $0 \le z \le b$ and the circular disks $x^2 + y^2 \le a^2$ one with $z = 0$ and the other with $z = b$. By transforming to a triple integral evaluate $I = \iint_S (x^3\,dy\,dz + x^2 y\,dz\,dx + x^2 z\,dx\,dy)$.

$\vec{F} = M\,\hat{i} + N\,\hat{j} + P\,\hat{k}$, where $M = x^3,\ N = x^2 y,\ P = x^2 z$. Then div $\vec{F} = 5x^2$. Let D be the solid bounded by S. In cylindrical coordinates, using Gauss' divergence theorem,

$$I = \iiint_D 5x^2 dV = 5\int_0^b \int_0^a \int_0^{2\pi} r^2 \cos^2\theta\, r\, dr\, d\theta\, dz = \frac{5}{4}\pi a^4 b.$$

Problem 3.14: Compute the flux of the vector field $\vec{F} = (z^2 + xy^2)\,\hat{i} + \cos(x+z)\,\hat{j} + (e^{-y} - zy^2)\,\hat{k}$ through the boundary of the surface given in the following figure:



$$\text{div }(F) = \frac{\partial}{\partial x}(z^2 + xy^2) + \frac{\partial}{\partial y}\cos(x+z) + \frac{\partial}{\partial z}(e^{-y} - zy^2) = 0.$$

Let D be the region enclosed by S. By the Divergence theorem,

$$\text{Flux through } S = \iiint_D \text{div } \vec{F}\, dV = 0.$$

Problem 3.15: Let a closed smooth surface S be such that any straight line parallel to the z-axis cuts it in no more than two points. Let n_3 denote the z-component of the unit outward normal $\hat{n}$ to the surface S. Then what is $\iint_S z n_3\, d\sigma$?

In this case, S has an upper part and a lower part. Suppose they are given, respectively, by the equations

$$z = f_u(x, y), \quad z = f_b(x, y).$$

Let D be the projection of S on the xy-plane. Then

$$\iint_S z\, n_3\, d\sigma = \iint_D f_u(x, y)\, dA - \iint_D f_b(x, y)\, dA.$$

This is equal to the volume of the solid B bounded by S.

Alternatively, take $\vec{F} = z\hat{k}$. Then div $\vec{F} = 1$. By the Divergence theorem,

$$\iint_S z\, n_3\, d\sigma = \iint_S \vec{F}\cdot\hat{n}\, d\sigma = \iiint_B \text{div } \vec{F}\, dV = \text{volume of } B.$$

Problem 3.16: Prove that the integral of the Laplacian over a planar region is the same as the integral, over the boundary curve, of the directional derivative in the direction of the unit normal to the boundary curve.

We rephrase: Let $f(x, y)$ be a function defined over a simply connected region D in the xy-plane.

Let C be the boundary curve of D. Denote by $D_n f(x,y)$ the directional derivative of f in the direction of the unit outer normal $\hat{n}$ to C. Show that $\iint_D (f_{xx}+f_{yy})dA = \int_C D_n f\, ds$.

Let θ be the angle between $\hat{n}$ and $\hat{i}$, the x-axis. Then $\hat{n} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$. If α is the angle between the tangent line to C and the x-axis, then $\cos\alpha = -\sin\theta$ and $\sin\alpha = \cos\theta$. Then

$$dx = \cos\alpha\, ds = -\sin\theta\, ds \text{ and } dy = \sin\alpha\, ds = \cos\theta\, ds.$$

Consequently, the directional derivative $D_n f$ is given by

$$D_n f(x,y) = (f_x\,\hat{i} + f_y\,\hat{j})\cdot\hat{n} = f_x\cos\theta + f_y\sin\theta.$$

For the vector function $\vec{F} = f_x\,\hat{i} + f_y\,\hat{j}$, by Green's theorem, we obtain

$$\iint_D (f_{xx}+f_{yy})dA = \int_C f_x dy - f_y dx = \int_C (f_x\cos\theta + f_y\sin\theta)ds = \int_C D_n f\, ds.$$

Problem 3.17: Let f and g be functions with continuous partial derivatives up to second order on a domain D in space, which has a smooth boundary ∂D. Denote by Δf and Δg their Laplacians. Prove the Green's formula:

$$\iiint_D (g\Delta f - f\Delta g)dV = \iint_{\partial D}\left(g\frac{\partial f}{\partial\,\hat{n}} - f\frac{\partial g}{\partial\,\hat{n}}\right)\, d\sigma.$$

Let $\vec{F} = M\,\hat{i} + N\,\hat{j} + P\,\hat{k}$. Gauss' divergence theorem says that

$$\iiint_D \operatorname{div}\vec{F}\, dV = \iint_{\partial D}\vec{F}\cdot\,\hat{n}\, d\sigma.$$

Suppose the unit normal $\hat{n}$ has the components a, b, c in the x, y, z-directions, respectively. Then

$$\iiint_D (M_x + N_y + P_z)\, dV = \iint_{\partial D}(aM + bN + cP)\, d\sigma.$$

Substitute $M = gf_x - fg_x,\ N = gf_y - fg_y,\ P = gf_z - fg_z$. Then

$$M_x + N_y + P_z = g(f_{xx}+f_{yy}+f_{zz}) - f(g_{xx}+g_{yy}+g_{zz}) = g\Delta f - f\Delta g.$$

$$aM + bN + cP = g(af_x + bf_y + cf_z) - f(ag_x + bg_y + cg_z) = g\frac{\partial f}{\partial\,\hat{n}} - f\frac{\partial g}{\partial\,\hat{n}}.$$

Now Green's formula follows from Gauss' divergence theorem.

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Appendix A

One Variable Summary

This appendix is devoted to summarizing some results and formulas from calculus of functions of one real variable that we may use in the class. For details, see *Functions of One Variable - A Survival Guide*.

A.1 Graphs of Functions

The **absolute value** of $x \in \mathbb{R}$ is defined as $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

Thus $|x| = \sqrt{x^2}$. And $|-a| = a$ or $a \geq 0$; $|x - y|$ is the distance between real numbers x and y. Moreover, if $a, b \in \mathbb{R}$, then

$$|-a| = |a|, \quad |ab| = |a|\,|b|, \quad \left|\frac{a}{b}\right| = \frac{|a|}{|b|} \text{ if } b \neq 0, \quad |a+b| \leq |a| + |b|, \quad |\,|a| - |b|\,| \leq |a - b|.$$

Let $x \in \mathbb{R}$ and let $a > 0$. The following are true:

1. $|x| = a$ iff $x = \pm a$.
2. $|x| < a$ iff $-a < x < a$ iff $x \in (-a, a)$.
3. $|x| \leq a$ iff $-a \leq x \leq a$ iff $x \in [-a, a]$.
4. $|x| > a$ iff $-a < x$ or $x > a$ iff $x \in (-\infty, -a) \cup (a, \infty)$ iff $x \in \mathbb{R} - [-a, a]$.
5. $|x| \geq a$ iff $-a \leq x$ or $x \geq a$ iff $x \in (-\infty, -a] \cup [a, \infty)$ iff $x \in \mathbb{R} - (-a, a)$.

Therefore, for $a \in \mathbb{R}, \delta > 0, \quad |x - a| < \delta$ iff $a - \delta < x < a + \delta$.

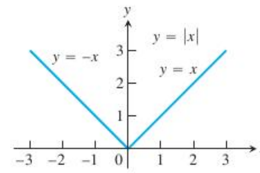
The following statements are useful in proving equalities from inequalities:

Let $a, b \in \mathbb{R}$.

