Real Analysis*

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Contents

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1 Set theoretic Preliminaries

1. For sets A and B,
$$A \subseteq B \iff \text{“}x \in A \Rightarrow x \in B\text{”},$$
$$A \cup B := \{x : x \in A \text{ or } x \in B\},$$
$$A \cap B := \{x : x \in A \text{ and r } x \in B\},$$

2. For sets $A_1, A_2, \ldots, A_n$,
$$\bigcup_{i=1}^{n} A_i := A_1 \cup \cdot \cup A_n := \{x : x \in A_i \text{ for some } i = 1, \ldots, n\},$$
$$\bigcap_{i=1}^{n} A_i := A_1 \cap \cdot \cap A_n := \{x : x \in A_i \text{ for every } i = 1, \ldots, n\},$$

3. Let Λ be a nonempty set. For each $\alpha \in \Lambda$, let A_α be a set. Then
$$\bigcup_{\alpha\in\Lambda}^{n} A_\alpha := \{x : x \in A_\alpha \text{ for some } \alpha \in \Lambda\}, \quad \bigcap_{\alpha\in\Lambda}^{n} A_\alpha := \{x : x \in A_\alpha \text{ for every } \alpha \in \Lambda\}.$$

4. Let S be a nonempty set. Any subset of $S \times S$ is called a *relation* on S.

5. Let R be a relation on S. If $(x, y) \in R$, then we write xRy and say x *is related to* y.

6. Let R be a relation on S.
   (a) R is said to be *reflexive* if xRx for every $x \in S$.
   (b) R is said to be *symmetric* if for $x, y \in S$, $\quad xRy \Rightarrow yRx$.
   (c) R is said to be *antisymmetric* if for $x, y \in S$, $\quad xRy \,\&\, yRx \Rightarrow x = y$.
   (d) R is said to be *transitive* if for $x, y, x \in S$, $\quad xRy \,\&\, yRz \Rightarrow xRz$.

7. Let R be a relation on S.
   (a) R is said to be an *equivalence relation* if it is reflexive, symmetric and transitive.
   (b) R is said to be a *partial order* if it is reflexive, antisymmetric and transitive.

8. An equivalence relation is usually denoted by $\sim$, and a partial order is usually denoted by $\preceq$.

9. Given a partial order $\preceq$ on S, and $x, y \in S$, we write $x \prec y$ if $x \prec y$ and $x \neq y$.

10. Let $(S, \preceq)$ be a partially ordered set and $A \subseteq S$. Then A is called a *totally ordered subset* if for every $x, y \in A$, either $x \preceq y$ or $y \preceq x$.

11. Given an equivalence relation $\sim$ on S and $x \in S$, the set
$$[x] := \{y \in S : y \sim x\}$$
is called the *equivalence class* of x.

12. Given an equivalence relation $\sim$ on S and $x, y \in S$,
$$[x] \neq [y] \iff [x] \cap [y] = \emptyset.$$

13. Given an equivalence relation $\sim$ on S, $S = \bigcup_{x \in S} [x]$. Thus, S is a disjoint union of equivalence classes.

14. Given a partial order $\preceq$ on S and $A \subseteq S$, and element $x \in S$,
    (a) x is called an *upper bound* of A if $a \preceq x$ for every $a \in A$.
    (b) A is said to be *bounded above* if an upper bound exists for A.
    (c) x is called a *maximal element* S if $y \in S$ and $x \preceq y$ implies $y = x$
    (d) An element $x \in S$ is called a *least upper bound* of A if x an upper bound of A and if y is any upper bound of A, then $x \preceq y$; equivalently, x an upper bound of A and if $v \prec x$, then there exists $w \in A$ such that $v \prec w$.

15. Given a partial order $\preceq$ on S and $A \subseteq S$, and element $x \in S$,
    (a) x is called a *lower bound* of A if $x \preceq a$ for every $a \in A$.
    (b) A is said to be *bounded below* if a lower bound exists for A.
    (c) x is called a *minimal element* S if $y \in S$ and $y \preceq x$ implies $y = x$
    (d) An element $x \in S$ is called a *greatest lower bound* of A if x a lower bound of A and if y is any lower bound of A, then $y \preceq x$; equivalently, x is a lower bound of A and and if $x \prec v$, then there exists $w \in A$ such that $w \prec v$.

16. A relation $\prec$ on a set S is called an *order relation* if it is transitive and for every $x, y \in S$, one and only one of the following true: $x \prec y$, $x = y$, $y \prec x$.

Exercise 1. Let $(S, \preceq)$ be a partially ordered set and $A \subseteq S$.

1. Least upper bound of A, if exists, is unique.
2. Greatest lower bound of A, if exists, is unique.
3. If $\prec$ is an order relation on S, then $\preceq$ defined by $x \preceq y$ for either $x \prec y$ or $x = y$, is a partial order.

$\diamondsuit$

Example 2. Let us consider a few examples:

1. Let $S = \mathbb{N} \times \mathbb{N}$ and for (m, n), (p, q) in S, define
$$(m, n) \sim (p, q) \iff mq = np.$$
Then $\sim$ is an equivalence relation on S. In this case the equivalence class of (m, n) is denoted by $\frac{m}{n}$. The set of all equivalence classes is th set of all *positive rational numbers.*

2. Let $n \in \mathbb{N}$. For $a, b \in \mathbb{N}$, define $a \sim b \iff a - b$ is a multiple of n. Then $\sim$ is an equivalence relation on $\mathbb{N}$.

3. Let G be a group and H be a subgroup of G. For $a, b \in G$, define $a \sim b \iff a - b \in H$. Then $\sim$ is an equivalence relation on S.

4. For a nonempty set X let $S = 2^X$. For A, B in S, define $A \preceq B \iff A \subseteq B$. Then $\preceq$ is a partial order on S.

5. For $a, b \in \mathbb{R}$, define $a \preceq b \iff a \leq b$.Then $\preceq$ is a partial order on $\mathbb{R}$. In this case $A := \{x \in \mathbb{R} : x \geq 0\}$ has no upper bound, but 0 and every negarive number is a lower bound.

6. Let $S = re^{i\theta} : 0 \leq \theta < 2\pi,\ 0 \leq r \leq 1\}$. For $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$ in S, define
$$z_1 \preceq z_2 \iff \theta_1 = \theta_2 \quad \text{and } r_1 \leq r_2.$$
Then $\preceq$ is a partial order on S. In this case,
   (a) for each θ, the set $A_\theta := \{re^{i\theta} : 0 \leq r \leq 1\}$ is a totally ordered subset for which $e^{i\theta}$ is an upper bounded.
   (b) Every $e^{i\theta}$ is a maximal element of S.

◇

2 Real Number System

We shall denote the set of real numbers by $\mathbb{R}$.

1. An *ordered field* $\mathbb{F}$ is a field along with an order relation $\prec$ such that
$$\forall\, x, y \in \mathbb{R}, \quad x \prec y \Rightarrow x + z \prec y + z \quad \forall z \in \mathbb{R},$$
$$\forall\, x, y \in \mathbb{R}, \quad x \prec y \Rightarrow xz \prec yz \quad \forall z \prec 0.$$
Recall that if $\prec$ is an order relation, then $\preceq$ defined by $x \preceq y \iff$ either $x \prec y$ or $x = y$ is a partial order.

2. An ordered set is said to have *least upper bound property* if every every nonempty subset of it which is bounded above has the least upper bound.

3. A *complete ordered field* is an ordered field $\mathbb{F}$ such that every subset of $\mathbb{F}$ which is bounded above has least upper bound.

THEOREM 3. *The field* $\mathbb{Q}$ *does not have least upper bound property.*

Proof. Consider the set $A = \{x \in \mathbb{Q} : x^2 < 2\}$. Clearly A is nonempty and it is bounded above. We show that for every $x \in A$, there exists $y \in A$ such that $x < y$.

Let $x \in A$ with $x > 0$, and let $y = \frac{2x+2}{x+2}$. Note that

$$x < y \iff x(x+2) < 2x+2 \iff x^2 < 2 \iff x \in A$$

and

$$2 - y^2 = 2 - \frac{(2x+2)^2}{(x+2)^2} = \frac{2(x+2)^2 - (2x+2)^2}{(x+2)^2} = \frac{2(2-x^2)}{(x+2)^2}$$

so that $y^2 < 2 \iff x^2 < 2 \iff x \in A$. Thus, we have proved that $x \in A$ implies $y \in A$ and $x < y$. □

Remark 4. The proof of Theorem Th-lub-1 shows that if we define $B = \{x \in \mathbb{Q} : x^2 > 2\}$, then B does not have a greatest lower bound in $\mathbb{Q}$. ♢

Exercise 5. For $x > 0$ in $\mathbb{R}$, let $y = f(x) := \frac{2x+2}{x+2}$. Prove the following:

(i) $y = x \iff x^2 = 2$.

(ii) Let $x_1 = 1$ and for $n \in \mathbb{N}$, let $x_{n+1} = f(x_n)$. Then $x_n < x_{n+1}$ for all $n \in \mathbb{N}$ and $x_n \to \sqrt{2}$. ♢

THEOREM 6. *There exists a complete ordered field containing $\mathbb{Q}$ as a subfield, and any two complete ordered fields are isomorphic.*

Here are a few important consequences of the lub property of $\mathbb{R}$.

THEOREM 7. (Archimedean property) *If $a, b \in \mathbb{R}$ with $a > 0$, then there exists $n \in \mathbb{N}$ such that $na > b$.*

Proof. Let $a > 0$, $b > 0$. Suppose there is no $n \in \mathbb{N}$ such that $na > b$. Then $na \leq b$ for all $n \in \mathbb{N}$. That is, $S := \{na : n \in \mathbb{N}\}$ bounded above by b. Let $\beta := \sup S$. Then $\beta - a < \beta$ so that there exists $n \in \mathbb{N}$ such that $\beta - a < na$. Thus, $\beta < (n+1)a$, which contradicts the definition of β. □

THEOREM 8. (Denseness of $\mathbb{Q}$) *If $a, b \in \mathbb{R}$ with $a < b$, then there exists $r \in \mathbb{Q}$ such that $a < r < b$.*

Proof. Let $a, b \in \mathbb{R}$ with $a < b$. Then $b - a > 0$. By the Archimedean property, there exists $n \in \mathbb{N}$ such that $n(b-a) > 1$. Hence there exists $m \in \mathbb{N}$ such that $na < m < nb$. Thus, $a < \frac{m}{n} < b$. □

3 Completeness of $\mathbb{R}$

Definition 9. A sequence (x_n) of real numbers is said to be convergent if there exists $x \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ for all $n \geq n_0$, and in that case, we write

$$x_n \to x \quad \text{as} \quad n \to \infty \quad \text{or} \quad x_n \to x \quad \text{or} \quad \lim_{n\to\infty} x_n = x.$$

♢

1. Every convergent sequence is bounded: if $x_n \to x$, then there exists $\alpha > 0$ such that $|x_n| \leq \alpha$ for all $n \in \mathbb{N}$.

Proof. Suppose $x_n \to x$. Let $n_0 \in \mathbb{N}$ be such that $|x_n - x| \le 1$ for all $n \ge n_0$. Then

$$|x_n| \le |x_n - x| + |x| \le \max\{1 + |x|, M\}, \quad M := \max\{|x_k| : k = 1, \ldots, n_0\}.$$

□

2. Every convergent sequence is a Cauchy sequence, i.e., for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|x_n - x_m| < \varepsilon$ for all $n, m \ge n_0$.

   Proof. Suppose $x_n \to x$ and $\varepsilon > 0$. Let $n_0 \in \mathbb{N}$ be such that $|x_n - x_m|\langle\varepsilon$ for all $n, m \ge n_0$. Then

   $$|x_n - x_m| \le |x_n - x| + |x - x_m| \le 2\varepsilon \quad \forall\, n, m \ge n_0.$$

   □

THEOREM 10. *Every Cauchy sequence in $\mathbb{R}$ converges.*

As steps towards proving the above theorem, we prove some lemmas.

LEMMA 11. *Suppose (x_n) is a Cauchy sequence in $\mathbb{R}$ and $S = \{x_n : n \in \mathbb{N}\}$. If S is a finite set, then (x_n) is eventually constant, and hence it converges.*

Proof. Suppose S is a finite set, say $S = \{\alpha_1, \ldots, \alpha_k\}$. Assume for a moment that (x_n) is not eventually constant. Then, for any $n_N \in \mathbb{N}$ there exists $n, m \ge N$ such that $x_n \ne x_m$. But, for such n, m,

$$|x_n - x_m| \ge \min\{|\alpha_i - \alpha_j| : i, j = 1, \ldots, k,\ i \ne j\} = d > 0.$$

This is not possible, because (x_n) is a Cauchy sequence. Thus, either (x_n) is eventually constant or else S is an infinite set. □

LEMMA 12. *Suppose (x_n) is a Cauchy sequence in $\mathbb{R}$. Then S is bounded.*

Proof. Let $n_0 \in \mathbb{N}$ be such that $|x_n - x_m| < 1$ for all $n, m \ge n_0$. Hence,

$$|x_n| \le |x_n - x_{n_0}| + |x_{n_0}| < 1 + |x_{n_0}| \quad \forall\, n \ge n_0.$$

Thus,

$$|x_n| \le \max\{1 + |x_{n_0}|, M\} + |x|, \quad M := \max\{|x_k| : k = 1, \ldots, n_0\}.$$

□

LEMMA 13. *Suppose (x_n) is a Cauchy sequence in $\mathbb{R}$. If (x_n) has a subsequence which converges to some $x \in \mathbb{R}$, then (x_n) converges to x.*

Proof. Suppose there exists a subsequence (x_{n_k}) of $(x - n)$ such that $x_{n_k} \to x$ as $k \to \infty$. Let $\varepsilon > 0$ be given, and let $k_0 \in \mathbb{N}$ be such that $|x_{n_k} - x| < \varepsilon$ for all $k \ge k_0$. Since $n_k \ge k$ for all $k \in \mathbb{N}$, we have

$$|x_k - x| \le |x_k - x_{n_k}| + |x_{n_k} - x| \le |x_k - x_{n_k}| + \varepsilon \quad \forall\, k \ge k_0.$$

Also, let k_1 be such that $|x_n - x_m| < \varepsilon$ for all $n, m \ge k_1$. Then $|x_k - x_{n_k}| < \varepsilon$ for all $k \ge k_1$. Thus,

$$|x_k - x| \le |x_k - x_{n_k}| + \varepsilon < 2\varepsilon \quad \forall\, k \ge \max\{k_0, k_1\}.$$

□

LEMMA 14. *Suppose (x_n) is a Cauchy sequence in $\mathbb{R}$. Then (x_n) has a convergent subsequence.*

Proof. Let $S = \{x_n : n \in \mathbb{N}\}$. If S is a finite set, then we know by Lemma 11 that (x_n) itself converges. Therefore, assume that S is an infinite set. By Lemma 12, S is bounded.

Let $S \subseteq I_0 := [a, b]$. Let $c = (a + b)/2$. Since S is an infinite set, either $S \cap [a, c]$ is an infinite set or $S \cap [c, b]$ is an infinite set. Let

$$I_1 := [a_1, b_1] = \begin{cases} [a, c] & \text{if } S \cap [a, c] \text{ is an infinite set} \\ [c, b] & \text{if } S \cap [a, c] \text{ is not an infinite set} \end{cases}.$$

Thus, $S \cap I_1$ is an infinite set. Next let $c_1 = (a_1 + b_1)/2$. Then either $S \cap [a_1, c_1]$ infinite set or $S \cap [c_1, b_1]$ is an infinite set. Let

$$I_2 := [a_2, b_2] = \begin{cases} [a_1, c_1] & \text{if } S \cap [a_1, c_1] \text{ is an infinite set} \\ [c_1, b_1] & \text{if } S \cap [a_1, c_1] \text{ is not an infinite set} \end{cases}.$$

Thus, $S \cap I_2$ is an infinite set. Continuing this process, we obtain closed intervals $I_1, I_2, \ldots$ such that

1. $I_1 \supset I_2 \supset I_3 \supset \cdots$, with $E \cap I_k$ is an infinite set for every $k \in \mathbb{N}$,
2. $\ell(I_{k+1}) = \ell(I_k)/2$ for all $k \in \mathbb{N}$ so that $\ell(I_n) = \ell(I_0)/2^n$, $n \in \mathbb{N}$.

Note also that

$$a \le a_k \le a_{k+1} \le b_{k+1} \le b_k \le b \quad \forall\, k \in \mathbb{N},$$

Thus (a_n) is bounded above and (b_n) is bounded below. Let

$$\alpha := \sup_n a_n, \quad \beta := \inf_n b_n.$$

Then $a_k \le \alpha \le \beta \le b_k$ or all $k \in \mathbb{N}$. Since

$$b_k - a_k = \ell(I_0)/2^k \to 0 \quad \text{as} \quad k \to \infty,$$

it follows that $\alpha = \beta =: \gamma$, say, and $a_k \to \gamma$ and $b_k \to \gamma$.

Since each $S \cap [a_k, b_k]$ is an infinite set, $x_{n_k} \in S \cap [a_k, b_k]$ can be chosen such that

$$n_k \le n_{k+1} \quad \text{and} \quad x_{n_{k+1}} \notin \{x_{n_1}, \ldots, x_{n_k}\} \quad \forall\, k \in \mathbb{N}.$$

Now, for $\varepsilon > 0$, let $k_0 \in \mathbb{N}$ such that $a_k, b_k \in (\gamma - \varepsilon, \gamma + \varepsilon)$ for all $k \ge k_0$. Then

$$x_{n_k} \in [a_k, b_k] \subset (\gamma - \varepsilon, \gamma + \varepsilon) \quad \forall\, k \ge k_0.$$

Thus $x_{n_k} \to \gamma$ as $k \to \infty$. □

Proof of Theorem 10. Follows by Lemma 13 and Lemma 14. □

A particular case the following result has already been used in proving Lemma 14

THEOREM 15. (Nested interval theorem) *If (I_n) is a decreasing sequence of closed and bounded intervals, then $\cap_{n=1}^{\infty} I_n$ is nonempty. Further, if $\ell(I_n) \to 0$, then $\cap_{n=1}^{\infty} I_n$ is a singleton set.*

Proof. Let $I_n = [a_n, b_n]$, $n \in \mathbb{N}$. Since $I_n \supseteq I_{n+1}$ for all $n \in \mathbb{N}$, $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$ for all $n \in \mathbb{N}$. In particular, (a_n) is bounded above by b_1 and (b_n) is bounded below by a_1. Let

$$a := \sup a_n, \quad b = \inf b_n.$$

Then $a_n \leq a_{n+1} \leq a \leq b \leq b_{n+1} \leq b_n$ for all $n \in \mathbb{N}$ (Why $a \leq b$?). Thus, $a, b \in \cap_{n=1}^{\infty} I_n$. Clearly, if $\ell(I_n) \to 0$, then $b_n - a_n \to 0$ so that $a = b$, and $a_n \to a$, $b_n \to b$. $\square$

Exercise 16. If $E \subseteq \mathbb{R}$ is a bounded infinite set, then E contains a convergent sequence with distinct entries. $\diamondsuit$

Exercise 17. Let (x_n) be a sequence in $\mathbb{R}$. Prove the following:

(i) If (x_n) is monotonically increasing and bounded above, and if $x := \sup\{x_n : n \in \mathbb{N}\}$, then (x_n) converges to x.

(ii) If (x_n) is a Cauchy sequence having a subsequence which converges to x, then (x_n) also converges to x.

(iii) if $a_n \to x$, $b_n \to x$ and if $a_n \leq x_n \leq b_n$ for all $n \in \mathbb{N}$, then $x_n \to x$.

(iv) If $a_n \leq b_n$ for all $n \in \mathbb{N}$ and $a_n \to a$ and $b_n \to b$, then $a \leq b$. $\diamondsuit$

Exercise 18. (i) Suppose (a_n) is such that $|a_{n+2} - a_{n+1}| \leq r|a_{n+1} - a_n|$ for all $n \in \mathbb{N}$, where $0 < r < 1$. Then (a_n) converges.

(ii) Suppose $a_1 = 1 = a_2$ and $a_{n+2} = a_{n+1} + a_n$ and $b_n = a_{n+1}/a_n$ for all $n \in \mathbb{N}$. Then (b_n) converges to $(1 + \sqrt{5})/2$. $\diamondsuit$

4 Metric spaces: Basic Concepts

4.1 Definition and Examples

Definition 19. Let X be a non-empty set. A function $d : X \times X \to \mathbb{R}$ is said to be a **metric** on X if it satisfies the following properties:

(i) $d(x, y) \geq 0 \quad \forall\, x, y \in X$;

(ii) $x, y \in X\; d(x, y) = 0 \;\Rightarrow\; x = y$;

(iii) $x, y \in X \;\Rightarrow\; d(x, y) = d(y, x)$;

(iii) $x, y, z \in X \;\Rightarrow\; (d(x, y) \leq d(x, z) + d(z, x)$.

A set together with a metric on it is called a *metric space.* $\diamondsuit$

Exercise 20. For every x, y, z is a metric space with metric d, $|d(x, z) - d(y, z)| \leq d(x, y)|$. $\diamondsuit$

Example 21. $(x, y) \mapsto d(x, y) := |x - y|$ is a metric on $\mathbb{R}$. $\diamondsuit$

Example 22. Let X be a set. For $x, y \in X$, let $d(x, y) = 0$. Then d is a metric on X, called the *indiscrete metric.* $\diamondsuit$

Example 23. Let X be a non-empty set. For $x, y \in X$, let $d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$ Then d is metric on X, called the *discrete metric*. ♢

In the following we shall denote by $\mathbb{K}$ the set $\mathbb{R}$ or $\mathbb{C}$.

Example 24. For $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{K}^n$, let

$$d(x,y) := \sum_{k=1}^{n} |x_k - y_k| \quad \text{and} \quad \tilde{d}(x,y) := \max\{|x_k - y_k| : k = 1, \ldots, k\}.$$

Then d and $\tilde{d}$ are metrics on $\mathbb{K}^n$. ♢

For the next example, we first prove:

THEOREM 25. (Cauchy-Schwarz inequality) *For* $(x_1, \ldots, x_n))$ *and* $(y_1, \ldots, y_n)$ *in* $\mathbb{K}^n$,

$$\sum_{k=1}^{n} |x_k y_k| \le \left(\sum_{k=1}^{n} |x_k|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} |y_k|^2\right)^{\frac{1}{2}}.$$

Proof. For $(x_1, \ldots, x_n))$, let us denote $\|x\| := \left(\sum_{k=1}^{n} |x_k|^2\right)^{\frac{1}{2}}$. Then to prove that

$$\sum_{k=1}^{n} |x_k y_k| \le \|x\| \, \|y\|. \tag{4.1}$$

Clearly, (4.1) holds if one of $\|x\|$ and $\|y\|$ is 0. So, assume that $\|x\| \neq 0 \neq \|y\|$. First we observe that for any $a, b \in \mathbb{R}$, $|ab| \le (a^2 + b^2)/2$. Hence, taking $a_k = |x_k|/\|x\|$ and $b_k = |y_k|/\|y\|$, we obtain

$$\sum_{k=1}^{n} \frac{|x_k y_k|}{\|x\| \, \|y\|} = \sum_{k=1}^{n} |a_k b_k| \le \frac{1}{2} \sum_{k=1}^{n} (a_k^2 + b_k^2) = 1.$$

□

From this we deduce:

THEOREM 26. (Triangle inequality) *For* $(x_1, \ldots, x_n))$ *and* $(y_1, \ldots, y_n)$ *in* $\mathbb{K}^n$,

$$\left(\sum_{k=1}^{n} |x_k + y_k|^2\right)^{\frac{1}{2}} \le \left(\sum_{k=1}^{n} |x_k|^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{n} |y_k|^2\right)^{\frac{1}{2}}.$$

Proof. For $(x_1, \ldots, x_n))$, let us denote $\|x\| := \left(\sum_{k=1}^{n} |x_k|^2\right)^{\frac{1}{2}}$. Then to prove that

$$\|x + y\| \le \|x\| + \|y\|.$$

Note that, using Cauchy-Schwarz inequality,

$$\begin{aligned}
\|x+y\|^2 &= \sum_{k=1}^{n}|x_k+y_k|^2 = \sum_{k=1}^{n}(|x_k|^2+|y_k|^2+2|x_ky_k| \\
&= \|x\|^2+\|y\|+2\sum_{k=1}^{n}|x_ky_k| = \|x\|^2+\|y\|+2\|x\|\,\|y\| \\
&= (\|x\|+\|y\|)^2.
\end{aligned}$$

Thus, $\|x+y\| \leq \|x\|+\|y\|$. □

Example 27. For $(x_1,\ldots,x_n)$ and $(y_1,\ldots,y_n)$ in $\mathbb{K}^n$, let

$$d(x,y) := \left(\sum_{k=1}^{n}|x_k-y_k|^2\right)^{\frac{1}{2}}.$$

The $(\mathbb{K}^n,d)$ is a metric space: First three conditions hold easily. To see the third condition, let $z=(z_1,\ldots,z_n)\in\mathbb{K}^n$. Then, by triangle inequality,

$$d(x,y)=\|x-y\|=\|(x-z)+(z-y)\|\leq\|x-z\|+\|z-y\|=d(x,z)+d(z,y).$$

♢

Example 28. Let $C[a,b]$ be the set of all real valued continuous functions defined on $[a,b]$. Recall that every $x\in C[a,b]$ is a bounded function and it attains a maximum at some point in $[a,b]$. For $x,y\in C[a,b]$, let

$$d(x,y) := \max_{a\leq t\leq b}|x(t)-y(t)|,$$

$$d_1(x,y) := \int_a^b |x(t)-y(t)|dt,$$

$$d_2(x,y) := \left(\int_a^b |x(t)-y(t)|^2dt\right)^{\frac{1}{2}}.$$

Then d,d_1,d_2 are metrics on $C[a,b]$: It can be easily verified that d,d_1 are metrics. The fact that d_2 is also a metric will follow from the Triangle inequality proved below. ♢

For obtaining Triangle inequality, one may use *Cauchy-Schwarz inequality.*

THEOREM 29. (Cauchy-Schwarz inequality) *For x,y in $C[a,b]$,*

$$\int_a^b |x(t)y(t)|\,dt \leq \left(\int_a^b |x(t)|^2dt\right)^{\frac{1}{2}}\left(\int_a^b |y(t)|^2dt\right)^{\frac{1}{2}}.$$

Proof. Use the arguments as in Theorem 25. □

THEOREM 30. (Triangle inequality) *For x,y in $C[a,b]$,*

$$\Big(\int_a^b |x(t)+y(t)|^2\,dt\Big)^{\frac{1}{2}} \leq \Big(\int_a^b |x(t)|^2dt\Big)^{\frac{1}{2}}+\Big(\int_a^b |y(t)|^2dt\Big)^{\frac{1}{2}}.$$

Proof. Use the arguments as in Theorem 26. □

From the above theorem, we deduce:

Example 31. For a non-empty subset, let $B(S)$ be the set of all real valued bounded functions defined on S. For x, y in $B(S)$, let

$$d(x,y) := \sup\{|x(s) - y(s)| : s \in S\}.$$

Then d is a metric on $B(S)$ (Exercise).

Taking $S = \mathbb{N}$, we see that $B(\mathbb{N})$ is the set of all bounded sequences. This metric space $B(\mathbb{N})$ is usually denoted by ℓ^∞. ♢

Example 32. Let

$$\ell^1(\mathbb{N}) := \{(a_n) \in \ell^\infty(\mathbb{N}) : \sum_{k=1}^{\infty} |a_k| \text{ converges}\},$$

$$\ell^2(\mathbb{N}) := \{(a_n) \in \ell^\infty(\mathbb{N}) : \sum_{k=1}^{\infty} |a_k|^2 \text{ converges}\}.$$

For $x, y \in \ell^1(\mathbb{N})$, let

$$d_1(x,y) := \sum_{k=1}^{\infty} |x_n - y_n|,$$

and for $x, y \in \ell^2(\mathbb{N})$, let

$$d_2(x,y) := \left(\sum_{k=1}^{\infty} |x_n - y_n|^2\right)^{\frac{1}{2}}.$$

Then $(\ell^1(\mathbb{N}), d_1)$ and $(\ell^2(\mathbb{N}), d_2)$ are metric spaces:

(i) Let $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots)$ and $z = (z_1, z_2, \ldots)$ be in ℓ^1. We know that, for every $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} |x_k - y_k| \le \sum_{k=1}^{n} (|x_k - z_k| + |z_k - y_k) \le \sum_{k=1}^{\infty} |x_k - z_k| + \sum_{k=1}^{\infty} |z_k - y_k|.$$

Letting $n \to \infty$, we obtain $d_1(x,y) \le d_1(x,z) + d_1(z,y)$. Other axioms for $d_1(\cdot,\cdot)$ to be a metric can be easily verified.

(ii) Let $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots)$ and $z = (z_1, z_2, \ldots)$ be in ℓ^2. Since $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n)$ and $(z_1, \ldots, z_n)$ are in $\mathbb{R}^n$, by the triangle inequality in Example 27, we have

$$\begin{aligned} \Big(\sum_{k=1}^{n} |x_k - y_k|^2\Big)^{\frac{1}{2}} &\le \Big(\sum_{k=1}^{n} (|x_k - z_k|^2\Big)^{\frac{1}{2}} + \Big(\sum_{k=1}^{n} |z_k - y_k|^2\Big)^{\frac{1}{2}} \\ &\le \Big(\sum_{k=1}^{\infty} (|x_k - z_k|^2\Big)^{\frac{1}{2}} + \Big(\sum_{k=1}^{\infty} |z_k - y_k|^2\Big)^{\frac{1}{2}}. \end{aligned}$$

This is true for all $n \in \mathbb{N}$. Consequently,

$$\Big(\sum_{k=1}^{\infty} |x_k - y_k|^2\Big)^{\frac{1}{2}} \le \Big(\sum_{k=1}^{\infty} (|x_k - z_k|^2\Big)^{\frac{1}{2}} + \Big(\sum_{k=1}^{\infty} |z_k - y_k|^2\Big)^{\frac{1}{2}}.$$

Thus, we have proved $d_2(x,y) \le d_2(x,z) + d_2(z,y)$. Other axioms for $d_2(\cdot,\cdot)$ to be a metric can be easily verified. ♢

Definition 33. Let X be a vector space over $\mathbb{R}$. A map $x \mapsto \|x\|$ from X to the set of all non-negative real numbers is called a **norm** on X if

1. for every $x \in X$, $\|x\| = 0 \iff x = 0$,
2. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$, and
3. $\|\alpha x\| = |\alpha| \, \|x\|$ for all $x \in X$ and $\alpha \in \mathbb{R}$.

A vector space together with a norm is called a **normed linear space**. ♢

The following result follows easily from the above definition.

THEOREM 34. *If X is a vector space with a norm $\| \cdot \|$, then*

$$d(x, y) := \|x - y\|$$

defines a metric on X, called the **metric induced by the norm**.

Example 35. (i) On the vector space $\mathbb{R}^n$

$$x := (x_1, \ldots, x_n) \mapsto \|x\|_1 := \sum_{i=1}^{n} |x_i|,$$

$$x := (x_1, \ldots, x_n) \mapsto \|x\|_2 := \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2},$$

$$x := (x_1, \ldots, x_n) \mapsto \|x\|_\infty := \max\{|x_i| : i = 1, \ldots, n\}$$

are norms.

(ii) For $x := (x_1, x_2, \ldots) \in \ell^1(\mathbb{N})$, let

$$\|x\|_1 := \sum_{i=1}^{\infty} |x_i|.$$

Then $\ell^1(\mathbb{N})$ is a vector space and $x \mapsto \|x\|_1$ is a norm on $\ell^1(\mathbb{N})$.

(iii) For $x := (x_1, x_2, \ldots) \in \ell^\infty(\mathbb{N})$, let

$$\|x\|_1 := \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{1/2}.$$

Then Then $\ell^2(\mathbb{N})$ is a vector space and $x \mapsto \|x\|_2$ is a norm on $\ell^1(\mathbb{N})$.

(iv) For a nonempty set S, $B(S)$ is a vector space and $x \mapsto \|x\| := \sup\{|x(s)| : s \in S\}$ is a norm on $B(S)$. ♢

4.2 Topological concepts

Definition 36. A sequence (x_n) in a metric space (X, d) is said to be **convergent** if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$, and in that case we say that (x_n) **converges to** x, and write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$, or simply $x_n \to x$.. $\diamondsuit$

- A sequence (x_n) in a metric space (X, d) converges to $x \in X$ iff for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that
$$d(x_n, x) < \varepsilon \quad \forall\, n \geq n_0.$$

Definition 37. Let (X, d) be a metric space. For $x \in X$ and $r > 0$, the set

$$B(x, r) := \{y \in X : d(y, x) < r\}$$

is called an **open ball** with centre x and radius $r > 0$. $\diamondsuit$

- A sequence (x_n) in a metric space (X, d) converges to $x \in X$ iff for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that
$$x_n \in B(x, \varepsilon) \quad \forall\, n \geq n_0.$$

Exercise 38. Let (x_n) be a sequence in a metric space X. If (x_n) converges in X, then the limit is unique. That is, If $x_n \to x$ and $x_n \to y$ for some $x, y \in X$, then $x = y$. $\diamondsuit$

Exercise 39. Let X be a metric space, (x_n) be a sequence in X and $x \in X$.

(i) $x_n \to x$ iff every subsequence of (x_n) converges to x.

(ii) If (x_n) is a sequence, then $x_n \to x$ iff (x_n) has a subsequence which converges to x. $\diamondsuit$

Definition 40. Let (X, d) be a metric space, $A \subseteq X$ and $x \in X$.

(i) x is called an **interior point** of A if there exists $r > 0$ such that $B(x, r) \subseteq A$.

(ii) x is called a **closure point** of A if for every $r > 0$, $B(x, r) \cap A \neq \emptyset$.

(iii) x is called a **limit point** of A if for every $r > 0$, $B(x, r) \cap (A \setminus \{x\}) \neq \emptyset$.

(iv) x is called a **boundary point** of A if for every $r > 0$, $B(x, r) \cap A \neq \emptyset$ and $B(x, r) \cap A^c \neq \emptyset$. $\diamondsuit$

Definition 41. Let (X, d) be a metric space and $A \subseteq X$.

(i) The set of all interior points of A is called the **interior** of A, and it is denoted by A^o or $int(A)$ or $int_X(A)$.

(ii) The set of all closure points of A is called the **closure** of A, and it is denoted by $\bar{A}$ or $cl(A)$ or $cl_X(A)$.

(iii) The set of all limit points of A is denoted by A'.

(iv) A is called a **perfect set** if $A = A'$. $\diamondsuit$

Definition 42. Let (X, d) be a metric space and $A \subseteq X$.

(i) A is called an **open set** if every point in A is an interior point of A.

(ii) A is called a **closed set** if A contains all its limit points.

(iii) A together with all its closure points is called the **closure** of A, and it is denoted by $\bar{A}$.

(iv) The set of all boundary points of A is called the **boundary** of A, and it is denoted by $\partial(A)$.
$\diamondsuit$

THEOREM 43. *Let (X, d) be a metric space and $A \subseteq X$. The following are equivalent:*

(i) A is closed.

(ii) A contains all its closure points.

(iii) A contains all its limit points.

(iv) A contains all its boundary points.

(v) A^c is open.

Proof. The implications (i) $\Rightarrow$ (ii)$\Rightarrow$ (iii) $\Rightarrow$ (i), and (ii) $\iff$ (iv) follow from the definitions (See exercise below). Now, we prove (i) $\iff$ (v):

Suppose A^c is not open. Then there exists $x \in A^c$ such that for every $r > 0$, $B(x, r)$ contains some point of A. Then x is a limit point of A - a contradiction to the fact that A contains all its limit points. Thus (i) $\Rightarrow$ (v). Conversely, suppose A^c is open, and x is a limit point of A. If x does not belong to A, then $x \in A^c$, and since A^c is open there exists $r > 0$ such that $B(x, r) \subseteq A^c$, i.e., $B(x, r) \cap A = \varnothing$ - a contradiction to the assumption that x is a limit point of A. Thus, (v) $\Rightarrow$ (1). $\square$

Exercise 44. Let (X, d) be a metric space and $A \subseteq X$.

(i) $\bar{A} = A \cup \partial A = A \cup A'$

(ii) A is open iff $A = A^o$.

(iii) A is closed iff $A = \bar{A}$. $\diamondsuit$

Exercise 45. Let (X, d) be a metric space and $A \subseteq X$.

(i) A^o is an open set.

(ii) A^o is the union of all open sets contained in A, that is,

$$A^o = \bigcup \{G \subseteq X : G \text{ open and } G \subseteq A\}.$$

(ii) $\bar{A}$ is the intersection of all open sets containing A, that is,

$$\bar{A} = \bigcap \{G \subseteq X : G \text{ open and } G \subseteq A\}.$$

$\diamondsuit$

Exercise 46. Let (X, d) be a metric space and $A \subseteq X$.

(i) $x \in \bar{A} \iff \exists (x_n)$ in A such that $x_n \to x$.

(ii) $x \in A' \iff \forall\, r > 0$, $B(x, r) \cap A$ is an infinite set. $\diamondsuit$

Exercise 47. Every finite subset of a metric space is closed, and it does not have any limit point. $\diamondsuit$

Example 48. Let $X = \mathbb{R}$ with usual metric $d(x, y) := |x - y|$, $x, y \in \mathbb{R}$. Let $A = (0, 1]$. Then we have the following (verify):

(i) $A^o = (0, 1)$ and $\bar{A} = [0, 1]$.

(ii) A is neither open nor closed.