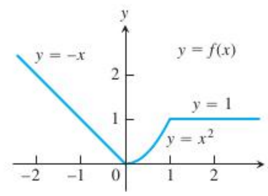
1. If for each $\epsilon > 0$, $|a| < \epsilon$, then $a = 0$.
2. If for each $\epsilon > 0$, $a < b + \epsilon$, then $a \leq b$.

Graphs of some known functions including $|\cdot|$, are as follows:

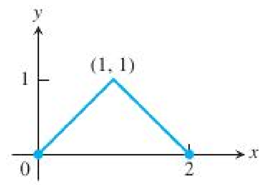
1. $y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$



2. $y = \begin{cases} -x & \text{if } x < 0 \\ x^2 & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$



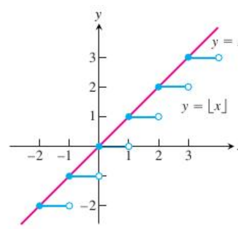
3. $y = f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 2 - x & \text{if } 1 < x \leq 2 \end{cases}$



4. $y = \lfloor x \rfloor = n$ if $n \leq x < n+1$ for $n \in \mathbb{N}$. It is the largest integer less than or equal to x.

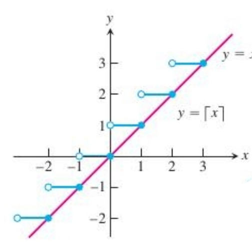
The largest integer function or the **floor** function.

Sometimes we write $\lfloor\ \rfloor$ as $[\ ]$.

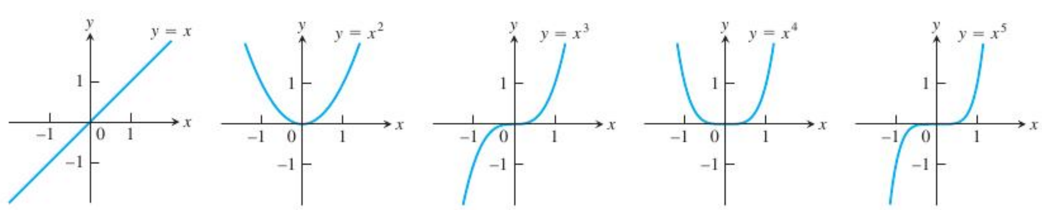


5. $y = \lfloor x \rfloor = n+1$ if $n < x \leq n+1$ for $n \in \mathbb{N}$. It is the smallest integer greater than or equal to x.

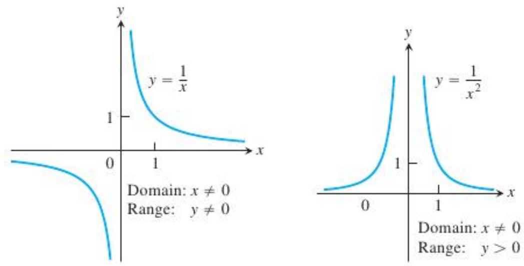
The smallest integer function or the **ceiling** function.



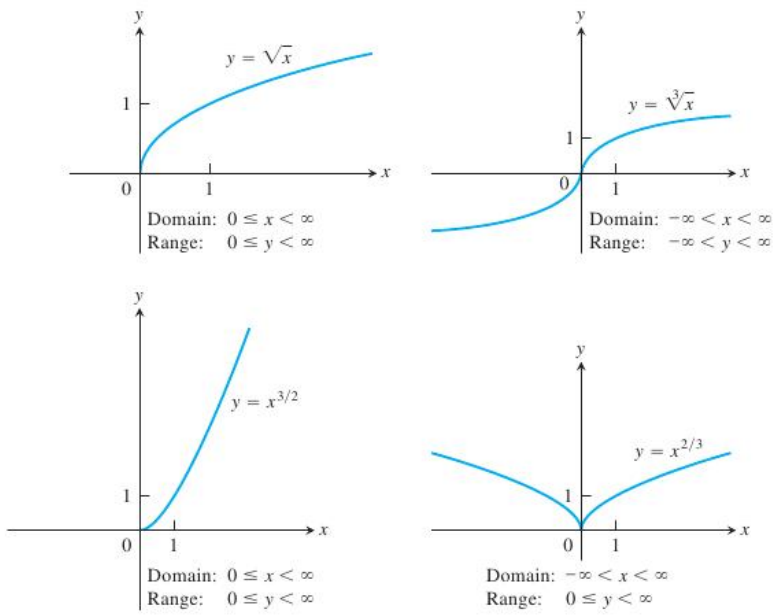
6. The **power function** $y = x^n$ for $n = 1, 2, 3, 4, 5$ look like



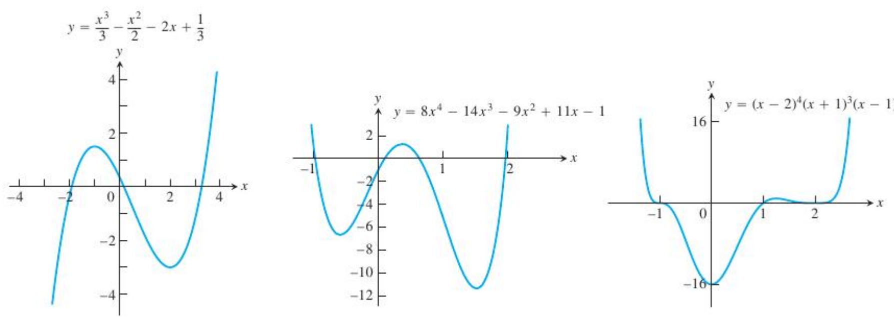
7. The power function $y = x^n$ for $n = -1$ and $n = -2$ look like



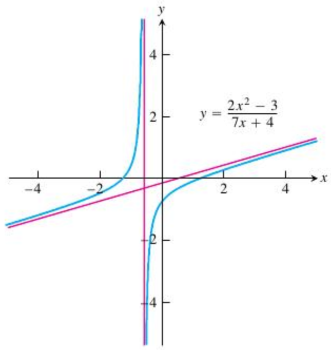
8. The graphs of the power function $y = x^a$ for $a = \frac{1}{2},\ \frac{1}{3},\ \frac{3}{2}$ and $a = \frac{2}{3}$ are

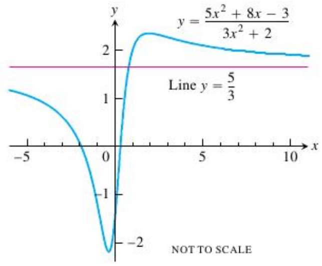


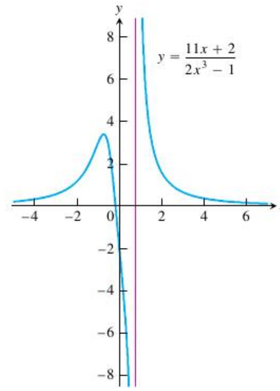
9. **Polynomial** functions are $y = f(x) = a_0 + a_1x + a_2x^2 + \cdots a_nx^n$ for some $n \in \mathbb{N} \cup \{0\}$. Here, the coefficients of powers of x are some given real numbers $a_0, \ldots, a_n$ and $a_n \neq 0$. The highest power n in the polynomial is called the **degree** of the polynomial. Graphs of some polynomial functions are as follows:



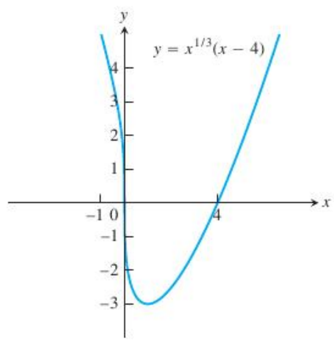
10. A **rational function** is a ratio of two polynomials; $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials, may or may not be of the same degree. Graphs of some rational functions are as follows:

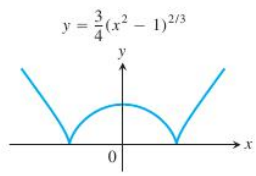


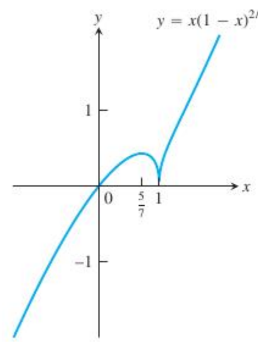




11. **Algebraic functions** are obtained by adding subtracting, multiplying, dividing or taking roots of polynomial functions. Rational functions are special cases of algebraic functions. Some graphs of alhebraic functions:







12. **Trigonometric functions** come from the ratios of sides of a right angled triangle. The angles are measured in radian. The trigonometric functions have a period. That is, $f(x + p) = f(x)$ happens for some $p > 0$. The **period** of $f(x)$ is the minimum of such p. The period for $\sin x$ is 2π.