(iii) $\partial A = \{0, 1\}$. $\diamondsuit$

Example 49. Let $X = \mathbb{R}$ with usual metric $d(x, y) := |x - y|$, $x, y \in \mathbb{R}$. Let $A = \mathbb{Z}$. Then we have the following (verify):

(i) $A^o = \varnothing$ and $\bar{A} = \mathbb{Z}$.

(ii) A is a closed set, not open.

(iii) $\partial A = \mathbb{Z}$. $\diamondsuit$

Example 50. Let X be with discrete metric and $A \subseteq X$. Then we have the following (verify):

(i) $A^o = A = \bar{A}$.

(ii) A is open and closed.

(iii) $\partial A = \varnothing$.

(iv) Every subset of X is open and closed.

(v) A sequence in X is a Cauchy sequence iff it is eventually constant. $\diamondsuit$

Example 51. Let $X = \mathbb{R}$ be with usual (Euclidian) metric and $A = (0, 1) \times \{0\}$. Then we have the following (verify):

(i) $A^o = \varnothing, \quad \bar{A} = [0, 1] \times \{0\}$.

(ii) A is closed but not open.

(iii) $\partial A = [0, 1] \times \{0\}$. $\diamondsuit$

Exercise 52. Let (X, d) be a metric space.

(i) Every open ball in X is an open set.

(ii) Union of every arbitrary collection of open sets in X is open in X.

(iii) Finite intersection of opens sets in X is open in X.

(iv) Intersection of every arbitrary collection of closed sets in X is closed in X.

(v) Finite union of of closed sets in X is closed in X. $\diamondsuit$

THEOREM 53. *If Y is a subset of a metric space X with metric d, then $(x, y) \mapsto d(x, y)$ defines a metric on Y.*

Proof. Exercise. $\square$

Definition 54. The metric on Y as in the above theorem is called the **metric induced by** d. $\diamondsuit$

Let (X, d) be a metric space and $Y \subseteq X$. Let $A \subseteq Y$. Then, as per the above definition, A is open in Y (w.r.t. the induced metric) iff for each $x \in A$, there exists $r > 0$ such that

$$B_Y(x, r) := \{y \in Y : d(x, y) < r\} \subseteq A.$$

THEOREM 55. *Let (X, d) be a metric space and $Y \subseteq X$. Let $A \subseteq Y$. Then A is open in Y iff there exists an open set G in X such that $A = G \cap Y$.*

Proof. Let A be open in Y. Then, for every $x \in A$, there exists $r_x > 0$ such that $B_Y(x, r_x) \subseteq A$. Hence, $A = \cup_{x \in A} B_Y(x, r_x)$. But, $B_Y(x, r_x) = B_X(x, r_x) \cap Y$. Hence,

$$A = \cup_{x \in A} B_Y(x, r_x) = (\cup_{x \in A} B_X(x, r_x)) \cap Y.$$

Thus, taking $G = \cup_{x \in A} B_X(x, r_x)$, we obtain $A = G \cap Y$.

Conversely, let $A = G \cap Y$ for some open set G in X. Let $x \in A$. Since B is open in X, there exists $r_x > 0$ such that $B_X(x, r_x) \subseteq G$. Hence, $B_Y(x, r_x) = B_X(x, r_x) \cap Y \subseteq G \cap Y = A$. □

Exercise 56. Let (X, d) be a metric space and $Y \subseteq X$. Let $A \subseteq Y$. Then the following are true.
(i) If A is open in X, then A is open in Y.
(ii), Converse of (i) holds if Y is open in X.
(iii), Converse of (i) need not hold if Y is not open in X. ♢

THEOREM 57. *Let (X, d) be a metric space and $Y \subseteq X$. Let $B \subseteq Y$. Then B is closed in Y iff there exists a closed set F in X such that $B = F \cap Y$.*

Proof. Let B be open in Y. Then, $Y \setminus B$ is open in Y. Hence, there exists an open set G in X such that $Y \setminus B = G \cap Y$, that is, $Y \cap B^c = G \cap Y$. This implies, $(Y \cap B^c)^c = (G \cap Y)^c$, that is, $Y^c \cup B = G^c \cup Y^c$, so that $Y \cap (Y^c \cup B) = Y \cap (G^c \cup Y^c)$, that is, $(Y \cap Y^c) \cup (Y \cap B) = (Y \cap G^c) \cup (Y \cap Y^c)$, that is, $B = Y \cap G^c$. Take $F = G^c$.

Conversely, suppose F is a closed set in X such that $B = F \cap Y$. Then,

$$Y \setminus B = Y \cap B^c = Y \cap (F \cap Y)^c = Y \cap (F^c \cup Y^c) = (Y \cap F^c) \cup (Y \cap Y^c) = Y \cap F^c.$$

Taking $G = F^c$, we see that $Y \setminus B$ is open in Y; hence B is closed in Y. □

Exercise 58. Let (X, d) be a metric space and $Y \subseteq X$. Let $B \subseteq Y$. Then the following are true.
(i) If B is closed in X, then B is closed in Y.
(ii), Converse holds if Y is closed in X.
(iii), Converse of (i) need not hold if Y is not closed in X. ♢

Exercise 59. Let $X = \mathbb{R}$ with usual metric.

(i) If $Y = \mathbb{Z}$, what are the open subsets of Y?

(ii) If $Y = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$, then what are the open subsets of Y?

(iii) If $Y = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$, then what are the open subsets of Y? Is $\{0\}$ open in Y? ♢

Exercise 60. Let X be a metric space and $Y \subseteq X$. If $E \subseteq Y$, then $cl_Y(E) \subseteq cl_X(E)$; and equality need not hold. ♢

Definition 61. Let X be a set and $\mathcal{T}$ be a family of subsets of X. Then $\mathcal{T}$ is called a **topology** on X if the following conditions are satisfied:

(i) $X \in \mathcal{T}$, $\varnothing \in \mathcal{T}$.

(ii) Union of every sub-collection of members of $\mathcal{T}$ is a member of $\mathcal{T}$.

(iii) Intersection of every finite sub-collection of members of $\mathcal{T}$ is a member of $\mathcal{T}$.

The pair $(X, \mathcal{T})$ is called a **topological space**. Members of $\mathcal{T}$ are called **open sets** in X and compliments of open sets are called **closed sets**. ♢

Example 62. Let X be a metric space. Then the family of all open sets w.r.t. the metric on X form a topology on X, and the concept of open sets and closed sets in the setting of metric space coincides with the corresponding concepts in the setting of topological space. $\diamondsuit$

Definition 63. Let d and ρ be metrics on a set X. Then d and ρ are said to be **equivalent** if there exists $C_1 > 0$, $C_2 > 0$ such that

$$C_1 d_1(x, y) \leq d_2(x, y) \leq C_2 d_1(x, y) \quad \forall\, x, y \in X.$$

$\diamondsuit$

Exercise 64. Suppose d and ρ are equivalent metric on X. Then (x_n) converges to x w.r.t. d iff (x_n) converges to x w.r.t. ρ; $\diamondsuit$

Exercise 65. For $x, y \in \mathbb{R}^k$, let $d_\infty(x, y) := \max\{|x_k - y_k| : k = 1, \ldots, k\}$. Then for every p with $1 \leq p < \infty$, d_p and d_∞ are equivalent metrics on $\mathbb{R}^k$. Further, for any p, r with $1 \leq p \leq \infty$ and $1 \leq r \leq \infty$, the metrics d_p and d_ρ on $\mathbb{R}^n$ are equivalent. $\diamondsuit$

Exercise 66. Let $X = C[a, b]$. Show that the metrics

$$d_1(x, y) := \int_a^b |x(t) - y(t)|dt, \quad d_\infty(x, y) = \sup_{a \leq t \leq b} |x(t) - y(t)|$$

are not equivalent. $\diamondsuit$

Definition 67. Consider a metric space (X, d).

(i) A subset D of X is said to be **dense** in X if $\bar{D} = X$.

(ii) X is said to be **separable** if it has a countable dense subset.

(iii) A set $E \subseteq X$ is said to be **nowhere dense** if $(\bar{A})^0 = \varnothing$. $\diamondsuit$

- A metric space (X, d) is separable iff there exists a countable set D such that for every $x \in X$ and for every $\varepsilon > 0$, there exists $y \in D$ such that $d(x, y) < \varepsilon$.

Example 68. Let $\mathbb{R}$ be with usual metric.

(i) $\mathbb{Q}$ and $\mathbb{Q}^c := \mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$.

(ii) $\mathbb{R}$ is separable.

(iii) $\mathbb{R} \setminus \mathbb{Z}$ is dense in $\mathbb{R}$.

(iv) $\mathbb{Z}$ is nowhere dense in $\mathbb{R}$. $\diamondsuit$

- Is $\mathbb{Q}^c$ separable?

Yes. To see this consider the set

$$D := \{r + \sqrt{2} : r \in \mathbb{Q}\}.$$

Then D is a countable dense subset of $\mathbb{Q}^c$: Clearly, D is countable. To see that it is dense in $\mathbb{Q}^c$, let $x \in \mathbb{Q}^c$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, for every $\varepsilon > 0$, there exists $r \in \mathbb{Q}$ such that $|(x - \sqrt{2}) - r| < \varepsilon$. Hence, $|x - (r + \sqrt{2})| < \varepsilon$. Thus, D is dense in $\mathbb{Q}^c$.

More generally, we have the following.

THEOREM 69. *Every subset of a separable metric space is separable.*

Proof. Let (X, d) be a separable metric space and Y be a subset of X. Let $E := \{x_1, x_2, \cdots\}$ be a countable dense subset of X. First we observe that, due to denseness of E in X, for each $k \in \mathbb{N}$, $X = \bigcup_{n=1}^{\infty} B(x_n, \frac{1}{k})$. Let $S := \{(k, n) \in \mathbb{N} \times \mathbb{N} : B(x_n, \frac{1}{k}) \cap Y \neq \varnothing\}$. For such $(k, n) \in S$, let $y_{k,n} \in B(x_n, \frac{1}{k}) \cap Y$. Clearly, $D := \{y_{k,n} : k, n \in \mathbb{N}\}$ is countable. We show that D is dense in Y. For this let $x \in Y$. Then, for each $k \in N$, there exists $n \in N$ such that $d(x, x_n) < 1$. Hence,

$$d(x, y_{k,n}) \leq d(x, x_n) + d(x_n, y_{k,n}) < \frac{1}{k} + \frac{1}{k} = \frac{2}{k}.$$

Now, let $\varepsilon > 0$ be given. Let $k \in \mathbb{N}$ be such that $\frac{2}{k} < \varepsilon$. Let n be as above. Then $d(x, y_{k,n}) < \frac{2}{k} < \varepsilon$.

Thus, we have shown that for every $\varepsilon > 0$, there exists $y \in D$ such that $d(x, y) < \varepsilon$. Consequently, D is dense in Y . □

5 Completeness

Definition 70. Let (X, d) be a metric space and (x_n) be a sequence in X. We say that (x_n) is a **Cauchy sequence** if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon \quad \forall n, m \geq N$. ♢

The following theorem is easy to prove.

THEOREM 71. *If a Cauchy sequence (x_n) in a metric space X has a subsequence which converges to a point $x \in X$, then $x_n \to x$.*

Definition 72. A metric space X is said to be a **complete metric space** if every Cauchy sequence in X converges to some point in X. ♢

Before giving some examples, let us observe the following result.

THEOREM 73. *If (x_n) and (y_n) are Cauchy sequences in a metric space (X, d), then the sequence $(d(x_n, y_n))$ converges.*

Proof. Note that

$$d(x_n, y_n) \leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n)$$

so that

$$d(x_n, y_n) - d(x_m, y_m) \leq d(x_n, x_m) + d(y_m, y_n).$$

Since (x_n) and (y_n) are Cauchy sequences, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon/2$ and $d(y_m, y_n) < \varepsilon/2$ for all $n, m \geq N$. Hence,

$$d(x_n, y_n) - d(x_m, y_m) \leq d(x_n, x_m) + d(y_m, y_n) < \varepsilon \quad \forall n, m \geq N.$$

Similarly, $d(x_m, y_m) - d(x_n, y_n) < \varepsilon$ for all $n, m \geq N$. Hence,

$$|d(x_n, y_n) - d(x_m, y_m)| < \varepsilon \quad \forall n, m \geq N.$$

Thus, $(d(x_n, y_n))$ is a cauchy sequence of real numbers. Since R is complete, $(d(x_n, y_n))$ converges. □

COROLLARY 74. *If (x_n) is a Cauchy sequence in a metric space (X, d), then for every $x \in X$, the sequence $(d(x_n, x))$ converges.*

Example 75. (i) $\mathbb{R}$ with $d(x,y) = |x-y|$ is complete.

(ii) $\mathbb{Q}$ with $(x,y) \mapsto |x-y|$ is not complete.

(iii) $\mathbb{R}^2$ is complete with the metrics $d_1(x,y) := |x_1 - y_1| + |x_2 - y_2|$ and $d_\infty(x,y) := \max\{|x_1 - y_1|, |x_2 - y_2|\}$.

(iv) $(0,1]$, $[0,1)$, $(0,1)$ with $d(x,y) = |x-y|$ are not complete. ♢

Exercise 76. Let X be with discrete metric.

(i) Every Cauchy sequence in X is eventually constant.

(ii) X is complete. ♢

Exercise 77. Suppose d and ρ are equivalent metric on X, and (x_n) is a sequence in X. Then (x_n) is a Cauchy sequence w.r.t. d iff (x_n) (x_n) is a Cauchy sequence w.r.t. ρ. ♢

THEOREM 78. *If d and ρ are equivalent metrics on a set X and if X is complete w.r.t. d, then it is complete w.r.t. ρ.*

Proof. Suppose d and ρ be equivalent metrics on X. Let $c_1, c_2 > 0$ such that

$$c_1 d(x,y) \le \rho(x,y) \le c_2 d(x,y) \quad \forall\, x, y \in X. \tag{$*$}$$

Suppose X is complete w.r.t. d. We show that X is complete w.r.t. ρ. For this, let (x_n) be a Cauchy sequence w.r.t. ρ. Then, by $(*)$, we see that (x_n) be a Cauchy sequence w.r.t. d. Hence, there exists $x \in X$ such that $d(x_n, x) \to 0$. Hence, again by $(*)$, $\rho(x_n, x) \to 0$.. □

Example 79. The metric space $(\mathbb{R}^k, d_\infty)$ is complete. Further, $\mathbb{R}^k$ with d_p is complete for any p with $1 \le p < \infty$. ♢

The above example is a a particular case of the following.

THEOREM 80. *Let S be a nonempty set. Then $B(S)$ is complete w.r.t. the sup-metric.*

Proof. Let (x_n) be a Cauchy sequence in $B(S)$. For $\varepsilon > 0$, let $N \in \mathbb{N}$ be such that

$$\sup_{t \in S} |x_n(t) - x_m(t)| = d_\infty(x_n, x_m) < \varepsilon \quad \forall\, n, m \ge N. \tag{$*$}$$

Hence, for each $t \in S$,

$$|x_n(t) - x_m(t)| = d_\infty(x_n, x_m) < \varepsilon \quad \forall\, n, m \ge N$$

so that $(x_n(t))$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $\lim n \to \infty x_n(t)$ exists for each $t \in S$. Let $x(t) := \lim\limits_{n\to\infty} x_n(t)$, $t \in S$. Note that

$$|x_n(t)| \le |x_n(t) - x_{n_0}(t)| + |x_{n_0}(t)| \le d_\infty(x_n, x_{n_0}) + \sup_{t\in S} |x_{n_0}(t)| \le 1 + \sup_{t \in S} |x_N(t)| = M,$$

where $n_0 \in \mathbb{N}$ is such that $d_\infty(x_n, x_m) < 1 \quad \forall\, n, m \ge n_0$. Hence, $|x(t)| = \lim\limits_{n\to\infty} |x_n(t)| \le M$ for all $t \in S$ so that $x \in B(S)$. Now, from $(*)$,

$$|x_n(t) - x_m(t)| = \lim_{m\to\infty} |x_n(t) - x_m(t)| \le \varepsilon \quad \forall\, n \ge N.$$

This is true for all $t \in S$. Hence, $d_\infty(x_n, x) \le \varepsilon$ for all $n \ge N$. Thus, we have shown that (x_n) converges to $x \in B(S)$. □

Example 81. As special cases of the above theorem, we have the following:

(i) Taking $S = \{1, \ldots, k\}$, we have $B(S) = \mathbb{R}^k$.

(ii) Taking $S = \mathbb{N}$, we have $B(S) = \ell^\infty$, the set set of all bounded sequences of real numbers. $\diamondsuit$

Definition 82. A normed linear space which is complete w.r.t. the metric induced by the norm is called a **Banach space**. $\diamondsuit$

THEOREM 83.

THEOREM 84. *The following hold.*

(i) *Every closed subset of a complete metric space is complete.*

(ii) *Every complete subset of a metric space is closed.*

Proof. Let X be a metric space with metric $d(\cdot,\cdot)$, and let $E \subseteq X$.

(i) Suppose X is complete and E is closed. Let (x_n) be a Cauchy sequence in E. Then (x_n) is a Cauchy sequence in E. Since X is complete, there exists $x \in X$ such that $x_n \to x$. Since E is closed, $x \in \bar{E} = E$.

(ii) Suppose E is a complete subset of X. Let $x \in \bar{E}$. Then there exists a sequence (x_n) in E such that $x_n \to x$. Hence, (x_n) is a Cauchy sequence in E. Since E is complete, $x \in E$. Thus, $\bar{E} = E$. $\square$

COROLLARY 85. *If $Y \subseteq X$ is not closed in X, then Y is not complete.*

THEOREM 86. *Let Ω be a closed and bounded subset of $\mathbb{R}$ and $C(\Omega)$ be the set of all real (or complex) valued continuous functions defined on Ω. Then $C(\Omega)$ is a closed subset of $B(\Omega)$ w.r.t. the sup-norm. In particular, $C(\Omega)$ is a complete metric space w.r.t. the sup-norm.*

Proof. Let $d(x,y) := \sup_{t\in\Omega} |x(t) - y(t)|$ for $x, y \in B(\Omega)$. Clearly $C(\Omega) \subseteq B(\Omega)$. Let (x_n) be a sequence in $C(\Omega)$ such that $d(x_n, x) \to 0$ for some $x \in B(\Omega)$. To prove that $x \in C(\Omega)$. For this, let $t_0 \in \Omega$ and let $\varepsilon > 0$ be given. Note that, for any $t \in \Omega$,

$$|x(t) - x(t_0)| \le |x(t) - x_n(t)| + |x_n(t) - x_n(t_0)| + |x_n(t_0) - x(t_0)|.$$

Hence,

$$|x(t) - x(t_0)| \le d(x, x_n) + |x_n(t) - x_n(t_0)| + d(x_n, x). \qquad (*)$$

Let $N \in \mathbb{N}$ be such that $d(x_n, x) < \varepsilon/3$ for all $n \ge N$, and let $\delta > 0$ be such that $|x_n(t) - x_n(t_0)| < \varepsilon/3$ whenever $t \in \Omega$ and $|t - t_0| < \delta$. Hence, from $(*)$, we obtain

$$\begin{aligned} |x(t) - x(t_0)| &\le d(x, x_N) + |x_N(t) - x_N(t_0)| + d(x_N, x) \\ &< \varepsilon \quad \text{whenever} \quad t \in \Omega,\ |t - t_0| < \delta. \end{aligned}$$

Thus, $x \in C(\Omega)$. $\square$

Example 87. $C[a,b]$ is not complete w.r.t the metric $d_1(x,y) := \int_a^b |x(t) - y(t)|dt$:

Let $a < c < b$, and let $x_n(t) = \begin{cases} 0, & a \le t < c - \frac{1}{n}, \\ 1 + n(t-c), & c - \frac{1}{n} \le t < c, \\ 1, & c \le t \le b, \end{cases}$ Also, let $x(t) = \begin{cases} 0, & a \le t < c, \\ 1, & c \le t \le b. \end{cases}$

Then it can be seen that

$$\int_a^b |x_n(t) - x(t)|dt = \int_{c-\frac{1}{n}}^c |x_n(t) - x(t)|dt = \frac{1}{2n}.$$

Hence,

$$d_1(x_n, x_m) = \int_a^b |x_n(t) - x_m(t)|dt \le \int_a^b |x_n(t) - x(t)|dt + \int_a^b |x(t) - x_m(t)|dt \le \frac{1}{2n} + \frac{1}{2m}.$$

Thus, (x_n) is a Cauchy sequence w.r.t. d_1. However, it does not converge to any function in $C[a,b]$. [You cannot conclude this by stating that "(x_n) converges to x but x is not continuous at c." – Why?]

To see this, suppose that $d_1(x_n, y) \to 0$ converges to some $y \in C[a,b]$. Then we have

$$\int_a^b |x(t) - y(t)|dt \le \int_a^b |x(t) - x_n(t)|dt + \int_a^b |x_n(t) - y(t)|dt.$$

$$\int_a^b |x(t) - x_n(t)|dt \le \frac{1}{2n} \to 0 \quad \text{and} \quad \int_a^b |x_n(t) - y(t)|dt = d_1(x_n, y) \to 0,$$

we obtain $\int_a^b |x(t) - y(t)|dt = 0$. Hence, $\int_a^c |x(t) - y(t)|dt = 0$ and $\int_c^b |x(t) - y(t)|dt = 0$. Since x is continuous on $[a,c)$ and on $(c,b]$, it follows that $y(t) = x(t) = \begin{cases} 0, & a \le t < c, \\ 1, & c < t \le b. \end{cases}$ This contradicts the continuity of y. In fact, there exists $t_n \in [a,c)$, $s_n \in (c,b]$ such that $t_n \to c$, $s_n \to c$, so that by continuity of y, $0 = y(t_n) \to y(c)$, $1 = y(s_n) \to y(c)$, which is a contradiction.

An alternative proof for the last part: Suppose that $d_1(x_n, y) \to 0$ converges to some $y \in C[a,b]$. Then

$$\begin{aligned} d_1(x_n, y) &= \int_a^{c-\frac{1}{n}} |x_n(t) - y(t)|dt + \int_{c-\frac{1}{n}}^c |y(t) - x_m(t)|dt + \int_c^b |x_n(t) - y(t)|dt \\ &= \int_a^{c-\frac{1}{n}} |y(t)|dt + \int_{c-\frac{1}{n}}^c |y(t) - x_m(t)|dt + \int_c^b |1 - y(t)|dt. \end{aligned}$$

Thus,

$$0 \le \int_a^{c-\frac{1}{n}} |y(t)|dt \le d_1(x_n, y), \quad 0 \le \int_c^b |1 - y(t)|dt \le d_1(x_n, y).$$

Since $d_1(x_n, y) \to 0$, we obtain

$$\int_a^c |y(t)|dt = 0 \quad \text{and} \quad \int_c^b |1 - y(t)|dt = 0.$$

Since y is continuous on $[a,c]$ and on $[c,b]$, we obtain $y(t) = 0$ for $a \le t \le c$ and $y(t) = 1$ for $c \le t \le b$. This contradicts the fact that y is continuous on $[a,b]$. ♢

Exercise 88. On the set $C[a,b]$, the metrics

$$d_1(x,y) := \int_a^b |x(t) - y(t)|dt, \quad d_\infty(x,y) := \sup_{a \le t \le b} |x(t) - y(t)|, \quad x, y \in C[a,b]$$

are not equivalent. ♢

Definition 89. Let X be a normed linear space with norm $\|\cdot\|$. Then the set

$$\{x \in X : \|x\| \le 1\}$$

is called the **closed unit ball** in X. ♢

THEOREM 90. *Let X be a normed linear space and let E be its closed unit ball. Then*

1. *E is closed in X,*
2. *E is complete $\iff$ X is complete.*

Proof. (i) let (x_n) be in E such that $x_n \to x$ for some $x \in X$. Then

$$\|x\| \leq \|x - x_n\| + \|x_n\| \leq \|x - x_n\| + 1 \quad \forall\, n \in \mathbb{N}.$$

Since $\|x - x_n\| \to 0$, it follows that $\|x\| \leq 1$. hence $x \in E$.

(ii) Suppose E is compete. Let (x_n) be a Cauchy sequence in X. Then there exists $M > 0$ such that $\|x_n\| \leq M$ for all $n \in \mathbb{N}$. Since $x_n/M \in E$ for all $n \in \mathbb{N}$ and since (x_n/M) is a Cauchy sequence, by completeness of E, there exists $u \in E$ such that $x_n/M \to u$. Hence, $x_n \to x := Mu$.

The reverse implication follows from (i) and Theorem 84. □

THEOREM 91. *Let (X, d) be a metric space. Then, for every Cauchy sequence (x_n) in X, there exists a metric space (Y, ρ) such that $X \subseteq Y$, d is the restriction of ρ and (x_n) converges in Y.*

Proof. Let (x_n) is a Cauchy sequence in X. Let $\tilde{x} := (x_n)$, $Y = X \cup \{\tilde{x}\}$ and

$$\tilde{\rho}(x, y) := \begin{cases} d(x, y), & x, y \in X, \\ 0, & x = y = \tilde{x}, \\ \lim\limits_{n\to\infty} d(x, x_n), & x \in X,\, y = \tilde{x}. \end{cases}.$$

Then ρ is a metric on Y and $\rho(x_n, \tilde{x}) \to 0$ as $n \to \infty$. [observe that for each $n \in \mathbb{N}$, $\rho(x_n, \tilde{x}) = \lim_{m\to\infty} d(x_n, x_m)$ so that $\rho(x_n, \tilde{x}) \to 0$ as $n \to \infty$.] □

THEOREM 92. *A metric space (X, d) is complete iff every sequence in X which converges in any super space of X also converges in X.*

Proof. Suppose (X, d) is complete and (x_n) is a sequence in X which converges to some x in a super space (Y, ρ). Since X is closed in Y, $x \in X$.

Conversely, suppose that every sequence in X which converges in any super space of X also converges in X. Let (x_n) is a Cauchy sequence in X. Take (Y, ρ) and $\tilde{x}$ as in Theorem 91. Then $\rho(x_n, \tilde{x}) \to 0$. By hypothesis, (x_n) converges in X. □

COROLLARY 93. *A metric space (X, d) is not complete iff there exists a sequence (x_n) and a super space Y of X such that (x_n) converges in Y, but not in X.*

THEOREM 94. (Completion) *Let (X, d) be a metric space. Then there exists a complete metric $(\tilde{X}, \rho)$ and an isometry $f : X \to \tilde{X}$, i.e., $\rho(f(x), f(y)) = d(x, y)$ for all $x, y \in X$, such that the range of F is dense in $\tilde{X}$. Further, if (Y, η) is any compete metric space and $g : X \to Y$ is an isometry with range of g is dense in Y, then Y is isometric with $\tilde{X}$.*

Sketch of the proof. Let $\mathcal{X}$ be the set of Cauchy sequences in X. recall that if (x_n), (y_n) are in $\mathcal{X}$, then $(d(x_n, y_n))$ converges. Define, $(x_n) \sim (y_n)$ iff $d(x_n, y_n) \to 0$. Then, it can be see that $\sim$ is an equivalence relation on $\mathcal{X}$. Let $\tilde{X}$ be the set of all equivalence classes. For $\tilde{x} = [(x_n)]$, $\tilde{y} = [y_n)]$ in $\tilde{X}$, define

$$\varphi(\tilde{x}, \tilde{y}) := \lim_{n\to\infty} d(x_n, y_n).$$

It can be seen that ρ is well-defined on $\tilde{X} \times \tilde{X}$ and it is a metric on $\tilde{X}$. Now, define $f : X \to \tilde{X}$ by

$$f(x) = [(x, x, \ldots)], \quad x \in X.$$

Then the following can be verified (do it!):

1. ρ is a complete metric;
2. f is an isometry, i.e., $\varphi(f(x), f(y) = d(x, y) \,\forall\, x, y \in X$;
3. range of f is dense in $\tilde{X}$;
4. if (Y, η) is any compete metric space and $g : X \to Y$ is an isometry with range of g is dense in Y, then Y is isometric with $\tilde{X}$.

□

Definition 95. The metric space $(\tilde{X}, \rho)$, or any complete metric space (Y, η) as in Theorem 94 is called the *completion* of (X, d). ♢

Following theorem describes one of the completions.

THEOREM 96. *Let (X, d) be a metric space. Let $x_0 \in X$. For each $u \in X$, let $f_x : X \to \mathbb{R}$ be defined by*

$$f_u(x) := d(x, x_0) - d(x, u), \quad x \in X.$$

Then $f_u \in B(X)$ for every $u \in X$, and the map $T : X \to B(X)$ defined by

$$T(u) = f_u, \quad u \in X$$

is an isometry. In particular, $\overline{R(T)}$ is a completion of X.

Proof. Note that for every $u \in X$,

$$f_u(x) := d(x, x_0) - d(x, u) \le d(u, x_0) \quad \forall\, x \in X.$$

Hence, $f_u \in B(X)$ for every $u \in X$. . Also, for $u, v, x \in X$,

$$f_u(x) - f_v(x) = [d(x, x_0) - d(x, u)] - [d(x, x_0) - d(x, v)] = d(x, v) - d(x, u) \le d(u, v).$$

Hence,

$$d_\infty(f_u, f_v) := \sup_{x \in X} |f_u(x) - f_v(x)| \le d(u, v).$$

Further,

$$f_u(u) - f_v(u) = [d(u, x_0) - d(u, u)] - [d(u, x_0) - d(u, v)] = d(u, v).$$

Hence (why?)

$$d_\infty(f_u, f_v) = d(u, v).$$

□

Remark 97. In the above theorem, the function f_u is not only bounded, but it is also continuous. Thus, we have prove that *every metric space Ω is isometric with a subset of $C_b(\Omega)$.* ♢

Example 98. With respect to the usual metric,

(i) $\mathbb{R}$ is the completion of $\mathbb{Q}$ and $\mathbb{Q}^c$;

(ii) $[0,1]$ is the completion of $(0,1]$, $[0,1)$, $(0,1)$, $[0,1]\cap\mathbb{Q}$, $[0,1]\cap\mathbb{Q}^c$. ♢

We shall see that

- $C[a,b]$ with sup-metric is the completion of the set of all polynomial functions on $[a,b]$.

6 Compactness

Let (X,d) be a metric space. Before introducing the concept of *compactness*, let us observe the following property.

THEOREM 99. *For every distinct points $x,y\in X$, there exists disjoint open sets and containing x and y, respectively.*

Proof. Let $x,y\in X$ with $x\neq y$. Let $0<r<\frac{1}{2}d(x,y)$. Then $B(x,r)$ and $B(y,r)$ are disjoint open sets containing x and y, respectively: Clearly, $x\in B(x,r)$ and $y\in B(y,r)$. Further, if $z\in B(x,r)\cap B(y,r)$, then

$$d(x,y)\leq d(x,z)+d(z,y)<r+r=2r<d(x,y),$$

which is a contradiction. □

- The property of of metric space stated in the above theorem is called the *Hausdorff property.*
- A general topological space need not have this property: For example, let X be a set with *indiscrete topology* $\mathcal{T}:=\{\varnothing,\mathcal{X}\}$. If $\#(X)\geq 2$, then X is not Hausdorff.

Definition 100. A family $\{A_\alpha:\alpha\in\Lambda\}$ of subsets of a set X is said to be a **cover** of a set $E\subseteq X$ if $E\subseteq\bigcup_{\alpha\in\Lambda}A_\alpha$. A cover $\{A_\alpha:\alpha\in\Lambda\}$ of E is called an **open cover** if each A_α is an open set. ♢

Definition 101. A subset E of X said to be a *compact subset* of X or **compact in** X if for every open cover of E has a finite subcover; that is, for family $\{V_\alpha:\alpha\in\Lambda\}$ of open sets in X with $E\subseteq\bigcup_{\alpha\in\Lambda}V_\alpha$, there exist $\alpha_1,\ldots,\alpha_n$ in Λ such that $E\subseteq\bigcup_{i=1}^n V_{\alpha_i}$. ♢

Example 102. Let $X=\mathbb{R}$ with usual metric. Then the following sets are not compact. You may find an open cover in each case so that it does not have a finite subcover.

(i) $E=(0,1]$: Take $V_n=(\frac{1}{n},2)$, $n\in\mathbb{N}$.

(ii) $E=[0,1)$: Take $V_n=(-1,\frac{n}{n+1})$, $n\in\mathbb{N}$.

(iii) $E=(0,1)$: Take $V_n=(\frac{1}{n},\frac{n}{n+1})$, $n\in\mathbb{N}$.

(iv) $E=[0,\infty)$: Take $V_n=(-1,n)$, $n\in\mathbb{N}$. ♢

THEOREM 103. *Compact subset of a metric space is closed.*

Proof. Let E be a compact subset of X. It is enough to show that $E^c:=X\setminus E$ is open in X. So, let $x\in E^c$. Then for every $y\in E$, there exists disjoint open sets U_y and V_y containing x and y respectively. Then $\{V_y:y\in E\}$ is an open cover of E. Since E is compact, there exists $y_1,\ldots,y_n$ in E such that $E\subseteq V:=\bigcup_{i=1}^n V_{y_i}$ and $x\in U:=\bigcap_{i=1}^n U_{\alpha_i}$. Note that U is open and $U\cap V=\varnothing$ so that $U\cap E=\varnothing$. Thus, $U\subseteq E^c$. This proves that E^c is open. □

By this theorem we can assert that the sets $(0,1]$, $[0,1)$, $(0,1)$ are not compact in $\mathbb{R}$, without actually constructing an open cover in each case so that it does not have a finite subcover.

THEOREM 104. *Closed subsets of a compact metric space are compact.*

Proof. Let X be a compact metric space and E be a closed subset of X. Let $\{V_\alpha : \alpha \in \Lambda\}$ be an open cover of E. Then $\{V_\alpha : \alpha \in \Lambda\} \cup \{E^c\}$ is an open cover of X. Since X is compact, there exist $\alpha_1, \ldots, \alpha_n$ in Λ such that $X = \{E^c\} \cup (\bigcup_{i=1}^n V_{\alpha_i})$. Then $E = E \cap X = E \cap (\bigcup_{i=1}^n V_{\alpha_i}) \subseteq \bigcup_{i=1}^n V_{\alpha_i}$. □

THEOREM 105. *Let $E \subseteq Y \subseteq X$. Then E compact in X iff E is compact in Y.*

Proof. Suppose E compact in X, and let $\{V_\alpha : \alpha \in \Lambda\}$ be a family of open sets in Y which covers E. Then there exist open sets G_α in X such that $V_\alpha = G_\alpha \cap Y$, $\alpha \in \Lambda$. Thus

$$E \subseteq \bigcup_{\alpha\in\Lambda} V_\alpha = \bigcup_{\alpha\in\Lambda} (G_\alpha \cap Y) \subseteq \bigcup_{\alpha\in\Lambda} G_\alpha.$$

Since E is compact in X, there exist $\alpha_1, \ldots, \alpha_n$ in Λ such that $E \subseteq \bigcup_{i=1}^n G_{\alpha_i}$. Hence,

$$E \subseteq \bigcup_{i=1}^n (Y \cap G_{\alpha_i}) = \bigcup_{i=1}^n V_{\alpha_i}.$$

Conversely, suppose E is compact in Y, and let $\{G_\alpha : \alpha \in \Lambda\}$ be a family of open sets in X which covers E. Then $\{Y \cap G_\alpha : \alpha \in \Lambda\}$ is a family of open sets in Y which covers E. Since E is compact in Y, there exist $\alpha_1, \ldots, \alpha_n$ in Λ such that $E \subseteq \bigcup_{i=1}^n (Y \cap G_{\alpha_i})$. Hence, $E \subseteq \bigcup_{i=1}^n G_{\alpha_i}$. □

THEOREM 106. *Every compact metric space is separable.*

Proof. Let (X, d) be a compact metric space. For each $n \in \mathbb{N}$, $\{B(x, \frac{1}{n}) : x \in X\}$ is an pen cover of X. Then there exists $x_1^{(n)}, \ldots, x_{k_n}^{(n)}$ in X such that $X = \bigcup_{i=1}^{k_n} \{B(x_i^{(n)}, \frac{1}{n})$. Consider the set

$$D := \bigcup_{n\in\mathbb{N}} \{x_i^{(n)} : i = 1, \ldots, k_n\}.$$

We show that D is dense in X. For this, let $x \in X$ and $\varepsilon > 0$ be given. Then, for each $n \in \mathbb{N}$, there exists $i_n \in \{1, \ldots, k_n\}$ such that $x \in B(x_{i_n}^{(n)}, \frac{1}{n})$. Take n large enough such that $\frac{1}{n} < \varepsilon$. Then we have $d(x, x_{i_n}^{(n)}) < \frac{1}{n} < \varepsilon$. Thus, the countable set D is dense in X. □

Definition 107. A subset of a metric space is said to be **bounded** if it is contained in a ball. ♢

- A subset E of a metric space X is bounded if and only if

  $$\text{daim}(E) := \sup\{d(x, y) : x, y \in E\} < \infty$$

  if and only if there exists $M > 0$ such that for every $x \in X$, $d(x, y) \le M$ for all $y \in E$.

- Every Cauchy sequence in a metric space is bounded. In particular, every convergent sequence is bounded: To see this, let (x_n) be a Cauchy sequence in a metric space X. Then there exists $N \in \mathbb{N}$ such that $d(x_n, x_N) < 1$ for all $n \ge N$. Therefore,

  $$d(x_n, x_N) < \rho := 1 + \max_{1\le i\le N} d(x_i, x_N) \quad \forall\, n \in \mathbb{N},$$

  i.e., $x_n \in B(x_N, \rho)$ for all $n \in \mathbb{N}$.