The functions $\cos x$ and $\sec x$ are **even functions** and all others are **odd functions**. Recall that $f(x)$ is even if $f(-x) = f(x)$ and it is odd if $f(-x) = -f(x)$ for each x in the domain of the function. Some of the useful inequalities are

$$-|x| \leq \sin x \leq |x| \text{ for all } x \in \mathbb{R}.$$

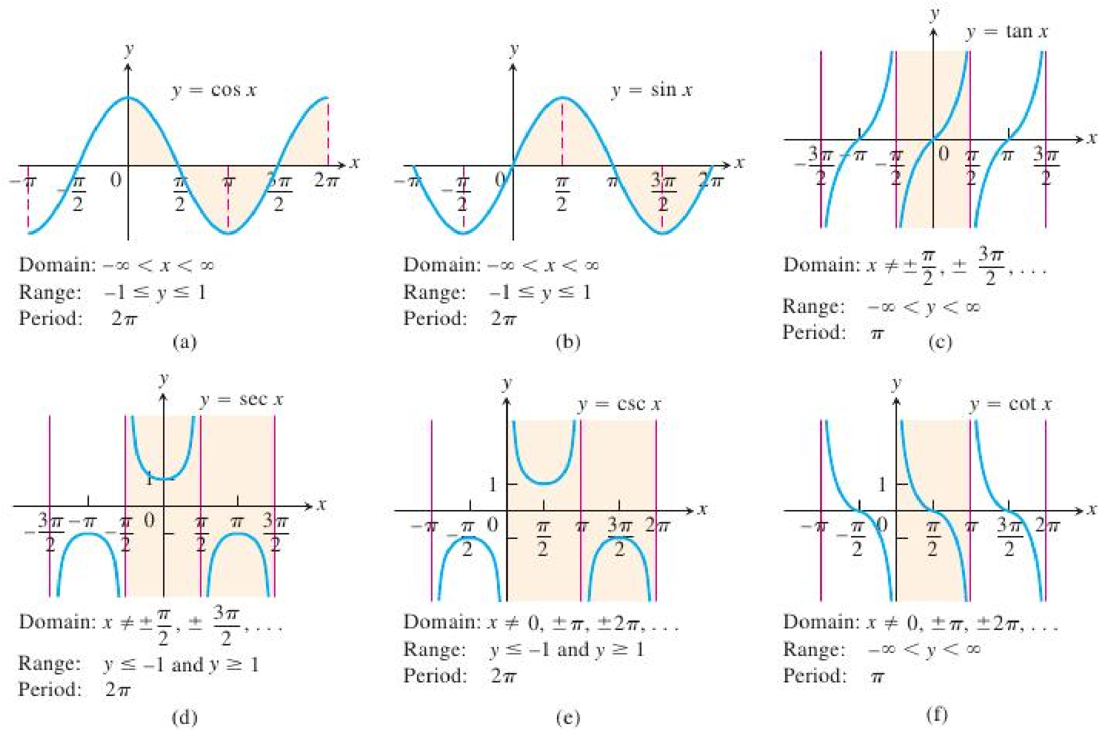
$$-1 \leq \sin x, \ \cos x \leq 1 \text{ for all } x \in \mathbb{R}.$$

$$0 \leq 1 - \cos x \leq |x| \text{ for all } x \in \mathbb{R}.$$

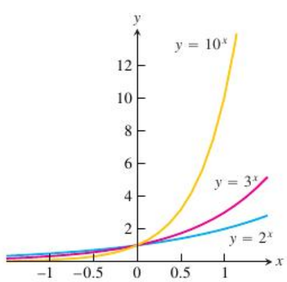
$$\sin x \leq x \leq \tan x \text{ for all } x \in (0, \pi/2).$$

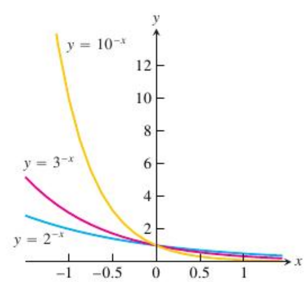
In fact, if $x \neq 0$, then $\sin x < |x|$.

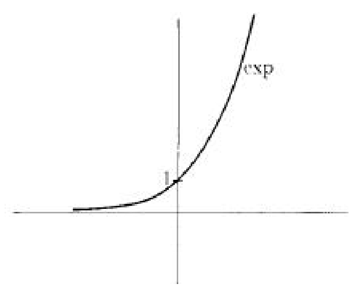
Graphs of the trigonomaetric functions are as follows:



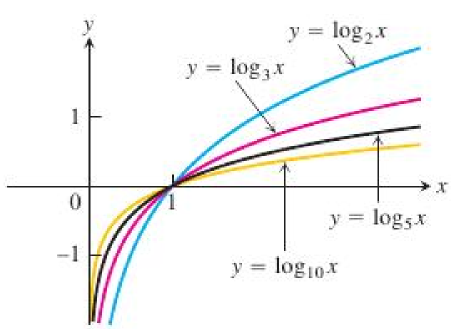
13. **Exponential functions** are in the form $y = a^x$ for some $a > 0$ and $a \neq 1$. All exponential functions have domain $(-\infty, \infty)$ and co-domain $(0, \infty)$. They never assume the value 0. Graphs of some exponential functions:

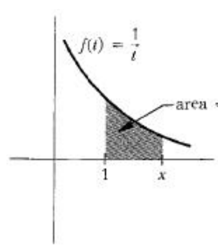


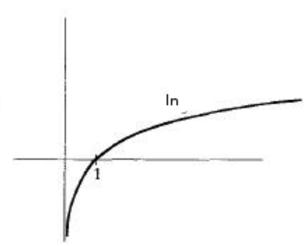




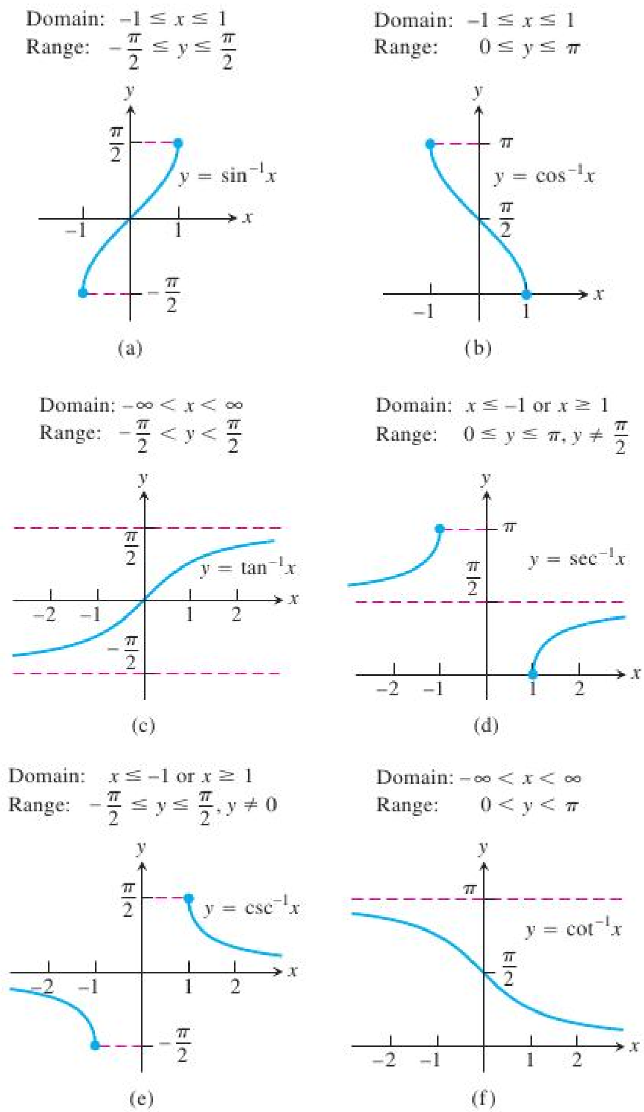
14. **Logarithmic functions** are inverse of exponential functions. That is,
$a^{\log_a x} = \log_a(a^x) = x$. Some examples:







15. **Trigonometric inverse functions**:



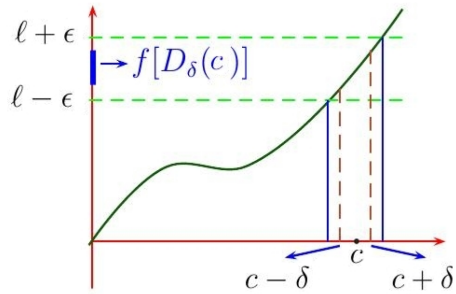
Functions that are not algebraic are called transcendental functions. Trigonometric functions, exponential functions, logarithmic functions and inverse trigonometric functions are examples of transcendental functions.

A.2 Concepts and Facts

Let $a < c < b$. Let $f : D \to R$ be a function whose domain D contains the union $(a, c) \cup (c, b)$. Let $\ell \in \mathbb{R}$. We say that **the limit of** $f(x)$ **as** x **approaches** c **is** ℓ and write it as

$$\lim_{x \to c} f(x) = \ell$$

iff for each $\epsilon > 0$, there exists a $\delta > 0$ such that for each $x \in (a, c) \cup (c, b)$ with $0 < |x - c| < \delta$, we have $|f(x) - \ell| < \epsilon$.



Limit Properties: Let k be a constant; or a constant function.

1. $\lim\limits_{x\to c} k = k$ and $\lim\limits_{x\to c} x = c$.
2. $\lim\limits_{x\to c}(f(x) \pm g(x)) = \lim\limits_{x\to c} f(x) \pm \lim\limits_{x\to c} g(x)$.
3. $\lim\limits_{x\to c} kf(x) = k \lim\limits_{x\to c} f(x)$.
4. $\lim\limits_{x\to c}[f(x)g(x)] = \lim\limits_{x\to c} f(x)\ \lim\limits_{x\to c} g(x)$.
5. $\lim\limits_{x\to c}[f(x)/g(x)] = \big[\lim\limits_{x\to c} f(x)\big] / \big[\lim\limits_{x\to c} g(x)\big]$ if $\lim\limits_{x\to c} g(x) \neq 0$.
6. $\lim\limits_{x\to c}(f(x))^r = (\lim\limits_{x\to c} f(x))^r$ if taking powers are meaningful.
7. $\lim\limits_{x\to c} f(x)$ is a unique real number if it exists.
8. If $\lim\limits_{x\to c} g(x) = 0$, and $\lim\limits_{x\to c}[f(x)/g(x)]$ exists, then $\lim\limits_{x\to c} f(x) = 0$.
9. (**Sandwich**) Let f, g, h be functions whose domain include $(a, c) \cup (c, b)$ for $a < c < b$. Suppose that $g(x) \leq f(x) \leq h(x)$ for all $x \in (a, c) \cup (c, b)$. If $\lim\limits_{x\to c} g(x) = \ell = \lim\limits_{x\to c} h(x)$, then $\lim\limits_{x\to c} f(x) = \ell$.
10. (**Domination**) Let f, g be functions whose domains include $(a, c) \cup (c, b)$ for $a < c < b$. Suppose that both $\lim\limits_{x\to c} f(x)$ and $\lim\limits_{x\to c} g(x)$ exist. If $f(x) \leq g(x)$ for all $x \in (a, c) \cup (c, b)$, then $\lim\limits_{x\to c} f(x) \leq \lim\limits_{x\to c} g(x)$.

Let I be (a, ∞) or $[a, \infty)$ for some $a \in \mathbb{R}$. Let $f : I \to \mathbb{R}$. Let $\ell \in \mathbb{R}$. We say that $\lim\limits_{x\to\infty} f(x) = \ell$ if for each $\epsilon > 0$, there exists an $m > 0$ such that if x is any real number greater then m, then $|f(x) - \ell| < \epsilon$.

Let $f(x)$ have a domain containing (a, c). Then $\lim\limits_{x\to c-} f(x) = \infty$ iff for each $m > 0$, there exists a $\delta > 0$ such that for every x with $c - \delta < x < c$, we have $f(x) > m$.

That is, $\lim\limits_{x\to c-} f(x) = \infty$ iff, "as x increases to c, $f(x)$ increases without bound".

Let $f : D \to \mathbb{R}$ be a function. Let c be an interior point of D. We say that $f(x)$ is **continuous at** c if $\lim\limits_{x\to c} f(x) = f(a)$.

If $D = [a, b)$ or $D = [a, b]$, then $f(x)$ is called continuous at the left end-point a if $\lim\limits_{x\to a+} f(x) = f(a)$.

If $D = (a, b]$ or $D = [a, b]$, then $f(x)$ is called continuous at the right end-point b if $\lim_{x \to b-} f(x) = f(a)$.

$f(x)$ is called **continuous** if it is continuous at each point of its domain D.

The sum, multiplication by a constant, and product of continuous functions is continuous. In addition, the following are some properties of continuous functions:

1. Let $f(x)$ be continuous at $x = c$, where the domain of $f(x)$ includes a neighborhood of c. If $f(c) > 0$, then there exists a neighborhood $(c - \delta, c + \delta)$ such that $f(x) > 0$ for each point $x \in (c - \delta, c + \delta)$.

2. Let $f(x)$ be a continuous function, whose domain contains $[a, b]$ for $a < b$. Then there exist $\alpha,\ \beta \in \mathbb{R}$ such that $\{f(x) : x \in [a, b]\} = [\alpha, \beta]$.

3. *(Extreme Value Theorem)* Let $f(x)$ be continuous on a closed bounded interval $[a, b]$. Then there exist numbers $c, d \in [a, b]$ such that $f(c) \le f(x) \le f(d)$ for each $x \in [a, b]$.

4. *(Intermediate Value Theorem)* Let $f(x)$ be continuous on a closed bounded interval $[a, b]$. Let d be a number between $f(a)$ and $f(b)$. Then there exists $c \in [a, b]$ such that $f(c) = d$.

Let $f(x)$ be a function whose domain includes an open interval (a, b). Let $c \in (a, b)$. If the limit

$$\lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$$

exists, we say that $f(x)$ is **differentiable at** $x = c$; and we write the limit as $f'(c)$ and call it the derivative of $f(x)$ at $x = c$. If $f'(c)$ exists for each $c \in (a, b)$, then we write $f'(x)$ as $\frac{df}{dx}$.

Also, derivative of f defined on a closed interval $[a, b]$ at the end-point a is taken as the left hand derivative, where in the defining limit of the derivative we take $h \to a-$. Similarly, derivative at b is taken as the limit of that ratio for $h \to 0+$.

Let $f(x)$ be a function defined on an interval I.
We say that $f(x)$ is **increasing on** I if for all $s < t \in I,\ f(s) < f(t)$.
Similarly, we say that $f(x)$ is **decreasing on** I if for all $s < t \in I,\ f(s) > f(t)$.
A **monotonic** function on I is one which either increases on I or decreases on I.

The sum, multiplication by a constant, and product of differentiable functions is differentiable. In addition, the following are some properties of differentiable functions:

1. Each function differentiable at $x = c$ is continuous at $x = c$.

2. Derivatives of Sum, product etc. are respectives equal to sum, product etc of derivatives.

3. *(Chain Rule)* $\frac{dg(f(x))}{dx} = \frac{dg(f(x))}{df(x)} \cdot \frac{df(x)}{dx}$.

4. *(Rolle's Theorem)* Suppose that $f : [a, b] \to \mathbb{R}$ is continuous, $f(x)$ is differentiable on (a, b), and $f(a) = f(b)$. Then $f'(c) = 0$ for some $c \in (a, b)$.

5. *(Mean value Theorem)* Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and $f(x)$ is differentiable on (a, b). Then there exists $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.

6. Let I be an interval containing at least two points. Let $f : I \to \mathbb{R}$ be differentiable. If $f'(x) = 0$ for each $x \in I$, iff $f(x)$ is a constant function.

7. *(Cauchy Mean Value Theorem)* Let $f(x)$ and $g(x)$ be continuous on $[a, b]$ and differentiable on (a, b). If $g'(x) \neq 0$ on (a, b), then there exists $c \in (a, b)$ such that $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$.

8. *(L'Hospital's Rule)* Let $f(x)$ and $g(x)$ be differentiable on a neighborhood of a point $x = a$. Suppose $f(a) = g(a) = 0$ but $g(x) \neq 0,\ g'(x) \neq 0$ in the deleted neighborhood of $x = a$. If $\displaystyle\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, then $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$.

9. Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b).

   (a) If $f'(x) > 0$ on (a, b), then $f(x)$ is increasing on $[a, b]$.

   (b) If $f'(x) < 0$ on (a, b), then $f(x)$ is decreasing on $[a, b]$.

Let a function $f(x)$ have domain D. The function $f(x)$ has a **local maximum** at a point $d \in D$ if $f(x) \leq f(d)$ for every x in some neighborhood of d contained in D. in such a case, we also say that the point $x = d$ is a **point of local maximum** of the function $f(x)$.

Similarly, $f(x)$ has an **local minimum** at $b \in D$ if $f(b) \leq f(x)$ for every x in some neighborhood of b contained in D. In this case, we say that the point $x = b$ is a **point of local minimum** of the function $f(x)$.

The points of local maximum and local minimum are commonly referred to as **local extremum points**; and the function is said to have **local extrema** at those points.

Let $f(x)$ have domain D. A point $c \in D$ is called a **critical point** of $f(x)$ if c is not an interior point of D, or if $f(x)$ is not differentiable at $x = c$, or if $f'(c) = 0$.

If $f(x)$ has an extremum at $x = c$, then c is a critical point of $f(x)$.

Test for Local Extrema:

Let c be an interior point of the domain of $f(x)$ with $f'(c) = 0$.

$f'(x)$ changes sign from $+$ to $-$ at $x = c$ iff $x = c$ is a point of local maximum of $f(x)$.

If $f''(c) < 0$, then $x = c$ is a point of local maximum of $f(x)$.

$f'(x)$ changes sign from $-$ to $+$ at $x = c$ iff $x = c$ is a point of local minimum of $f(x)$.

If $f''(c) > 0$, then $x = c$ is a point of local minimum of $f(x)$.

Let $x = c$ be a left end-point of the domain of $f(x)$.

$f'(x) < 0$ on the immediate right of $x = c$ iff $x = c$ is a point of local maximum of $f(x)$.

$f'(x) > 0$ on the immediate right of $x = c$ iff $x = c$ is a point of local minimum of $f(x)$.

Let $x = c$ be a right end-point of the domain of $f(x)$.

$f'(x) > 0$ on the immediate left of $x = c$ iff $x = c$ is a point of local maximum of $f(x)$.

$f'(x) < 0$ on the left of $x = c$ iff $x = c$ is a point of local minimum of $f(x)$.

The graph of a function $y = f(x)$ is **concave up** on an open interval I if $f'(x)$ is increasing on I. The graph of $y = f(x)$ is **concave down** on an open interval I if $f'(x)$ is decreasing on I. A **point of inflection** is a point where $y = f(x)$ has a tangent and the concavity changes.

Second derivative test for concavity:

Let $y = f(x)$ be twice differentiable on an interval I.

If $f''(x) > 0$ on I, then the graph of $y = f(x)$ is concave up over I.

If $f''(x) < 0$ on I, then the graph of $y = f(x)$ is concave down over I.

If $f''(x)$ is positive on one side of $x = c$ and negative on the other side, then the point $(c, f(c))$ on the graph of $y = f(x)$ is a point of inflection.

Let $f : [a, b] \to \mathbb{R}$. Divide $[a, b]$ into smaller sub-intervals by choosing the break points as

$$a = x_0 < x_1 < \ldots < x_n = b.$$

The set $P = \{x_0, x_1, \ldots, x_n\}$ is called a **partition of** $[a, b]$.

Now P divides $[a, b]$ into n sub-intervals: $[x_0, x_1], \cdots, [x_{n-1}, x_n]$. Here, the kth sub-interval is $[x_{k-1}, x_k]$. The area under the curve $y = f(x)$ raised over the kth sub-interval is approximated by $f(c_k)(x_k - x_{k-1})$ for some choice of the point $c_k \in [x_{k-1}, x_k]$.

Write the choice points (also called **sample points**) as a set $C = \{c_1, \ldots, c_n\}$.

Then the **Riemann sum**

$$S(f, P, C) = \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1})$$

is an approximation to the whole area raised over $[a, b]$ and lying between the curve $y = f(x)$ and the x-axis. By taking the **norm** of the partition as $\|P\| = \max_k (x_k - x_{k-1})$, we would say that when the norm of the partition approaches 0, the Riemann sum would approach the required area. Thus, we define the area of the region bounded by the lines $x = a,\ x = b,\ y = 0$, and $y = f(x)$ as

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1})$$

provided that this limit exists. We define this limit (which is the mentioned area here) as the definite integral of f on the interval $[a, b]$. That is,

$$\int_a^b f(x)\,dx = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1}).$$

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then $\int_a^b f(x)\,dx$ exists.