THEOREM 108. *Every compact subset of a metric space is bounded.*

Proof. Let E be a compact subset of a metric space. If $E = \varnothing$, then it is bounded. So, assume that $E \neq \varnothing$. Let $x_0 \in E$. Then $E \subseteq \bigcup_{n=1}^{\infty} B(x_0, n)$; because, if $y \in E$, then there exists $n \in \mathbb{N}$ such that $d(y, x_0) < n$. Since E is compact, there exists $N \in \mathbb{N}$ such that $E \subseteq \bigcup_{n=1}^{N} B(x_0, n)$. Since $B(x_0, n) \subseteq B(x_0, N)$ for every $n \leq N$, we have $E \subseteq B(x_0, N)$. □

Proof. (Aliter) Note that, for each $r > 0$, $\{B(x, r) : x \in E\}$ is an open cover of E. Then there exists $x_1, \ldots, x_k$ in E such that $\{B(x_i, r) : i = 1, \ldots, k\}$ is also a cover of E. Note that (verify) for any $x \in X$ there exists $j_x \in \{1, \ldots, k\}$ such that $d(x, x_{j_x}) < r$. Hence

$$d(x, x_1) \leq d(x, x_{j_x}) + d(x_{j_x}, x_1) \leq r + \max_{1 \leq j \leq k} d(x_j, x_1) = \rho, \text{ say.}$$

Thus, $E \subseteq B(x, \rho)$ so that E is bounded. □

By this theorem we can assert that the sets $(0, \infty)$, $[0, \infty)$, $(-\infty, 1]$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$ are not compact in $\mathbb{R}$, without actually constructing an open cover in each case so that it does not have a finite subcover.

- A closed and bounded set need be compact. For example, an infinite set with discrete metric is closed and bounded, but not compact.

THEOREM 109. (Finite intersection property) *Suppose $\{K_\alpha : \alpha \in \Lambda\}$ is a family of compact sets in a metric space such that every finite subfamily of it has nonempty intersection. Then $\bigcap_{\alpha \in \Lambda} K_\alpha \neq \varnothing$.*

Proof. Suppose $\bigcap_{\alpha \in \Lambda} K_\alpha = \varnothing$. Let $\alpha_0 \in \Lambda$. Then there exists $\alpha \neq \alpha_0$ such that $K_{\alpha_0} \cap K_\alpha = \varnothing$. Hence, $K_{\alpha_0} \subseteq \bigcup_{\alpha \neq \alpha_0} K_\alpha^c$. Since K_{α_0} is compact, there exists $\alpha_1, \ldots, \alpha_k$ different from α_0 such that $K_{\alpha_0} \subseteq \bigcup_{i=1}^{k} K_{\alpha_i}^c$. This implies that $K_{\alpha_0} \cap \left(\bigcap_{i=1}^{k} K_{\alpha_i}^c\right)^c = \varnothing$, i.e., $K_{\alpha_0} \cap \left(\bigcap_{i=1}^{k} K_{\alpha_i}^c\right) \varnothing$. This is a contradiction. □

COROLLARY 110. *Suppose (K_n) is a sequence of compact sets in a metric space such that $K_1 \supseteq K_2 \supseteq \cdots$. Then $\bigcap_{n=1}^{\infty} K_n \neq \varnothing$.*

THEOREM 111. *Every compact metric space is complete.*

Proof. Let (X, d) be a compact metric space. Assume for a moment that it is not complete. Then there exists a Cauchy sequence (x_n) which does not converge in X. This also implies that (x_n) does not have any convergent subsequence[1]. Hence, for any given $x \in X$, there exists $r_x > 0$ such that $B(x, r_x)$ contains only a finite number of terms from (x_n). Since $\{B(x, r_x) : x \in X\}$ is an open cover of X and X is compact, there exist $x^{(1)}, \ldots, x^{(k)}$ in X such that $X = \bigcup_{i=1}^{k} B(x^{(i)}, r_i)$, $r_i := r_{x^{(i)}}$. This implies that X contains only a finite number of terms from $\{x_n : n \in \mathbb{N}\}$ - a contradiction. □

Definition 112. A subset E of a metric space is said to be **totally bounded** if for every $\varepsilon > 0$, there exists a finite number points $x_1, \ldots, x_n$ in X such that $E \subseteq \bigcup_{i=1}^{n} B(x_i, \varepsilon)$. ♢

- Every totally bounded set is bounded (verify). In particular, $\mathbb{R}$ with usual metric is not totally bounded.

[1] Recall that if a Cauchy sequence has a convergent subsequence, then the sequence itself will converge.

- A bounded set need not be totally bounded: For example, if X is an infinite set with discrete metric, then every subset of it is bounded, but infinite subsets are not totally bounded (Why?)
- A set is totally bounded iff its closure is totally bounded.

Exercise 113. Every totally bounded metric space is separable.
(Hint: Look at the proof of Theorem 106). ♢

THEOREM 114. *Every bounded subset of $\mathbb{R}$, with usual metric, is totally bounded (verify).*

Proof. Let E be a bounded subset of $\mathbb{R}$. Then $E \subseteq [a,b]$ for some $a, b \in \mathbb{R}$ with $a < b$. Let $a_0 = a, a_i = a + \frac{i(b-a)}{n}$, $i = 1, \ldots, n$. Then $[a,b] \subseteq \bigcup_{i=1}^n B_{\mathbb{R}}(a_i, \frac{2i(b-a)}{n})$. Let

$$\Lambda_n := \{i \in \{1, \ldots, n\} : B_{\mathbb{R}}(a_i, \frac{2i(b-a)}{n}) \cap E \neq \varnothing\}.$$

Then $[a,b] \subseteq \bigcup_{i\in\Lambda_n} B_{\mathbb{R}}(a_i, \frac{2i(b-a)}{n})$. For $i \in \Lambda_n$, let $x_i \in B_{\mathbb{R}}(a_i, \frac{2i(b-a)}{n}) \cap E$. Then

$$E \subseteq \bigcup_{i\in\Lambda_n} B_{\mathbb{R}}(a_i, \frac{4i(b-a)}{n}).$$

Now, given $\varepsilon > 0$, we may choose n such that $\frac{4i(b-a)}{n}) < \varepsilon$. Then we have $E \subseteq \bigcup_{i\in\Lambda_n} B_{\mathbb{R}}(x_i, \varepsilon)$. □

- Every subset of a totally bounded set is totally bounded.

THEOREM 115. *Every compact subset of a metric space is totally bounded.*

Proof. Let E be a compact subset of a metric space X. For $\varepsilon > 0$, $\{B(x, \varepsilon) : x \in E\}$ is an open cover of E. Then there exists $x_1, \ldots, x_k$ in E such that $\{B(x_i, \varepsilon) : i = 1, \ldots, k\}$ is also a cover of E. □

THEOREM 116. *A metric space is compact iff it is complete and totally bounded.*

Proof. Let (X, d) be a metric space.

$\Rightarrow$): Suppose E is a compact compact subset of X. We have already seen that X is complete and totally bounded.

$\Leftarrow$): Suppose X is complete and totally bounded. To show that it is compact. Assume for a moment that X is not compact. Then there exists an open cover $\mathcal{G}$ of X which does not have any finite subcover.

Let (r_n) be a sequence of positive reals such that $r_n > r_{n+1}$ for all $n \in \mathbb{N}$ and $r_n \to 0$. Since X is totally bounded, there exist $x_1^{(1)}, x_2^{(1)} \ldots, x_{k_1}^{(1)}$ such that $X = \bigcup_{i=1}^{k_1} B(x_i^{(1)}, r_1)$. Since X is not covered by any finite sub-collection of $\mathcal{G}$, atleast one of $B(x_1^{(1)}, r_1), B(x_2^{(1)}, r_1), \ldots, B(x_{k_1}^{(1)}, r_1)$ is not covered by a finite number of members from $\mathcal{G}$. W.l.g, assume that $B(x_1^{(1)}, r_1)$ is not covered by a finite number of members from $\mathcal{G}$. Since $B(x_1^{(1)}, r_1)$ is totally bounded, there exist $x_1^{(2)}, x_2^{(2)} \ldots, x_{k_2}^{(2)}$ in $B(x_1^{(1)}, r_1)$ such that $B(x_1^{(1)}, r_1) \subseteq \bigcup_{i=1}^{k_2} B(x_i^{(2)}, r_2)$. Since $B(x_1^{(1)}, r_1)$ is not covered by any finite sub-collection of $\mathcal{G}$, atleast one of $B(x_1^{(2)}, r_2), B(x_2^{(2)}, r_2), \ldots, B(x_{k_2}^{(2)}, r_2)$ is not covered by any finite sub-collection of $\mathcal{G}$. W.l.g, assume that $B(x_2^{(2)}, r_2)$ is not covered by any finite sub-collection of $\mathcal{G}$. Continuing this, we obtain a sequence $(x_n^{(n)})$ in X such that for each $n \in \mathbb{N}$, $B(x_n^{(n)}, r_n)$ is not covered by any finite sub-collection of $\mathcal{G}$.

Note that, for each $m \in \mathbb{N}$, $x_n^{(n)} \in B(x_m^{(m)}, r_m)$ for all $n \geq m$. Hence $(x_n^{(n)})$ is a Cauchy sequence in X. Since X is complete, there exits $x \in X$ such that $x_n^{(n)} \to x$, and[2] $d(x_m^{(m)}, x) \leq r_m$. Let $G \in \mathcal{G}$ be such that $x \in G$, and let $\varepsilon > 0$ be such that $B(x, \varepsilon) \subseteq G$. We show that $B(x_m^{(m)}, r_m) \subseteq B(x, \varepsilon)$ for all large enough m. For this, let $y \in B(x_m^{(m)}, r_m)$. Then

$$d(y, x) \leq d(y, x_m^{(m)}) + d(x_m^{(m)}, x) < 2r_m.$$

Hence, taking m sufficiently large such that $2r_m < \varepsilon$, we obtain $B(x_m^{(m)}, r_m) \subseteq B(x, \varepsilon) \subseteq G$. This contradicts the fact that $B(x_n^{(n)}, r_n)$ is not covered by any finite sub-collection of $\mathcal{G}$. □

THEOREM 117. *A subset E of a metric space is totally bounded iff every sequence in E has a Cauchy subsequence.*

Proof. Let (X, d) be a metric space and $E \subseteq X$.

$\Rightarrow$): Suppose E is totally bounded. Let (x_n) be a sequence in E, and let $\varepsilon > 0$ be given. Since E is totally bounded, there are $x^{(1)}, x^{(2)} \ldots, x^{(k)}$ in E such that $E \subseteq \bigcup_{i=1}^{k} B(x^{(i)}, \frac{\varepsilon}{2})$. Hence, there exists $\ell \in \{1, \ldots, k\}$ such that $B(x^{(\ell)}, \frac{\varepsilon}{2})$ contains infinite number of terms of (x_n), say $x_{n_j} \in B(x^{(\ell)}, \frac{\varepsilon}{2})$ for $j \in \mathbb{N}$. Thus, for every $i, j \in \mathbb{N}$, $d(x_{n_i}, x_{n_j}) \leq d(x_{n_i}, x^{(\ell)}) + d(x^{(\ell)},, x_{n_j}) < \varepsilon$. W.l.g we can assume that (n_j) is strictly increasing. Hence, we have a Cauchy subsequence (x_{n_j}).

$\Leftarrow$): Assume for a moment that E is not totally bounded. Then there exists $\varepsilon > 0$ such that E cannot be covered by a finite number of ε-balls. Let $x_0 \in E$. Since E cannot be covered by $B(x_0, \varepsilon)$, there exists $x_1 \in E \setminus B(x_0, \varepsilon)$, $x_2 \in E \setminus B(x_0, \varepsilon) \cup (B(x_1, \varepsilon)$, $x_3 \in E \setminus B(x_0, \varepsilon) \cup B(x_1, \varepsilon) \cup B(x_2, \varepsilon)$ etc. $x_{n+1} \in E \setminus \bigcup_{i=1}^{n} B(x_i, \varepsilon)$. From this it follows that $d(x_n, x_m) \geq \varepsilon$ for every distinct $n \neq m$ in $\mathbb{N}$. Thus, the sequence (x_n) does not have any Cauchy subsequence. □

As a corollary we obtain the following:

THEOREM 118. *A subset E of a metric space is compact iff every sequence in E has a subsequence which converges in E.*

Proof. Let X be a metric space and $E \subseteq X$. Suppose E is compact in X. Let (x_n) be a sequence in E. By Theorem 118, (x_n) has a Cauchy subsequence. Since E is complete, this Cauchy sequence converges to a point in E.

Conversely, every sequence in E has a subsequence which converges in E. Then by Theorem 118, E is totally bounded. Now, let (x_n) be any Cauchy sequence in E. By assumption, (x_n) has a subsequence which converges to some point $x \in E$. Since, (x_n) is a Cauchy sequence, it itself converges to x. Thus, E is complete as well. Therefore, E is complete and totally bounded, and hence compact. □

THEOREM 119. *If d and ρ are equivalent metrics on a set X and if X is compact w.r.t. d, then it is compact w.r.t. ρ.*

Proof. Suppose X is compact w.r.t. d. Then by Theorem 116, it is complete and totally bounded w.r.t. d. Hence, by Theorem 78, it is complete w.r.t. ρ. Now, let $c_1, c_2 > 0$ such that

$$c_1 d(x, y) \leq \rho(x, y) \leq c_2 d(x, y) \quad \forall\, x, y \in X. \tag{$*$}$$

[2] If (a_n) in X such that $a_n \to a$ and if $b \in X$, then $d(a_n, b) \to d(a, b)$. Also, if $d(a_n, b) < \varepsilon$ for all $n \in \mathbb{N}$, then $d(a, b) \leq \varepsilon$.

Since X is totally bounded w.r.t. d, for every $\varepsilon > 0$, there exist $x_1, \dots, x_k$ in X such that $X = \bigcup_{i=1}^k B_d(x_i, \varepsilon)$. But, by $(*)$, $B_d(x, \varepsilon) \subseteq B_\rho(x, c_2\varepsilon)$ for every $x \in X$. Hence, $X = \bigcup_{i=1}^k B_\rho(x_i, c_2\varepsilon)$. Hence, X is totally bounded w.r.t. ρ. Thus, by Theorem 116, X is compact w.r.t. ρ. □

Definition 120. A subset E os a metric space is called a **relatively compact set** if its closure is compact. ♢

- If X is a complete metric space, then a subset of X is relatively compact iff it is totally bounded.

7 Connectedness

Definition 121. Let X be a metric space.

1. X is said to be **disconnected** if there exist nonempty sets A and B such that
$$X = A \cup B, \quad \bar{A} \cap B = \varnothing, \quad A \cap \bar{B} = \varnothing.$$
2. X is said to be **connected** if it is not disconnected.
3. A subset E of X is **disconnected** if it is disconnected with respect to the induced metric.
4. A subset E of X is **connected** if it is connected with respect to the induced metric.

♢

THEOREM 122. *A metric space X is disconnected iff there exist nonempty disjoint open sets A and B such that $X = A \cup B$.*

Proof. $\Rightarrow$) : Suppose X is disconnected. Let A and B be nonempty sets such that $X = A \cup B$, $\bar{A} \cap B = \varnothing$ and $A \cap \bar{B} = \varnothing$. We show that A and B are open: Let $x \in A$. Since $A \cap \bar{B} = \varnothing$, $x \notin \bar{B}$. Hence there exists $r > 0$ such that $B(x, r) \cap B = \varnothing$. Thus, $B(x, r) \subseteq B^c = A$. Thus, we have prove that every point of A is an interior point of A so that A is open. Similarly, using the fact that $\bar{A} \cap B = \varnothing$, it can be shown that B is open.

$\Leftarrow$) : Suppose there exists nonempty disjoint open sets A and B such that $X = A \cup B$. Then $B = A^c$ is closed so that $B = \bar{B}$ and hence $A \cap \bar{B} = A \cap B = \varnothing$. Similarly, using the fact that B is open, we obtain $\bar{A} \cap B = A \cap B = \varnothing$. □

The reader may verify the following:

THEOREM 123. *Let E be a subset of a metric space X. Then the following are equivalent.*

(i) *There exist nonempty sets A and B such that*
$$E = A \cup B, \quad \bar{A} \cap B = \varnothing, \quad A \cap \bar{B} = \varnothing,$$

(ii) $E = V_1 \cup V_2$, *where V_1 and V_2 are nonempty disjoint sets which are open in E.*

(iii) *There exist open sets G_1 and G_2 in X such that*
$$E \subseteq G_1 \cup G_2, \quad E \cap G_1 \neq \varnothing, \quad E \cap G_2 \neq \varnothing, \quad E \cap (G_1 \cap G_2) = \varnothing.$$

Proof. (i) $\Rightarrow$ (ii): By (i), $A \subseteq (\bar{B})^c$ and $B \subseteq (\bar{A})^c$. Take $V_1 = E \cap (\bar{B})^c$ and $V_2 = E \cap (\bar{A})^c$. Then $A \subseteq V_1$, $B \subseteq V_2$, and $E = V_1 \cup V_2$, since

$$E = E \cap [A \cup B] \subseteq E \cap [(\bar{B})^c \cup (\bar{A})^c] = V_1 \cup V_2 \subseteq E.$$

Thus, (ii) holds.

(ii) $\Rightarrow$ (iii): Let V_1 and V_2 be as in (ii). Let $V_1 = E \cap G_1$ and $V_2 = E \cap G_2$, where G_1 and G_2 are open in X. Then $E \cap G_1 = V_1$ and $E \cap G_2 = V_1$ are nonempty, and $E \subseteq G_1 \cup G_2$. Further, $E \cap (G_1 \cap G_2) = V_1 \cap V_2 = \varnothing$. This proves (iii).

(iii) $\Rightarrow$ (i): Let G_1 and G_2 be as in (iii). Take $A = E \cap G_1$ and $B = E \cap G_2$. Then A and B are nonempty, and $E = A \cup B$. Now, suppose that $\bar{A} \cap B \neq \varnothing$. Then there exists $x \in B = E \cap G_2$ and $x \in \bar{A}$ so that there exists $r > 0$ such that $B(x,r) \subseteq G_2$ and $B(x,r) \cap A \neq \varnothing$. Let $y \in B(x,r) \cap A$. Hence, $y \in B(x,r) \subseteq G_2$ and $y \in A = E \cap G_1$ so that $y \in E \cap G_1 \cap G_2$, which is a contradiction to the hypothesis in (iii). □

In view of Theorem 122 and Theorem 123,

THEOREM 124. *A subset E of a metric space is disconnected iff any of the three conditions in Theorem 123 is satisfied.*

What are connected subsets of $\mathbb{R}$ w.r.t the usual metric?

Recall that a subset E of $\mathbb{R}$ is an **interval** if for every $x, y \in E$ and $z \in \mathbb{R}$, $x < z < y$ implies $z \in E$.

THEOREM 125. *A subset of $\mathbb{R}$, which is not singleton, is connected iff it is an interval.*

Proof. Let $E \subseteq \mathbb{R}$. Suppose E is not an interval. Then, there exists $x, y \in E$ and $z \in \mathbb{R}$ with $x < z < y$, but $z \notin E$. Then $E \subseteq A_z \cup B_z$, where $A_z := (-\infty, z)$ and $B_z := (z, \infty)$ are disjoint open sets in X with $x \in A_z$, $y \in B_z$. Hence E is not connected.