The definite integral has the following properties:

(Properties of Definite Integral)

1. Let $f(x)$ have domain $[a, b]$. Let $c \in (a, b)$. Then $f(x)$ is integrable on $[a, b]$ iff $f(x)$ is integrable on both $[a, c]$ and $[c, b]$. In this case,
$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

2. Let $f(x)$ and $g(x)$ be integrable on $[a, b]$. Then $(f + g)(x)$ is integrable on $[a, b]$ and
$$\int_a^b (f+g)(x)dx = \int_a^b (f(x) + g(x))\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx.$$

3. Let $f(x)$ be integrable on $[a, b]$. Let $c \in \mathbb{R}$. Then $(cf)(x)$ is integrable on $[a.b]$ and
$$\int_a^b (cf)(x)\, dx = \int_a^b cf(x)\, dx = c\int_a^b f(x)\, dx.$$

4. Let $f(x)$ and $g(x)$ be integrable on $[a, b]$. If for each $x \in [a, b],\ f(x) \le g(x)$, then
$$\int_a^b f(x)\, dx \le \int_a^b g(x)\, dx.$$

5. Let $f(x)$ be integrable on $[a, b]$. If $m \le f(x) \le M$ for all $x \in [a, b]$, then
$$m(b-a) \le \int_a^b f(x)\, dx \le M(b-a).$$

6. (*Average Value Theorem*) Let $f(x)$ be continuous on $[a, b]$. Then there exists $c \in [a, b]$ such that
$$f(c) = \frac{1}{b-a}\int_a^b f(x)\, dx.$$

7. Let $f(x)$ be continuous on $[a, b]$. If $f(x)$ has the same sign on $[a, b]$ and $\displaystyle\int_a^b f(x)\, dx = 0$, then $f(x)$ is the zero function, i.e., $f(x) = 0$ for each $x \in [a, b]$.

We extend the integral even when $a \not< b$ by the following:

If $a = b$, then we take $\displaystyle\int_a^b f(x)\, dx = 0.$

If $a > b$, then we take $\displaystyle\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx.$

Also, for any real number c; even when c is outside the interval (a, b) we have
$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$
In all these extensions, we assume that the definite integrals exist.

The main result that shows that differentiation and integration are reverse processes is the following:

(Fundamental Theorem of Calculus) Let $f(x)$ be continuous on $[a, b]$.

1. If $F(x)$ is an antiderivative of $f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

2. The function $g(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b). Moreover, $g'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

The chain rule for differentiation is translated to integration as follows:

(Substitution)

1. Let $u = g(x)$ be a differentiable function whose range is an interval I. Let $f(x)$ be continuous on I. Then
$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

2. Let $u = g(x)$ be a continuously differentiable function on $[a, b]$ whose range is an interval I. Let $f(x)$ be continuous on I. Then
$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

The rpoduct rule for differntiation gives the *integration by parts* formula.

$$\int f(x)h(x)\,dx = f(x)\int h(x)\,dx - \int \left[f'(x)\int h(x)\,dx\right]\,dx + C.$$

We remember it as follows (Read F as first and S as second):

Integral of $F \times S = F\times$ integral of $S -$ integral of (derivative of $F \times$ integral of S).

The *natural logarithm* $\ln x$ is defined as follows:

$$\ln x = \int_1^x \frac{1}{t}\,dt \quad \text{for } \ x > 0.$$

The *exponential function* is the inverse of the natural logarithm. That is,

$$\exp : R \to (0, \infty); \quad y = \exp(x) \quad \text{iff} \quad x = \ln y.$$

Since $\exp(x)\ \exp(y) = \exp(x + y)$ and $\exp(0) = 1$, we write

$$\exp(x) = e^x, \quad \text{where} \quad e = \exp(1).$$

Then hyperbolic functions are defined by

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}, \quad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}), \quad \cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}), \quad \tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right).$$

Notice that $\cosh^{-1}$ has domain as $x \geq 1$ and $\tanh^{-1}$ has domain as $-1 < x < 1$.

Let C be a curve given parametrically by $x = f(t), y = g(t),\ a \leq t \leq b$. Assdume that both $f(t)$ and $g(t)$ are continuously differentiable.

$$\text{Length of the curve} = L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

If the curve is given as a function $y = f(x),\ a \leq x \leq b$, then take $x = t$ and $y = f(t)$ as its parameterization. We then have the length as

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + (y')^2}\, dx.$$

Notice that this formula is applicable when $f'(x)$ is continuous on $[a, b]$.

We write $L = \int_a^b ds$ with limits a and b for the variable of integration, which may be $x,\ y$ or t. Here,

$$ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

Suppose that a curve is given in polar coordinates by $r = f(\theta)$ for a continuous function $f(\theta)$, where $\alpha \leq \theta \leq \beta$. Then the area of the sector and the arc length of the curve are

$$\text{Area } = \int_\alpha^\beta r^2\, d\theta, \quad \text{Length } = \int_\alpha^\beta \sqrt{r^2 + (r')^2}\, d\theta.$$

A.3 Formulas

Here are some formulas for the exponential and the logarithm functions:

$$\lim_{t\to\infty} \frac{t^p}{a^t} = 0 \text{ for } p \in \mathbb{N} \text{ and } a > 1.$$

$$\ln e = 1 = e^0, \quad e^{\ln x} = x, \quad \ln(e^x) = x, \quad a^x = e^{x \ln a},$$

$$\lim_{x\to\infty} \ln x = \infty, \quad \lim_{x\to 0+} \ln x = -\infty, \quad \lim_{x\to\infty} e^x = \infty, \quad \lim_{x\to -\infty} e^x = 0,$$

$$(\ln x)' = \frac{1}{x}, \quad (e^x)' = e^x, \quad (a^x)' = (\ln a)a^x, \quad \int_1^e \frac{1}{t}\, dt = 1, \quad \int e^x\, dx = e^x.$$

$$\lim_{h\to 0} \frac{\ln(1 + xh)}{xh} = 1 \text{ for } x \neq 0, \quad \lim_{h\to 0} \frac{e^h - 1}{h} = 1, \quad \lim_{h\to 0}(1 + xh)^{1/h} = e^x,$$

Below are given some integrals, from which you should get the derivatives by following the simple rule that if $\int f(x)\, dx = g(x) + c$, then $g'(x) = f(x)$.

1. $\displaystyle\int u\, dv = uv - \int v\, du$

2. $\displaystyle\int a^u\, du = \frac{a^u}{\ln a} + C, \quad a \neq 1, \quad a > 0$

3. $\displaystyle\int \cos u\, du = \sin u + C$

4. $\displaystyle\int \sin u\, du = -\cos u + C$

5. $\displaystyle\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C, \quad n \neq -1$

6. $\displaystyle\int (ax+b)^{-1}\,dx = \frac{1}{a}\ln|ax+b| + C$

7. $\displaystyle\int x(ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a^2}\left[\frac{ax+b}{n+2} - \frac{b}{n+1}\right] + C, \quad n \neq -1, -2$

8. $\displaystyle\int x(ax+b)^{-1}\,dx = \frac{x}{a} - \frac{b}{a^2}\ln|ax+b| + C$

9. $\displaystyle\int x(ax+b)^{-2}\,dx = \frac{1}{a^2}\left[\ln|ax+b| + \frac{b}{ax+b}\right] + C$

10. $\displaystyle\int \frac{dx}{x(ax+b)} = \frac{1}{b}\ln\left|\frac{x}{ax+b}\right| + C$

11. $\displaystyle\int (\sqrt{ax+b})^n\,dx = \frac{2}{a}\frac{(\sqrt{ax+b})^{n+2}}{n+2} + C, \quad n \neq -2$

12. $\displaystyle\int \frac{\sqrt{ax+b}}{x}\,dx = 2\sqrt{ax+b} + b\int \frac{dx}{x\sqrt{ax+b}}$

13. a) $\displaystyle\int \frac{dx}{x\sqrt{ax-b}} = \frac{2}{\sqrt{b}}\tan^{-1}\sqrt{\frac{ax-b}{b}} + C$

b) $\displaystyle\int \frac{dx}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}}\ln\left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right| + C$

14. $\displaystyle\int \frac{\sqrt{ax+b}}{x^2}\,dx = -\frac{\sqrt{ax+b}}{x} + \frac{a}{2}\int \frac{dx}{x\sqrt{ax+b}} + C$