Conversely, suppose E is not connected. Then there exist nonempty sets A and B such that $E = A \cup B$, $\bar{A} \cap B = \varnothing$, $A \cap \bar{B} = \varnothing$. let $x \in A$ and $y \in B$. W.l.g., assume that $x < y$. Let $z = \sup(A \cap [x,y])$. Then $z \in \bar{A}$. Hence, $z \notin B$ so that $x \leq z < y$. Let us consider the two cases (i) $z \notin A$ and (ii) $z \in A$.

(i) $z \notin A$ implies $x < z < y$, and we are done.

(ii) if $z \in A$, then $z \notin \bar{B}$ so that there exists an open interval containing z which does not contain any point from B. Hence, there exists z' such that $z < z' < y$. Note that $z' \notin A$ and $z' \notin B$ so that $z' \notin E$.

Thus we have proved that there exists $\alpha \in \mathbb{R} \setminus E$ such that $x < \alpha < y$, so that E is not an interval. □

Exercise 126. A metric space X is disconnected iff there exist disjoint nonempty closed sets A and B such that $X = A \cup B$. ♢

Exercise 127. A metric space X is connected iff there is no nonempty set which is both open and closed in X. ♢

Definition 128. A metric space X is said to be **totally disconnected** if singleton sets are the only nonempty connected sets. ♢

Example 129. The set $\mathbb{Q}$ is totally disconnected w.r.t. the usual metric. To see this suppose E is a nonempty subset of $\mathbb{Q}$, which is not singleton. Let $x, y \in E$ with $x < y$. Let $z \in \mathbb{Q}^c$ such that $x < z < y$. Then $E = A \cup B$, where $A := E \cap (-\infty, z)$ and $B := E \cap (z, \infty)$ are disjoint nonempty open subsets of E. ♢

THEOREM 130. *Let X be a metric space and $E \subseteq X$. If E is connected then $\bar{E}$ is connected; and the converse is not true.*

Proof. Suppose $\bar{E}$ is not connected. Then there exist disjoint nonempty open subsets V_1, V_2 in $\bar{E}$ such that $\bar{E} = V_1 \cup V_2$. Then $V_1 = \bar{E} \cap G_1$, $V_2 = \bar{E} \cap G_2$ for some open sets G_1, G_2 in X.

Let $x \in V_1 = \bar{E} \cap G_1$. Then there exists $r > 0$ such that $B(x, r) \subseteq G_1$; also, $B(x, r) \cap E \neq \varnothing$. Hence, $E \cap G_1 \neq \varnothing$. Similarly, $E \cap G_1 \neq \varnothing$. Thus, $E = (E \cap G_1) \cup (E \cap G_2)$. Note that $U_1 = E \cap G_1$, $U_2 = E \cap G_2$ are open sets in E with $U_1 \neq \varnothing$, $U_2 \neq \varnothing$ and $U_1 \cap U_2 = (E \cap G_1) \cap (E \cap G_2) \subseteq (\bar{E} \cap G_1) \cap (\bar{E} \cap G_2) = \varnothing$. Thus, we have proved that $\bar{E}$ connected implies E disconnected.

To see that the converse is not true, consider $X = \mathbb{R}$ and $E = \mathbb{Q}$. □

8 Continuity

8.1 Definition and characterizations

Definition 131. Let X be a metric space with metric d_X and Y a metric space with metric d_Y. Let $E \subseteq X$ and let $f : E \to Y$ be a function.

(i) f is said to be continuous at a point $a \in E$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x \in E, \quad d_X(x, a) < \delta \quad \Rightarrow \quad d_Y(f(x), f(a)) < \varepsilon.$$

(ii) f is said to be continuous on $E \subseteq X$ if it is continuous at every point in E. ♢

Observe that

- f is continuous at a point $a \in E$ iff for every $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in B_E(a, \delta)$ implies $f(x) \in B_Y(f(a), \varepsilon)$.

Example 132. Let $X = [0, 1]$ with usual metric. Let $f(x) = \begin{cases} 0, & 0 \leq x < \frac{1}{2}, \\ 1, & \frac{1}{2} \leq x \leq 1. \end{cases}$ Then f is continuous at every point except at $\frac{1}{2}$. ♢

Example 133. Let $X = [0, 1]$ with usual metric. Let $f(x) = \begin{cases} 0, & x \in \mathbb{Q} \cap [0, 1], \\ 1, & x \in \mathbb{Q}^c \cap [0, 1]. \end{cases}$ Then $f : X \to \mathbb{R}$ is not continuous at any point. ♢

THEOREM 134. *A function $f : E \to Y$ is continuous at a point $a \in E$ iff for every sequence (x_n) in E,*

$$d_X(x_n, a) \to 0 \quad \Rightarrow \quad d_Y(f(x_n), f(a)) \to 0.$$

Proof. Suppose $f : E \to Y$ is continuous at $a \in E$. Let (x_n) be in E such that $d_X(x_n, a) \to 0$. To show that $d_Y(f(x_n), a) \to 0$.

Let $\varepsilon > 0$ be given. Since f is continuous at $a \in E$, there exists $\delta > 0$ such that $x \in E$, $d_X(x,a) < \delta \Rightarrow d_Y(f(x), f(a)) < \varepsilon$. Since $d_X(x_n, a) \to 0$, there exists $n_0 \in \mathbb{N}$ such that $d_X(x_n, a) < \delta$ for all $n \geq n_0$. Hence, $d_Y(f(x_n), f(a)) < \varepsilon$ for all $n \geq n_0$. Thus, $d_Y(f(x_n), f(a)) \to 0$.

Conversely, suppose that for every sequence (x_n) in E, $d_X(x_n, a) \to 0 \Rightarrow d_Y(f(x_n), f(a)) \to 0$. Suppose f is not continuous at a. Then there exists $\varepsilon > 0$ such that for any $\delta > 0$ there is $x \in E$ with $d_X(x,a) < \delta$, but $d_Y(f(x), f(a)) \geq \varepsilon$. In particular, there exists $\varepsilon > 0$ such that for any $n \in \mathbb{N}$ there is $x_n \in E$ with $d_X(x_n, a) < \frac{1}{n}$, but $d_Y(f(x_n), f(a)) \geq \varepsilon$. Thus, we obtain a sequence (x_n) in E such that $d_X(x_n, a) \to 0$, but $d_Y(f(x_n), a) \not\to 0$. This is a contradiction to the hypothesis. □

THEOREM 135. *A function $f : E \to Y$ is continuous at a point $a \in E$ iff for every open set V in Y containing $f(a)$, there exists an open set U in E containing a such that $f(U) \subseteq V$.*

Proof. $\Rightarrow)$: Suppose $f : E \to Y$ is continuous at $a \in E$. Let V be an open set in Y containing $f(a)$. Then there exists $\varepsilon > 0$ such that $B_Y(f(a), \varepsilon) \subseteq V$. Since f is continuous at a, there exists $\delta > 0$ such that $x \in E$ and $d_X(a,x) < \delta$ implies $d_Y(f(x), f(a)) < \varepsilon$. That is, $x \in B_E(a, \delta)$ implies $f(x) \in d_Y(f(a), \varepsilon) \subseteq V$. Thus, taking $U := B_E(a, \delta)$, $f(U) \subseteq V$.

$\Leftarrow)$: Let $\varepsilon > 0$ be given. Consider the open ball $V := B_Y(f(a), \varepsilon)$. Let U be an open set in E containing a such that $f(U) \subseteq V$. Let $\delta > 0$ be such that $B_E(x, \delta) \subseteq U$. Then we have $x \in B_E(u, \delta)$ implies $f(x) \in V := B_Y(f(a), \varepsilon)$. □

THEOREM 136. *A function $f : E \to Y$ is continuous on E iff for every open set V in Y, $f^{-1}(V)$ is open in E.*

Proof. $\Rightarrow)$: Suppose $f : E \to Y$ is continuous on E. Let V be an open set in Y. Let $x \in f^{-1}(V)$. Then $f(x) \in V$. Hence, there exists open set U in E containing x such that $f(U) \subseteq V$. Hence, $U \subseteq f^{-1}(V)$. Thus, we have proved that $f^{-1}(V)$ is open in E.

$\Leftarrow)$: Suppose $f^{-1}(V)$ is open in E for every open set V in Y. Let $a \in E$ and let $\varepsilon > 0$ be given. Consider the open ball $V := B_Y(f(a), \varepsilon)$. Since $f^{-1}(V)$ is open in E, there exists $\delta > 0$ such that $B_E(a, \delta) \subseteq f^{-1}(V)$. That is, $x \in B_E(a, \delta)$ implies $f(x) \in V := B_Y(f(a), \varepsilon)$. □

8.2 Relation with compactness and connectedness

THEOREM 137. *Let $f : X \to Y$ be a continuous function and $E \subseteq X$.*

(i) If E is compact in X, then $f(E)$ is compact in Y.

(ii) If E is connected in X, then $f(E)$ is connected in Y.

Proof. (i) Suppose E is compact in X. To show that $f(E)$ is compact in Y. For this, let $\{V_\alpha\}_{\alpha \in \Lambda}$ be an open cover of $f(E)$. Then $\{f^{-1}(V_\alpha)\}_{\alpha \in \Lambda}$ is a cover of E. Further, since f is continuous, for each $\alpha \in \Lambda$, $f^{-1}(V_\alpha) \cap E = \{x \in E : f(x) \in V_\alpha\}$ is open in E. Since E is compact, there exist $\alpha_1, \ldots, \alpha_n$ in Λ such that $E \subseteq \bigcup_{i=1}^n \big(f^{-1}(V_{\alpha_i}) \cap E\big)$. Hence, $E \subseteq \bigcup_{i=1}^n f^{-1}(V_{\alpha_i})$, and hence[3] $f(E) \subseteq \bigcup_{i=1}^n f^{-1}(V_{\alpha_i})$.

(ii) Suppose E is connected in X. Assume for a moment that $f(E)$ is disconnected. Then, by Theorem 123, there exist disjoint open sets V_1 and V_2 in Y such that

$$f(E) \subseteq V_1 \cup V_2, \quad f(E) \cap V_1 \neq \varnothing, \quad f(E) \cap V_2 \neq \varnothing.$$

[3]using $f(\cup_\alpha A_\alpha) \subseteq \cup_\alpha f(A_\alpha)$ and $f(f^{-1}(A)) \subseteq A$.

Since f is continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are open sets such that

$$E \subseteq f^{-1}(V_1) \cup f^{-1}(V_2), \quad E \cap f^{-1}(V_1) \neq \varnothing, \quad E \cap f^{-1}(V_2) \neq \varnothing.$$

Thus, again by Theorem 123, E is disconnected; a contradictions to the hypothesis. $\square$

Following are important corollaries of the above theorem.

THEOREM 138. (Existence of maxima and minima) *Let E be a compact subset of a metric space X and $f : E \to \mathbb{R}$ be a continuous function. Then there exists $u, v \in E$ such that*

$$f(u) = \sup_{x \in E} f(x), \quad f(v) = \inf_{x \in E} f(x).$$

Proof. Since f is continuous and E is a compact subset of X, $f(E)$ is compact in $\mathbb{R}$. In particular, $f(E)$ is bounded above and bounded below. Let

$$\alpha := \sup f(E), \quad \beta := \inf f(E).$$

Since $f(E)$ is compact, there exist (u_n) and (v_n) in E such that $f(u_n) \to \alpha$ and $f(v_n) \to \beta$ as $n \to \infty$. Since E is compact, (u_n) and (v_n) have subsequences, $(\tilde{u}_n)$ and $(\tilde{v}_n)$ such that $\tilde{u}_n \to u$ and $\tilde{v}_n \to v$ for some $u, v \in E$. Continuity of f implies, $f(\tilde{u}_n) \to f(u)$ and $f(\tilde{v}_n) \to f(v)$. Hence, $f(u) = \alpha$ and $f(v) = \beta$. $\square$

Alternate proof. Since E compact and f continuous, $f(E)$ is compact, and hence it is closed and bounded. Hence, $\alpha, \beta \in \mathbb{R}$, and

$$\alpha := \sup f(E) \in \overline{f(E)} = f(E) \quad \text{and} \quad \beta := \inf f(E) \in \overline{f(E)} = f(E).$$

Thus, there exists $u, v \in E$ such that $f(u) = \alpha$, $f(v) = \beta$. $\square$

THEOREM 139. (General intermediate value theorem) *Let E be a connected subset of a metric space X and $f : E \to \mathbb{R}$ be a continuous function. Let $u, v \in E$ and $\gamma \in \mathbb{R}$ be such that $f(u) \leq \gamma \leq f(v)$. Then there exists $x \in E$ such that $f(x) = \gamma$.*

Proof. Since E is a connected subset of X, $f(E)$ is connected in $\mathbb{R}$. Since connected subsets of $\mathbb{R}$ are either singleton sets or intervals, $[f(u), f(v)] \subseteq f(E)$. Hence, there exists $x \in E$ such that $f(x) = \gamma$. $\square$

THEOREM 140. (Intermediate value theorem) *Let I be an interval and let $f : I \to \mathbb{R}$ be a continuous function. Let $u, v \in I$ and $\gamma \in \mathbb{R}$ be such that $f(u) \leq \gamma \leq f(v)$. Let $a := \min\{u, v\}$ and $y = \max\{u, v\}$. Then there exists $x \in [a, b]$ such that $f(x) = \gamma$.*

Proof. Since $E := [a, b]$ is a connected subset of $\mathbb{R}$, $f(E)$ is connected in $\mathbb{R}$. Since $f(u), f(v) \in f(E)$, and since connected subsets of $\mathbb{R}$ are either singleton sets or intervals, $[f(a), f(b)] \subseteq f(E)$. Hence, there exists $x \in E$ such that $f(x) = \gamma$. $\square$

THEOREM 141. *A metric space X is disconnected iff there exists continuous surjective function $f : X \to \{0, 1\}$, where $\{0, 1\}$ is endowed with discrete metric.*

Proof. Suppose X is disconnected, and let A and B be disjoint open sets A and B such that $X = A \cup B$. Let $f : X \to \{0, 1\}$ be defined by $f(x) = 0$ for every $x \in A$ and $f(x) = 1$ for every $x \in B$. Then f is continuous (why?) and onto.

Conversely, suppose there is a continuous surjective function $f : X \to \{0, 1\}$. Then $A := f^{-1}(\{0\})$ and $B := f^{-1}(\{1\})$ are nonempty disjoint closed sets such that $X = A \cup B$, so that X is disconnected. □

8.3 Uniform continuity

Recall that a function $f : X \to Y$ is continuous on $E \subseteq X$ iff for every $\varepsilon > 0$ and for every $a \in E$, there exists $\delta > 0$ such that $x \in E$, $d_X(x, a) < \delta$ implies $d_Y(f(x), f(a)) < \varepsilon$. Note that, the δ above may depend not only on ε, but also on the point a.

Definition 142. Let X and Y be metric spaces. A function $f : X \to Y$ is said to be uniformly continuous on $E \subseteq X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x, y \in E, \quad d_X(x, y) < \delta \quad \Rightarrow \quad d_Y(f(x), f(y)) < \varepsilon.$$

♢

THEOREM 143. *Let X and Y be metric spaces and $f : X \to Y$. Then f is uniformly continuous on $E \subseteq X$ iff for every sequence (x_n) and (y_n) in E satisfying $d_X(x_n, y_n) \to 0$, we have $d_Y(f(x_n), f(y_n)) \to 0$.*

Proof. $\Rightarrow$): Suppose $f : X \to Y$ is uniformly continuous on $E \subseteq X$ and (x_n) and (y_n) are sequences in E such that $d_X(x_n, y_n) \to 0$. Let $\varepsilon > 0$ be given and let $\delta > 0$ be such that $x, y \in E$ with $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \varepsilon$. Let $N \in \mathbb{N}$ be such that $d_X(x_n, y_n) < \delta$ for all $n \geq N$. Then, it follows that $d_Y(f(x_n), f(y_n)) < \varepsilon$ for all $n \geq N$.

$\Leftarrow$): Suppose f is not uniformly continuous. Then there exists $\varepsilon > 0$ such that for every $\delta > 0$, there exist $x, y \in E$ (depending on δ) such that $d_X(x, y) < \delta$, but $d_Y(f(x), f(y)) \geq \varepsilon$. In particular, for every $n \in \mathbb{N}$, there exist $x_n, y_n \in E$ such that $d_X(x_n, y_n) < \frac{1}{n}$, but $d_Y(f(x_n), f(y_n)) \geq \varepsilon$. Thus, we have proved that, if f is not uniformly continuous, then there exist sequences (x_n) and (y_n) in E satisfying $d_X(x_n, y_n) \to 0$, but $d_Y(f(x_n), f(y_n)) \not\to 0$. □

Example 144. Consider $f : (0, 1] \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$. This function is continuous on $(0, 1]$, but not uniformly continuous. To see this consider the sequences $(\frac{1}{n})$ and $(\frac{1}{n^2})$. We see that $|\frac{1}{n} - \frac{1}{n^2}| \to 0$, but $|f(\frac{1}{n}) - f(\frac{1}{n^2})| = |n - n^2| \not\to 0$. ♢

THEOREM 145. *Let X and Y be metric spaces and $f : X \to Y$ be continuous. If $E \subseteq X$ is compact, then f is uniformly continuous on E.*

Proof. Let E be a compact subset of X. Let $\varepsilon > 0$ be given. Then for every $u \in E$, there exists $\delta_u > 0$ such that $x \in E$ and $d_X(x, u) < \delta_u$ implies $d_Y(f(x), f(u)) < \frac{\varepsilon}{2}$. Since $\{B_X(u, \frac{\delta_u}{2}) : u \in E\}$ is an open cover of E and since E is compact, there exist $u_1, \ldots, u_k$ in E such that $E \subseteq \bigcup_{i=1}^k B_X(u_i, \frac{\delta_i}{2})$, where $\delta_i := \delta_{u_i}$. Now, let $x, y \in E$ be such that

$$d_X(x, y) < \delta := \min\{\frac{\delta_i}{2} : i = 1, \ldots, k\}.$$

Let $i \in \{1, \ldots, k\}$ be such that $d_X(x, u_i) < \frac{\delta_i}{2}$. Then we have

$$d_X(y, u_i) \leq d_X(y, x) + d_X(x, u_i) < \delta + \frac{\delta_i}{2} < \delta_i.$$

Hence,

$$d_Y(f(x), f(y)) \le d_Y(f(x), f(u_i)) + d_Y(f(u_i), f(y)) < \varepsilon.$$

□

Alternate proof. Let E be compact. Suppose f is not uniformly continuous on E. Then, by Theorem 143, there exist sequences (x_n) and (y_n) in E such that $d_X(x_n, y_n) \to 0$, but $d_Y(f(x_n), f(y_n)) \not\to 0$. Then, there exist $\varepsilon > 0$ such that $d_Y(f(x_n), f(y_n)) \ge \varepsilon$ for infinitely many n's. Thus, (x_n) and (y_n) have subsequences $(\tilde{x}_n)$ and $(\tilde{y}_n)$, respectively, such that

$$d_Y(f(\tilde{x}_n), f(\tilde{y}_n)) \ge \varepsilon \quad \forall\, n \in \mathbb{N}. \tag{$*$}$$

Since E is compact, $(\tilde{x}_n)$ and $(\tilde{y}_n)$ have convergent subsequences, say $(\hat{x}_n)$ and $(\hat{y}_n)$, respectively. Let $\hat{x}_n \to x$ and $\hat{y}_n \to y$. Since $d_X(\hat{x}_n, \hat{y}_n) \to 0$, it follows that $x = y$. Consequently, $d_Y(f(\hat{x}_n), f(\hat{y}_n)) \to 0$. This is a contradiction to $(*)$. □

Remark 146. Note that, in Example 144, $X := (0, 1]$ is not compact. ♢

Remark 147. Theorem 145 is important in the context of Riemann integration, where to show that every continuous function $f : [a, b] \to \mathbb{R}$ is Riemann integral, what we essentially use is the uniform continuity of f. ♢

Exercise 148. Let X and Y be metric spaces. If $f : X \to Y$ is uniformly continuous, then for every Cauchy sequence (x_n) in X, the sequence $(f(x_n))$ is Cauchy in Y. Give an example to show that the assumption of uniform continuity of f cannot be dropped. ♢

The proof of the following theorem is left as an exercise.

THEOREM 149. *Let X and Y be metric spaces and $f : X \to Y$ be such that there exists $\kappa > 0$ satisfying*

$$d_Y(f(x), f(y)) \le \kappa\, d_X(x, y) \quad \forall\, x, y \in X.$$

Then f is uniformly continuous on X.

Definition 150. Let X and Y be metric spaces. A function $f : X \to Y$ is said to be **Lipschitz continuous** if there exists $\kappa > 0$ such that

$$d_Y(f(x), f(y)) \le \kappa\, d_X(x, y) \quad \forall\, x, y \in X.$$

The number κ above is called a **Lipschitz constant**.

If $f : X \to X$ is a Lipschitz continuous function with Lipschitz constant $\kappa < 1$, then f is called a **contraction**. ♢

- Every Lipschitz continuous function is uniformly continuous.

THEOREM 151. *Let (X, d) be a metric space. For $x_0 \in X$, the map $x \mapsto d(x, x_0)$ is Lipschitz continuous from X to $\mathbb{R}$ (with usual metric), with Lipschitz constant $\kappa = 1$.*

Proof. Note that for $x, y \in X$,

$$d(x, x_0) - d(y, x_0) \le d(x, y), \quad d(y, x_0) - d(x, x_0) \le d(x, y).$$

Hence,

$$|d(x, x_0) - d(y, x_0)| \le d(x, y).$$

□

THEOREM 152. *Let (X, d) be a metric space. For $E \subseteq X$ and $x \in X$, let*

$$d(x, E) := \inf\{(d(x, y) : y \in E\}.$$

(i) *For $x \in X$, $d(x, E) = 0$ iff $x \in \bar{E}$.*

(ii) *The map $x \mapsto d(x, E)$ is Lipschitz continuous from X to $\mathbb{R}$ (with usual metric), with Lipschitz constant $\kappa = 1$*

Proof. (i) Let $x \in X$. Then $d(x, E) := \inf\{(d(x, y) : y \in E\} = 0$ iff for every $\varepsilon > 0$, there exists $y \in E$ such that $d(x, y) < \varepsilon$ iff $x \in \bar{E}$.

(ii) Let $x, y \in E$. Then for every $u \in E$,

$$d(x, u) \leq d(x, y) + d(y, u).$$

Hence,

$$d(x, E) \leq d(x, E) := \inf_{v \in E} d(x, v) \leq d(x, y) + d(y, u),$$

$$d(x, E) \leq d(x, y) + \inf_{u \in E} d(y, u) = d(x, y) + \inf_{u \in E} d(y, u) = d(x, y) + d(y, E).$$

Thus, $d(x, E) \leq d(x, y) + d(y, E)$. Similarly, $d(y, E) \leq d(x, y) + d(x, E)$. Thus,

$$|d(x, E) - d(y, E)| \leq d(x, y).$$

□

Exercise 153. Let X and Y be metric spaces and D be a dense subset of X. If $f : D \to Y$ is uniformly continuous and if Y is complete, then there exists a unique uniformly continuous function $\varphi : X \to Y$ such that $\varphi_{|D} = f$.

Hint: Steps involved:

1. For $x \in X$, take a sequence (x_n) in D such that $x_n \to x$.
2. Observe that $f(x_n)$ is a Cauchy sequence in Y.
3. Write $F(x) = \lim_{n \to \infty} f(x_n)$.
4. Observe that if (u_n) in D with $u_n \to x$, then $\lim_{n \to \infty} f(u_n) = \lim_{n \to \infty} f(x_n)$.
5. Observe that $F : X \to Y$ is well-defined.
6. Observe that $F(u) = f(u)$ for every $u \in D$.
7. Show that F is uniformly continuous.

♢

9 A Few Important Theorems

9.1 Contraction mapping theorem

Definition 154. Let X be a nonempty set and let $f : X \to X$. A point $x_0 \in X$ is said to be a **fixed point** of f if $f(x_0) = x_0$. $\diamondsuit$

Existence of a fixed point of a function is important in the context of solving equations. For example, $F : X \to X$ is a given function and $y \in X$. Suppose we want to solve the equation

$$F(x) = y.$$

This problem can be converted into a problem of finding fixed point for the function $f : X \to X$ defined by

$$f(x) = x + F(x) - y.$$

THEOREM 155. *Let X be a complete metric space and $f : X \to X$ be a contraction. Then f has a unique fixed point. In fact, if $x_0 \in X$, and for $n \in \mathbb{N}$, if $x_n = f(x_{n-1})$, then (x_n) converges to a fixed point of f, and it is the only fixed point of f.*

Proof. Note that, for $n \in \mathbb{N}$,

$$d(x_n, x_{n+1}) = d(f(x_{n-1}), f(x_n)) \le \kappa d(x_{n-1}, x_n).$$

Hence, it follows that $d(x_n, x_{n+1}) \le \kappa^n d(x_0, x_1)$. Thus, for any $n, m \in \mathbb{N}$ with $n > m$,

$$\begin{aligned} d(x_m, x_n) &\le d(x_m, x_{m+1}) + \cdots + d(d(x_{n-1}, x_n) \\ &\le (\kappa^m + \kappa^{m+1} + \cdots + \kappa^{n-1}) d(x_1, x_0) \\ &\le \kappa^m (1 + \kappa + \cdots + \kappa^{n-1-m}) d(x_1, x_0) \\ &\le \frac{\kappa^m}{1-\kappa} d(x_1, x_0). \end{aligned}$$

Since $0 < \kappa < 1$, it follows that (x_n) is a Cauchy sequence in X. Since X is complete, there exists $x \in X$ such that $x_n \to x$. Now, since

$$d(x_n, f(x_n)) = d(x_n, x_{n+1}) \quad \forall\, n \in \mathbb{N}.$$

Hence, taking limit as $n \to \infty$, $d(x, f(x)) = d(x, x) = 0$. Thus, x is a fixed point of f. If $u \in X$ is another fixed pint of f. Then we get

$$0 \le d(x, u) = d(f(x), f(u)) \le \kappa d(x, u).$$

Since $0 < \kappa < 1$, the above inequality implies that $x = u$. $\square$

9.2 Baire category theorem

We know that intersection of two dense sets in a metric space need not be dense. But, intersection of two dense open sets will be dense.

THEOREM 156. *Suppose A and B be dense opens sets in a metric space (X, d). Then $A \cap B$ is dense.*

Proof. Let $D = A \cap B$. Let $x \in X$ and $r > 0$. To prove that $B(x,r) \cap D \neq \varnothing$. Since A is dense, $B(x,r) \cap A \neq \varnothing$. Since $B(x,r) \cap A$ is open, there exists $x_1 \in X$ and $0 < r_1 < r$ such that $B(x_1, r_1) \subseteq B(x,r) \cap A$. Since B is dense, $B(x_1, r_1) \cap B \neq \varnothing$. Since $B(x_1, r_1) \cap B$ is open, there exists $x_2 \in X$ and $0 < r_2 < r_1$ such that $B(x_2, r_2) \subseteq B(x_1, r_1) \cap B$. Thus,

$$B(x_2, r_2) \subseteq B(x_1, r_1) \cap B \subseteq [B(x,r) \cap A] \cap B = B(x,r) \cap (A \cap B).$$

In particular, $B(x,r) \cap D \neq \varnothing$. □

in the above theorem, there is nothing special about two sets. It can be easily generalized to any finite number of sets. Thus, we have

THEOREM 157. *Suppose $A_1, A_2, \ldots, A_n$ be dense opens sets in a metric space (XS, d). Then $\bigcap_{k=1}^n A_k$ is dense.*

Proof. Left as an exercise. □

Can we generalize the above theorem into infinitely many sets?

THEOREM 158. (Baire category theorem) *Suppose (X, d) is a complete metric space and $A_1, A_2, \ldots,$ are dense opens sets in X. Then $\bigcap_{n=1}^\infty A_k$ is dense.*

Proof. Let $D = \bigcap_{n=1}^\infty A_k$. Let $x \in X$ and $r > 0$. To prove that $B(x,r) \cap D \neq \varnothing$.

Since A_1 is dense, $B(x,r) \cap A_1 \neq \varnothing$. Since $B(x,r) \cap A_1$ is open, there exists $x_1 \in X$ and $0 < r_1 < r$ such that $\overline{B(x_1, r_1)} \subseteq B(x,r) \cap A_1$. Since A_2 is dense, $B(x_1, r_1) \cap A_2 \neq \varnothing$. Since $B(x_1, r_1) \cap A_2$ is open, there exists $x_2 \in X$ and $0 < r_2 < r_1$ such that $\overline{B(x_2, r_2)} \subseteq B(x_1, r_1) \cap A_2$. Continuing this, we obtain $x_1, x_2, \ldots$ and $r > r_1 > r_2 \cdots$ such that

$$\overline{B(x_n, r_n)} \subseteq B(x_{n-1}, r_{n-1}) \cap A_n \quad \forall n \in \mathbb{N} \quad \text{with} \quad x_0 = x, \quad r_0 = r.$$

In particular,

$$x_n \in B(x_n, r_n) \subseteq B(x_m, r_m) \forall n \geq m.$$

Without loss of generality, we may assume that $r_n \to 0$ as $n \to \infty$ (e.g., we may take $r_{n+1} < r_n/2$ for all $n \in \mathbb{N}$). Hence, (x_n) is a Cauchy sequence in X. Since X is complete there exists $\tilde{x} \in X$ such that $x_n \to \tilde{x}$.

Now, since $d(x_n, x_m) < r_m$ for all $n \geq m$, letting $n \to \infty$, $d(\tilde{x}, x_m) \leq r_m$ for all $m \in \mathbb{N}$. In other words, $\tilde{x} \in \overline{B(x_m, r_m)}$. But,

$$B(x_n, r_n) \subseteq B(x_{n-1}, r_{n-1}) \quad \forall\, n \in \mathbb{N}$$

so that

$$\tilde{x} \in \overline{B(x_m, r_m)} \subseteq B(x_{m-1}, r_{m-1}) \cap A_m \subseteq B(x_0, r_0) \cap A_m \quad \forall\, m \in \mathbb{N}.$$

Thus,

$$\tilde{x} \in B(x_0, r_0) \bigcap_{m=1}^{\infty} A_m.$$

In particular, $B(x,r) \cap D \neq \varnothing$. □

Why the above theorem is called Baire category theorem?

Definition 159. A metric space X is said to be of **first category** if it can be represented as a countable union of nowhere dense sets. It is said to be of **second category** if it is not of first category. $\diamondsuit$

LEMMA 160. *Let A be a subset of a metric space. Then A is nowhere dense iff $(\bar{A})^c$ is dense.*

THEOREM 161. *A compete metric space is of second category.*

Proof. Let (X, d) is a complete metric space. Suppose it is of first category. Then there exists nowhere dense sets $B_1, B_2, \ldots$ such that $X = \bigcup_{n=1}^{\infty} B_n$. Then, $X = \bigcup_{n=1}^{\infty} \bar{B}_n$ and hence, $\bigcap_{n=1}^{\infty} (\bar{B}_n)^c = \varnothing$. But, since B_n is nowhere dense, $(\bar{B}_n)^c$ is open and dense. Hence, by Theorem 158, we arrive at a contradiction. $\square$

9.3 Arzela-Ascoli's theorem

We know that closed and bounded subset of a metric space need not be compact. For example, if X is an infinite set with discrete metric, then it is closed and bounded, but not compact. More generally, if E is a *discrete subset* of a metric space, then it cannot be compact unless it is a finite set. We may recall that a subset of metric space is called a *discrete set* if every singleton subset of it is an open set. For instance, if $\mathbb{R}$ is with usual metric and $E = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots\}$, then E is not compact. In $\mathbb{R}^n$ with usual metric, a subset is compact iff it is closed and bounded. This result is called the *Heine-Borel theorem*. In particular, the closed unit ball in $\mathbb{R}^n$ is compact. Let us consider a few more examples of metric spaces in which closed and bounded sets need not be compact. For $1 \le p \le \infty$, let $X = \ell^p(\mathbb{N})$ with metric d defined by $d(x, y) := \|x - y\|_p$ for $x, y \in \ell^p(\mathbb{N})$, where we used the notation

$$\|x\|_p := \begin{cases} \left(\displaystyle\sum_{i=1}^{\infty} |x(i)|^p \right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \sup\{|x(i) : i \in \mathbb{N}\}, & p = \infty. \end{cases}$$

For $n \in \mathbb{N}$, let $e_n(j) := \delta_{nj}$, $j \in \mathbb{N}$. Then e_n belongs to the unit circle $S := \{x \in \ell^p(\mathbb{N}) : \|x\|_p = 1\}$. Note also that $\|e_n - e_m\|_p = 1$ for every $n \neq m$. Hence, (u_n) does not have a Cauchy subsequence. Thus, S is not compact.

In this section we consider a characterization of compact subsets of $C(\Omega)$ w.r.t. the sup-metric when Ω is a compact metric space. Essentially we would like to give a characterization for the totally bounded subsets of $C(\Omega)$, which is known as the *Arzela-Ascoli's theorem*.

Clearly, if Ω is a finite set with k elements, then $C(\Omega)$ can be identified with $\mathbb{R}^k$ with sup-metric. Hence, in this case, a subset of $C(\Omega)$ is compact iff it is closed and bounded.

What about if Ω with infinite number of elements?

In this case, a closed and bounded set in $C(\Omega)$ need not be compact. Consider, for example, $\Omega = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$. Then the unit circle $S := \{x \in C(\Omega) : \|x\|_\infty = 1\}$ in $C(\Omega)$ is closed and bounded. It is not compact. To see this, let $u_n(\frac{1}{j}) = \delta_{jn}$ for $j, n \in \mathbb{N}$ and $u_n(0) = 0$. Then $u_n \in S$ for all $n \in \mathbb{N}$ and $\|u_n - u_m\|_\infty = 1$ for every $n \neq m$. Thus, (u_n) does not have a Cauchy sequence, and hence S is not compact.

Also, consider the case $\Omega = [0, 1]$ and $X = C(\Omega)$ with sup-metric. Let (t_n) be a sequence in $(0, 1)$ such that $t_1 > t_2 > \cdots$ (for example $t_n = \frac{1}{n+1}$, $n \in \mathbb{N}$). Let u_n be the hat function centered at $t_n + 1$, that is,

$$u_n(t) := \begin{cases} \dfrac{t - t_{n+2}}{t_{n+1} - t_{n+2}}, & t_{n+2} \leq t \leq t_{n+1}, \\ \\ \dfrac{t_n - t}{t_n - t_{n+1}}, & t_{n+1} \leq t \leq t_n, \\ \\ 0, & \text{elesewhere} \end{cases}$$

Then, it can be easily seen that $\|u_n\|_\infty = 1$ for all $n \in \mathbb{N}$ and $\|u_n - u_m\|_\infty = 1$ for all $n \neq m$, so that (u_n) is a bounded sequence in X which does not have a convergent subsequence. Consequently, The set $\{u \in C[0,1] : \|u\|_\infty = 1\}$ is not compact.

The Arzela-Ascoli's theorem gives a characterization of compact subsets of $C(\Omega)$ with sup-metric whenever Ω is a compact metric space.

Definition 162. Let (Ω, δ) be a metric space and $\mathcal{F}$ be a family of functions from Ω to $\mathbb{R}$. The family $\mathcal{F}$ is said to be **equicontinuous** if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x, y \in \Omega, \quad d(x,y) < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \varepsilon \quad \forall\, f \in \mathcal{F}.$$

$\diamondsuit$

Exercise 163. Let $\mathcal{F}$ be a family of functions from Ω to $\mathbb{R}$. If $\mathcal{F}$ is equicontinuous, then every $f \in \mathcal{F}$ is uniformly continuous. $\diamondsuit$

THEOREM 164. (Arzela-Ascoli's theorem) *Let (Ω, d) be a compact metric space, $X := C(\Omega)$ be with sup-metric and $E \subseteq C(\Omega)$. Then E is totally bounded iff it is bounded and equicontinuous.*

Proof. $\Rightarrow$): Assume that E is totally bounded. Clearly, E is bounded. For showing its equicontinuity, let $\varepsilon > 0$ be given. Since E is totally bounded, there exist $f_1, \ldots, f_k$ in E such that $E \subseteq \bigcup_{i=1}^k B_X(f_i, \varepsilon)$. Let f be any arbitrary element in E and let $s, t \in \Omega$. Then there exists $j \in \{1, \ldots, k\}$ such that $f \in B_X(f_j, \varepsilon)$. Thus,

$$\begin{aligned} |f(s) - f(t)| &\leq |f(s) - f_j(s)| + |f_j(s) - f_j(t)| + |f_j(t) - f(t)| \\ &\leq \|f - f_j\|_\infty + |f_j(s) - f_j(t)| + \|f - f_j\|_\infty \\ &\leq 2\varepsilon + |f_j(s) - f_j(t)|. \end{aligned}$$

Since f_j is continuous and Ω is compact, it is uniformly continuous. Hence, there exists $\delta_j > 0$ such that $|f_j(s) - f_j(t)| < \varepsilon$ for every $s, t \in \Omega$ with $d(s,t) < \delta_j$. Thus, $s, t \in \Omega$ with $d(s,t) < \delta_j$ implies $|f(s) - f(t)| < 3\varepsilon$ for all $f \in E$.