15. $\displaystyle\int \frac{dx}{x^2\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{bx} - \frac{a}{2b}\int \frac{dx}{x\sqrt{ax+b}} + C$

16. $\displaystyle\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

17. $\displaystyle\int \frac{dx}{(a^2+x^2)^2} = \frac{x}{2a^2(a^2+x^2)} + \frac{1}{2a^3}\tan^{-1}\frac{x}{a} + C$

18. $\displaystyle\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right| + C$

19. $\displaystyle\int \frac{dx}{(a^2-x^2)^2} = \frac{x}{2a^2(a^2-x^2)} + \frac{1}{4a^3}\ln\left|\frac{x+a}{x-a}\right| + C$

20. $\displaystyle\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\frac{x}{a} + C = \ln(x+\sqrt{a^2+x^2}) + C$

21. $\displaystyle\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\ln(x+\sqrt{a^2+x^2}) + C$

22. $\displaystyle\int x^2\sqrt{a^2+x^2}\,dx = \frac{x}{8}(a^2+2x^2)\sqrt{a^2+x^2} - \frac{a^4}{8}\ln(x+\sqrt{a^2+x^2}) + C$

23. $\displaystyle\int \frac{\sqrt{a^2+x^2}}{x}\,dx = \sqrt{a^2+x^2} - a\ln\left|\frac{a+\sqrt{a^2+x^2}}{x}\right| + C$

24. $\displaystyle\int \frac{\sqrt{a^2+x^2}}{x^2}\,dx = \ln(x+\sqrt{a^2+x^2}) - \frac{\sqrt{a^2+x^2}}{x} + C$

25. $\displaystyle\int \frac{x^2}{\sqrt{a^2+x^2}}\,dx = -\frac{a^2}{2}\ln(x+\sqrt{a^2+x^2}) + \frac{x\sqrt{a^2+x^2}}{2} + C$

26. $\displaystyle\int \frac{dx}{x\sqrt{a^2+x^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2+x^2}}{x}\right| + C$

27. $\displaystyle\int \frac{dx}{x^2\sqrt{a^2+x^2}} = -\frac{\sqrt{a^2+x^2}}{a^2x} + C$

28. $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$

29. $\displaystyle\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

30. $\displaystyle\int x^2\sqrt{a^2-x^2}\,dx = \frac{a^4}{8}\sin^{-1}\frac{x}{a} - \frac{1}{8}x\sqrt{a^2-x^2}(a^2-2x^2) + C$

31. $\displaystyle\int \frac{\sqrt{a^2-x^2}}{x}\,dx = \sqrt{a^2-x^2} - a\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right| + C$

32. $\displaystyle\int \frac{\sqrt{a^2-x^2}}{x^2}\,dx = -\sin^{-1}\frac{x}{a} - \frac{\sqrt{a^2-x^2}}{x} + C$

33. $\displaystyle\int \frac{x^2}{\sqrt{a^2-x^2}}\,dx = \frac{a^2}{2}\sin^{-1}\frac{x}{a} - \frac{1}{2}x\sqrt{a^2-x^2} + C$

34. $\displaystyle\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right| + C$

35. $\displaystyle\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x} + C$

36. $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} + C = \ln\left|x+\sqrt{x^2-a^2}\right| + C$

37. $\displaystyle\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + C$

38. $\int (\sqrt{x^2-a^2})^n\,dx = \frac{x(\sqrt{x^2-a^2})^n}{n+1} - \frac{na^2}{n+1}\int (\sqrt{x^2-a^2})^{n-2}dx, \quad n \neq -1$

39. $\int \frac{dx}{(\sqrt{x^2-a^2})^n} = \frac{x(\sqrt{x^2-a^2})^{2-n}}{(2-n)a^2} - \frac{n-3}{(n-2)a^2}\int \frac{dx}{(\sqrt{x^2-a^2})^{n-2}}, \quad n \neq 2$

40. $\int x(\sqrt{x^2-a^2})^n\,dx = \frac{(\sqrt{x^2-a^2})^{n+2}}{n+2} + C, \quad n \neq -2$

41. $\int x^2\sqrt{x^2-a^2}\,dx = \frac{x}{8}(2x^2-a^2)\sqrt{x^2-a^2} - \frac{a^4}{8}\ln\left|x+\sqrt{x^2-a^2}\right| + C$

42. $\int \frac{\sqrt{x^2-a^2}}{x}\,dx = \sqrt{x^2-a^2} - a\sec^{-1}\left|\frac{x}{a}\right| + C$

43. $\int \frac{\sqrt{x^2-a^2}}{x^2}\,dx = \ln\left|x+\sqrt{x^2-a^2}\right| - \frac{\sqrt{x^2-a^2}}{x} + C$

44. $\int \frac{x^2}{\sqrt{x^2-a^2}}\,dx = \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + \frac{x}{2}\sqrt{x^2-a^2} + C$

45. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C = \frac{1}{a}\cos^{-1}\left|\frac{a}{x}\right| + C$

46. $\int \frac{dx}{x^2\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{a^2x} + C$

47. $\int \frac{dx}{\sqrt{2ax-x^2}} = \sin^{-1}\left(\frac{x-a}{a}\right) + C$

48. $\int \sqrt{2ax-x^2}\,dx = \frac{x-a}{2}\sqrt{2ax-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + C$

49. $\int (\sqrt{2ax-x^2})^n dx = \frac{(x-a)(\sqrt{2ax-x^2})^n}{n+1} + \frac{na^2}{n+1}\int (\sqrt{2ax-x^2})^{n-2}\,dx$

50. $\int \frac{dx}{(\sqrt{2ax-x^2})^n} = \frac{(x-a)(\sqrt{2ax-x^2})^{2-n}}{(n-2)a^2} + \frac{n-3}{(n-2)a^2}\int \frac{dx}{(\sqrt{2ax-x^2})^{n-2}}$

51. $\int x\sqrt{2ax-x^2}\,dx = \frac{(x+a)(2x-3a)\sqrt{2ax-x^2}}{6} + \frac{a^3}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + C$

52. $\int \frac{\sqrt{2ax-x^2}}{x}\,dx = \sqrt{2ax-x^2} + a\sin^{-1}\left(\frac{x-a}{a}\right) + C$

53. $\int \frac{\sqrt{2ax-x^2}}{x^2}\,dx = -2\sqrt{\frac{2a-x}{x}} - \sin^{-1}\left(\frac{x-a}{a}\right) + C$

54. $\int \frac{x\,dx}{\sqrt{2ax-x^2}} = a\sin^{-1}\left(\frac{x-a}{a}\right) - \sqrt{2ax-x^2} + C$

55. $\int \frac{dx}{x\sqrt{2ax-x^2}} = -\frac{1}{a}\sqrt{\frac{2a-x}{x}} + C$

56. $\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$

57. $\int \cos ax\,dx = \frac{1}{a}\sin ax + C$

58. $\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a} + C$

59. $\int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a} + C$

60. $\int \sin^n ax\,dx = -\frac{\sin^{n-1} ax\cos ax}{na} + \frac{n-1}{n}\int \sin^{n-2} ax\,dx$

61. $\int \cos^n ax\,dx = \frac{\cos^{n-1} ax\sin ax}{na} + \frac{n-1}{n}\int \cos^{n-2} ax\,dx$

62. a) $\int \sin ax\cos bx\,dx = -\frac{\cos(a+b)x}{2(a+b)} - \frac{\cos(a-b)x}{2(a-b)} + C, \quad a^2 \neq b^2$

b) $\int \sin ax\sin bx\,dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)} + C, \quad a^2 \neq b^2$

c) $\int \cos ax\cos bx\,dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)} + C, \quad a^2 \neq b^2$

63. $\int \sin ax \cos ax\, dx = -\frac{\cos 2ax}{4a} + C$

64. $\int \sin^n ax \cos ax\, dx = \frac{\sin^{n+1} ax}{(n+1)a} + C, \quad n \neq -1$

65. $\int \frac{\cos ax}{\sin ax}\, dx = \frac{1}{a} \ln |\sin ax| + C$

66. $\int \cos^n ax \sin ax\, dx = -\frac{\cos^{n+1} ax}{(n+1)a} + C, \quad n \neq -1$

67. $\int \frac{\sin ax}{\cos ax}\, dx = -\frac{1}{a} \ln |\cos ax| + C$

68. $\int \sin^n ax \cos^m ax\, dx = -\frac{\sin^{n-1} ax \cos^{m+1} ax}{a(m+n)} + \frac{n-1}{m+n} \int \sin^{n-2} ax \cos^m ax\, dx, \quad n \neq -m \quad (\text{reduces } \sin^n ax)$

69. $\int \sin^n ax \cos^m ax\, dx = \frac{\sin^{n+1} ax \cos^{m-1} ax}{a(m+n)} + \frac{m-1}{m+n} \int \sin^n ax \cos^{m-2} ax\, dx, \quad m \neq -n \quad (\text{reduces } \cos^m ax)$

70. $\int \frac{dx}{b + c \sin ax} = \frac{-2}{a\sqrt{b^2 - c^2}} \tan^{-1} \left[\sqrt{\frac{b-c}{b+c}} \tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) \right] + C, \quad b^2 > c^2$

71. $\int \frac{dx}{b + c \sin ax} = \frac{-1}{a\sqrt{c^2 - b^2}} \ln \left| \frac{c + b \sin ax + \sqrt{c^2 - b^2} \cos ax}{b + c \sin ax} \right| + C, \quad b^2 < c^2$

72. $\int \frac{dx}{1 + \sin ax} = -\frac{1}{a} \tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) + C$

73. $\int \frac{dx}{1 - \sin ax} = \frac{1}{a} \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right) + C$

74. $\int \frac{dx}{b + c \cos ax} = \frac{2}{a\sqrt{b^2 - c^2}} \tan^{-1} \left[\sqrt{\frac{b-c}{b+c}} \tan \frac{ax}{2} \right] + C, \quad b^2 > c^2$

75. $\int \frac{dx}{b + c \cos ax} = \frac{1}{a\sqrt{c^2 - b^2}} \ln \left| \frac{c + b \cos ax + \sqrt{c^2 - b^2} \sin ax}{b + c \cos ax} \right| + C, \quad b^2 < c^2$

76. $\int \frac{dx}{1 + \cos ax} = \frac{1}{a} \tan \frac{ax}{2} + C$

77. $\int \frac{dx}{1 - \cos ax} = -\frac{1}{a} \cot \frac{ax}{2} + C$

78. $\int x \sin ax\, dx = \frac{1}{a^2} \sin ax - \frac{x}{a} \cos ax + C$

79. $\int x \cos ax\, dx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax + C$

80. $\int x^n \sin ax\, dx = -\frac{x^n}{a} \cos ax + \frac{n}{a} \int x^{n-1} \cos ax\, dx$

81. $\int x^n \cos ax\, dx = \frac{x^n}{a} \sin ax - \frac{n}{a} \int x^{n-1} \sin ax\, dx$

82. $\int \tan ax\, dx = \frac{1}{a} \ln |\sec ax| + C$

83. $\int \cot ax\, dx = \frac{1}{a} \ln |\sin ax| + C$

84. $\int \tan^2 ax\, dx = \frac{1}{a} \tan ax - x + C$

85. $\int \cot^2 ax\, dx = -\frac{1}{a} \cot ax - x + C$

86. $\int \tan^n ax\, dx = \frac{\tan^{n-1} ax}{a(n-1)} - \int \tan^{n-2} ax\, dx, \quad n \neq 1$

87. $\int \cot^n ax\, dx = -\frac{\cot^{n-1} ax}{a(n-1)} - \int \cot^{n-2} ax\, dx, \quad n \neq 1$

88. $\int \sec ax\, dx = \frac{1}{a} \ln |\sec ax + \tan ax| + C$

89. $\int \csc ax\, dx = -\frac{1}{a} \ln |\csc ax + \cot ax| + C$

90. $\int \sec^2 ax\, dx = \frac{1}{a} \tan ax + C$

91. $\int \csc^2 ax\, dx = -\frac{1}{a} \cot ax + C$

92. $\int \sec^n ax\, dx = \frac{\sec^{n-2} ax \tan ax}{a(n-1)} + \frac{n-2}{n-1} \int \sec^{n-2} ax\, dx, \quad n \neq 1$

93. $\int \csc^n ax\, dx = -\frac{\csc^{n-2} ax \cot ax}{a(n-1)} + \frac{n-2}{n-1} \int \csc^{n-2} ax\, dx, \quad n \neq 1$

94. $\int \sec^n ax \tan ax\, dx = \frac{\sec^n ax}{na} + C, \quad n \neq 0$

95. $\int \csc^n ax \cot ax\, dx = -\frac{\csc^n ax}{na} + C, \quad n \neq 0$

96. $\int \sin^{-1} ax\, dx = x \sin^{-1} ax + \frac{1}{a}\sqrt{1 - a^2x^2} + C$

97. $\int \cos^{-1} ax\, dx = x \cos^{-1} ax - \frac{1}{a}\sqrt{1 - a^2x^2} + C$

98. $\int \tan^{-1} ax\,dx = x\tan^{-1} ax - \frac{1}{2a}\ln(1+a^2x^2) + C$

99. $\int x^n \sin^{-1} ax\,dx = \frac{x^{n+1}}{n+1}\sin^{-1} ax - \frac{a}{n+1}\int \frac{x^{n+1}\,dx}{\sqrt{1-a^2x^2}}, \quad n \neq -1$

100. $\int x^n \cos^{-1} ax\,dx = \frac{x^{n+1}}{n+1}\cos^{-1} ax + \frac{a}{n+1}\int \frac{x^{n+1}\,dx}{\sqrt{1-a^2x^2}}, \quad n \neq -1$

101. $\int x^n \tan^{-1} ax\,dx = \frac{x^{n+1}}{n+1}\tan^{-1} ax - \frac{a}{n+1}\int \frac{x^{n+1}\,dx}{1+a^2x^2}, \quad n \neq -1$

102. $\int e^{ax}dx = \frac{1}{a}e^{ax} + C$

103. $\int b^{ax}dx = \frac{1}{a}\frac{b^{ax}}{\ln b} + C, \quad b > 0,\ b \neq 1$

104. $\int xe^{ax}dx = \frac{e^{ax}}{a^2}(ax-1) + C$

105. $\int x^n e^{ax}dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx$

106. $\int x^n b^{ax}dx = \frac{x^n b^{ax}}{a\ln b} - \frac{n}{a\ln b}\int x^{n-1}b^{ax}\,dx, \quad b > 0, b \neq 1$

107. $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C$

108. $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C$

109. $\int \ln ax\,dx = x\ln ax - x + C$

110. $\int x^n(\ln ax)^m\,dx = \frac{x^{n+1}(\ln ax)^m}{n+1} - \frac{m}{n+1}\int x^n(\ln ax)^{m-1}dx, \quad n \neq -1$

111. $\int x^{-1}(\ln ax)^m\,dx = \frac{(\ln ax)^{m+1}}{m+1} + C, \quad m \neq -1$

112. $\int \frac{dx}{x\ln ax} = \ln|\ln ax| + C$

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