$\Leftarrow$): Assume that E is bounded and equicontinuous. To show that it is totally bounded. For this, let $\varepsilon > 0$ be given. Let $\delta > 0$be such that $|f(t) - f(s)| < \varepsilon$ whenever $s, t \in \Omega$ and $d(s,t) < \delta$. Since Ω is totally bounded, there exist $t_1, \ldots, t_n \in \Omega$ such that $\Omega = \bigcup_{i=1}^k B_\Omega(t_i, \delta)$. For each $i \in \{1, \ldots, n\}$, let $A_i := \{f(t_i) : f \in E\}$. Since E is bounded, there exists $M > 0$ such that $\|f\|_\infty \leq M$ for all $f \in E$. In particular, $|f(t_i)| \leq M$ for all $f \in E$. Thus, A_i is bounded in $\mathbb{R}$ for every $i = 1, \ldots, n$. Since every bounded subset of $\mathbb{R}$ is totally bounded, there exist $f_1^{(i)}, \ldots, f_{m_i}^{(i)}$ in E such that $A_i \subseteq \bigcup_{j=1}^{m_i} B_\mathbb{R}(f_j^{(i)}(t_i), \varepsilon)$. Now, let f be an arbitrary element in E and $t \in \Omega$. Let $i \in \{1, \ldots, n\}$ be such that $d(t, t_i) < \delta$ and . Then we have

$$|f(t) - f_j^{(i)}(t)| \leq |f(t) - f(t_i)| + |f(t_i) - f_j^{(i)}(t_i)| + |f_j^{(i)}(t_i) - f_j^{(i)}(t)|.$$

Let $j \in \{1, \ldots, m_i\}$ be such that $|f(t_i) - f_j^{(i)}(t_i)| < \varepsilon$. Since E is equicontinuous, there exists $\delta > 0$ such that $|f(t) - f(s)| < \varepsilon$ whenever $s, t \in \Omega$ and $d(s, t) < \delta$. Then, we have

$$|f(t) - f_j^{(i)}(t)| \leq |f(t) - f(t_i)| + |f(t_i) - f_j^{(i)}(t_i)| + |f_j^{(i)}(t_i) - f_j^{(i)}(t)| < 3\varepsilon.$$

This is true for all $t \in \Omega$. Hence, $\|f - f_j^{(i)}\|_\infty < 3\varepsilon$ so that $E \subseteq \bigcup_{i=1}^n \bigcup_{j=1}^{m_i} B_X(f_j^{(i)}, 3\varepsilon)$. □

COROLLARY 165. *Let Ω be a compact metric space, $C(\Omega)$ be with sup-metric, and let E be a closed and bounded subset of $C(\Omega)$. Then E is compact iff it is equicontinuous.*

9.4 Weierstrass approximation theorem

THEOREM 166. *For $f \in C[0,1]$ and $n \in \mathbb{N}$, let*

$$(B_n f)(x) := \sum_{k=1}^{n} f(\frac{k}{n}) \binom{n}{k} x^k (1-x)^{n-k}, \quad x \in [0,1].$$

Then

$$\|f - B_n f\|_\infty \to 0 \quad as \quad n \to \infty.$$

In particular, the set of polynomial functions on $[0,1]$ is dense in $C[0,1]$ with respect tot he sup-metric.

Proof. Let $r_k(x) = \binom{n}{k} x^k (1-x)^{n-k}$ for $0 \leq x \leq 1$. We observe that $\sum_{k=0}^n r_k(x) = 1$. Also, we require the identity

$$\sum_{k=0}^{n} (k - nx)^2 r_k(x) = nx(1-x). \qquad (*)$$

Let $p_n(x) = B_n f(x)$. Then we have

$$f(x) - p_n(x) = \sum_{k=0}^{n} [f(x) - f(\frac{k}{n})] r_k(x).$$

Let $\varepsilon > 0$ be given. Since f is uniformly continuous on $[0,1]$, there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon \quad \text{whenver} \quad |x - y| < \delta.$$

Let

$$A_x := \{k : |x - \frac{k}{n}| < \delta\}, \quad B_x := \{k : |x - \frac{k}{n}| \geq \delta\}.$$

Thus, $k \in A_x$ implies $|f(x) - f(\frac{k}{n})| < \varepsilon$. Also, $|f(x) - f(\frac{k}{n})| \leq 2\|f\|_\infty$. Thus, we have

$$\begin{aligned} |f(x) - p_n(x)| &\leq \sum_{k \in A_x} |f(x) - f(\frac{k}{n})| r_k(x) + \sum_{k \in B_x} |f(x) - f(\frac{k}{n})| r_k(x) \\ &\leq \varepsilon + 2\|f\|_\infty \sum_{k \in B_x} r_k(x). \end{aligned}$$

Note that

$$x \in B_x \Rightarrow |x - \frac{k}{n}| \geq \delta \iff |nx - k| \geq n\delta.$$

Hence from $(*)$,

$$nx(1-x) = \sum_{k=0}^{n}(k-nx)^2 r_k(x) \geq (n\delta)^2 \sum_{k\in B_x} r_k(x).$$

Thus,

$$\sum_{k\in B_x} r_k(x) \leq \frac{nx(1-x)}{n^2\delta^2} \leq \frac{1}{4n\delta^2}.$$

Therefore,

$$|f(x)-p_n(x)| \leq \varepsilon + 2\|f\|_\infty \sum_{k\in B_x} r_k(x) \leq \varepsilon + \frac{2\|f\|_\infty}{4n\delta^2}.$$

Let N be such that $\frac{2\|f\|_\infty}{4n\delta^2} < \varepsilon$ for all $n \geq N$. Then we obtain

$$|f(x)-p_n(x)| \leq 2\varepsilon \quad \forall\, n \geq N.$$

It remains to prove $(*)$:

$$\begin{aligned}
\sum_{k=0}^{n}(k-nx)^2 r_k(x) &= \sum_{k=0}^{n}(k^2-2knx+n^2x^2)r_k(x) \\
&= \sum_{k=0}^{n} k^2 r_k(x) - \sum_{k=0}^{n} 2knx r_k(x) + \sum_{k=0}^{n} n^2x^2 r_k(x) \\
&= \sum_{k=0}^{n} k^2 r_k(x) - 2nx \sum_{k=0}^{n} k\, r_k(x) + n^2x^2 \sum_{k=0}^{n} r_k(x) \\
&= \sum_{k=0}^{n} k^2 r_k(x) - 2nx \sum_{k=0}^{n} k\, r_k(x) + n^2x^2.
\end{aligned}$$

Note that for any $x, y \in \mathbb{R}$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Differentiating w.r.t. x,

$$n(x+y)^{n-1} = \sum_{k=1}^{n} k\binom{n}{k} x^{k-1} y^{n-k},$$

$$n(n-1)(x+y)^{n-2} = \sum_{k=2}^{n} k(k-1)\binom{n}{k} x^{k-2} y^{n-k}.$$

Evaluating the above at $y = 1-x$,

$$n = \sum_{k=1}^{n} k\binom{n}{k} x^{k-1}(1-x)^{n-k},$$

$$n(n-1) = \sum_{k=2}^{n} k(k-1)\binom{n}{k} x^{k-2}(1-x)^{n-k}.$$

$\Rightarrow$

$$nx = \sum_{k=1}^{n} k\binom{n}{k} x^k (1-x)^{n-k} = \sum_{k=0}^{n} k r_k(x),$$

$$\begin{aligned}
n(n-1)x^2 &= \sum_{k=0}^{n} k(k-1)\binom{n}{k} x^k (1-x)^{n-k} \\
&= \sum_{k=0}^{n} k(k-1) r_k(x) \\
&= \sum_{k=0}^{n} k^2 r_k(x) - \sum_{k=2}^{n} k\, r_k(x) \\
&= \sum_{k=0}^{n} k^2 r_k(x) - nx.
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{k=0}^{n} (k-nx)^2 r_k(x) &= \sum_{k=0}^{n} k^2 r_k(x) - 2nx \sum_{k=0}^{n} k\, r_k(x) + n^2x^2 \\
&= [n(n-1)x^2 + nx] - 2nx(nx) + n^2x^2 \\
&= nx(1-x).
\end{aligned}$$

□

10 Power Series

Let (a_n) be a sequence in $\mathbb{R}$. Consider the **power series** $\sum_{n=0}^{\infty} a_n x^n$.

THEOREM 167. (Abel's theorem) *Suppose the power series $\sum_{n=0}^{\infty} a_n x^n$ converges for some $x_0 \neq 0$. Then it converges absolutely at every x with $|x| < |x_0|$.*

Proof. Note that

$$|a_n x^n| = |a_n x_0^n| \left|\frac{x}{x_0}\right|^n \quad \forall\, n \in \mathbb{N}.$$

If $|x| < |x_0|$, then $\sum_{n=0}^{\infty} \left|\frac{x}{x_0}\right|^n$ converges. Hence, by comparison test, $\sum_{n=0}^{\infty} |a_n x^n|$ converges. □

COROLLARY 168. *Suppose the power series $\sum_{n=0}^{\infty} a_n x^n$ diverges at $x_1 \neq 0$. Then it diverges at every x with $|x| > |x_1|$.*

Let

$$D := \{x \in \mathbb{R} : \sum_{n=0}^{\infty} a_n x^n \text{ converges}\},$$

and let

$$R := \sup\{|x| : x \in D\}.$$

Then we have the following:

1. If $R = 0$, then $\sum_{n=0}^{\infty} a_n x^n$ only at $x = 0$.
2. If $0 < R \leq \infty$, then $\sum_{n=0}^{\infty} a_n x^n$ at every x with $|x| < R$.

Definition 169. In the above,

(i) the set D is called the **domain of convergence** of $\sum_{n=0}^{\infty} a_n x^n$,

(ii) R is called the **radius of convergence** of $\sum_{n=0}^{\infty} a_n x^n$, and

(iii) if $R > 0$, then $(-R, R)$ is called the **interval of convergence** of $\sum_{n=0}^{\infty} a_n x^n$. ♢

We may recall the following:

THEOREM 170. (d'Alembert's ratio test) *Suppose $b_n \neq 0$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$ such that either $\left(\left|\frac{b_{n+1}}{b_n}\right|\right)$ converges or diverges to ∞. Let $\ell := \lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right|$.*

(i) If $\ell < 1$, then $\sum_{n=1}^{\infty} b_n$ converges absolutely.

(ii) If $\ell > 1$, then $\sum_{n=1}^{\infty} b_n$ diverges.

THEOREM 171. (Cauchy's root test) *Let (b_n) be a sequence in $\mathbb{R}$ such that either $(|b_n|^{1/n})$ converges or diverges to ∞. Let $\ell := \lim_{n\to\infty} |b_n|^{1/n}$.*

(i) If $\ell < 1$, then $\sum_{n=1}^{\infty} b_n$ converges absolutely.

(ii) If $\ell > 1$, then $\sum_{n=1}^{\infty} b_n$ diverges.

The following two theorems are immediate consequences of Theorems 170 and 171.

THEOREM 172. *Suppose $a_n \neq 0$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$ such that either $\left(\left|\frac{a_{n+1}}{a_n}\right|\right)$ converges or diverges to ∞. Let $\ell := \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. Then we have the following:*

1. *If $\ell = 0$, then the power series $\sum_{n=0}^{\infty} a_n x^n$ converges for every $x \in \mathbb{R}$.*
2. *If $\ell = \infty$, then the power series $\sum_{n=0}^{\infty} a_n x^n$ converges only at $x = 0$.*
3. *If $0 < \ell < \infty$, then the power series $\sum_{n=0}^{\infty} a_n x^n$ converges for every x with $|x| < 1/\ell$ and diverges for every x with $|x| > 1/\ell$.*

THEOREM 173. *Let (a_n) be a sequence in $\mathbb{R}$ such that either $(|a_n|^{1/n})$ converges or diverges to ∞. Let $\ell := \lim_{n\to\infty} |a_n|^{1/n}$. Then we have the following:*

1. *If $\ell = 0$, then the power series $\sum_{n=0}^{\infty} a_n x^n$ converges for every $x \in \mathbb{R}$.*
2. *If $\ell = \infty$, then the power series $\sum_{n=0}^{\infty} a_n x^n$ converges only at $x = 0$.*
3. *If $0 < \ell < \infty$, then the power series $\sum_{n=0}^{\infty} a_n x^n$ converges for every x with $|x| < 1/\ell$ and diverges for every x with $|x| > 1/\ell$.*

There are situation in which the limits in the above theorems may not exist or if exists then the limit can be 1.

Example 174. Let $b_n = \begin{cases} 1, & n \text{ even}, \\ 2, & n \text{ odd}. \end{cases}$ Then $\left|\frac{b_{n+1}}{b_n}\right| = \begin{cases} 2, & n \text{ even}, \\ 1, & n \text{ odd} \end{cases}$ so that $\left(\left|\frac{b_{n+1}}{b_n}\right|\right)$ is bounded but $\lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right|$ does not exist. Note that $|b_n|^{1/n} = \begin{cases} 1, & n \text{ even}, \\ 2^{1/n}, & n \text{ odd} \end{cases}$ so that $\lim_{n\to\infty} |b_n|^{1/n} = 1$. Thus, Theorem 172 cannot be applied, whereas Theorem 173 can be applied. ♢

LEMMA 175. *Suppose (b_n) is a sequence of non-negative real numbers and $\alpha := \limsup_n b_n$, i.e., $\alpha := \lim_{k\to\infty} \sup_{n\geq k} b_n$. Then we have the following:*

(i) If $\alpha < \beta$, then there exists $N \in \mathbb{N}$ such that $b_n \leq \beta$ for all $n \geq N$.

(ii) If $\alpha > \beta$ then there exists a subsequence (b_{k_n}) such that $b_{k_n} \geq \beta$ for all $n \in \mathbb{N}$.

Proof. Let $c_k = \sup_{n\geq k} b_n$.

(i) Suppose $\alpha := \lim_{k\to\infty} \sup_{n\geq k} b_n = \lim_{k\to\infty} c_k < \beta$. Then there exists k_0 such that $c_k := \sup_{n\geq k} b_n < \beta$ for all $k \geq k_0$. Hence, $b_k < \beta$ for all $k \geq k_0$.

(ii) Suppose $\alpha := \lim_{k\to\infty} \sup_{n\geq k} b_n = \lim_{k\to\infty} c_k > \beta$. Then Then there exists k_0 such that $c_k := \sup_{n\geq k} b_n > \beta$ for all $k \geq k_0$. Hence, for each $k \geq k_0$, there exists n_k such that $b_{n_k} > \beta$. $\square$

The following two theorem can be proved (Exercise) using the above lemma.

THEOREM 176. (Cauchy's Root test) *Let $\ell := \limsup_n |b_n|^{1/n}$.*

1. *If $\ell < 1$, then $\sum_{n=1}^{\infty} b_n$ converges absolutely.*
2. *If $\ell > 1$, then $\sum_{n=1}^{\infty} b_n$ diverges.*

The following theorem is an immediate consequence of Theorem 176.

THEOREM 177. *Let $\ell := \limsup_n |a_n|^{1/n}$ and $R := 1/\ell$. Then we have the following:*

1. *If $\ell = 0$, then the power series $\sum_{n=0}^{\infty} a_n x^n$ converges for every $x \in \mathbb{R}$.*
2. *If $\ell = \infty$, then the power series $\sum_{n=0}^{\infty} a_n x^n$ converges only at $x = 0$.*
3. *If $0 < \ell < \infty$, then the power series $\sum_{n=0}^{\infty} a_n x^n$ converges for every x with $|x| < R$ and diverges for every x with $|x| > R$.*

Using the convention $1/0 = \infty$ and $1/\infty = 0$, $R := \dfrac{1}{\limsup_n |a_n|^{1/n}}$ is the radius of convergence of the series $\sum_{n=0}^{\infty} a_n x^n$.

Example 178. Let $a_n = \begin{cases} 1, & n \text{ even}, \\ 2^n, & n \text{ odd}. \end{cases}$ Then $|a_n|^{1/n} = \begin{cases} 1, & n \text{ even}, \\ 2, & n \text{ odd} \end{cases}$ so that $\lim_{n\to\infty} |a_n|^{1/n}$ does not exist. Thus, Theorem 173 cannot be applied. But, $\limsup_n |a_n|^{1/n} = 2$. Hence Theorem 177 can be applied. $\diamondsuit$

THEOREM 179. *Let $R > 0$ be the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$. Let $f(x) := \sum_{n=0}^{\infty} a_n x^n$ for $x \in (-R, R)$. Then for $0 < r < R$, $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-r, r]$, and f is continuous on $(-R, R)$.*

Proof. Let $0 < r < \rho < R$. Then for $|x| \leq r$,

$$|a_n x^n| = |a\rho^n| \left|\frac{x}{\rho}\right|^n \leq |a\rho^n| \left|\frac{r}{\rho}\right|^n \leq M \left|\frac{r}{\rho}\right|^n,$$

where M is a bound for $(|a_n\rho^n|)$. Since $\left|\frac{r}{\rho}\right| < 1$, it follows that $\sum_{n=0}^{\infty}|a_nx^n|$ converges uniformly on $[-r,r]$. Hence, f is continuous on $[-r,r]$. Now, let $x_0 \in (-R,R)$. Taking $r > 0$ such that $x_0 \in [-r,r] \subseteq (-R,R)$, f is continuous at x_0. □

THEOREM 180. *Let $R > 0$ be the radius of convergence of $\sum_{n=0}^{\infty} a_nx^n$. Let $f(x) := \sum_{n=0}^{\infty} a_nx^n$ for $x \in (-R,R)$. Then we have the following.*

1. *For $[a,b] \subseteq (-R,R)$, $\int_a^b f(x)dx = \sum_{n=0}^{\infty} a_n \frac{b^{n+1}-a^{n+1}}{b-a}$*
2. *f is differentiable on $(-R,R)$, and its derivative is given by $f'(x) = \sum_{n=1}^{\infty} na_nx^{n-1}$ on $(-R,R)$.*

We make use of the following:

PROPOSITION 181. *Let (f_n) be a sequence of continuous functions on $[a,b]$.*

1. *Suppose (f_n) converges uniformly on $[a,b]$ and $f(x) := \lim_{n\to\infty} f_n(x)$, $x \in [a,b]$. Then f is continuous and $\int_a^b f_n(x)dx \to \int_a^b f(x)dx$.*
2. *Suppose f_n is differentiable for each $n \in \mathbb{N}$ with f_n' continuous on $[a,b]$, (f_n') converges uniformly on $[a,b]$ and (f_n) converges at some point $x_0 \in [a,b]$. Then (f_n) converges uniformly on $[a,b]$, the function f defined by $f(x) := \lim_{n\to\infty} f_n(x)$, $x \in [a,b]$, is differentiable and $f' = g$, where $g(x) := \lim_{n\to\infty} f_n'(x)$, $x \in [a,b]$.*

Proof. 1. Since $f_n \in C[a,b]$ for all $n \in \mathbb{N}$ and (f_n) converges uniformly on $[a,b]$ and since $C[a,b]$ is a closed subset of $B[a,b]$ with respect to $\|\cdot\|_\infty$, $f \in C[a,b]$. Hence,

$$\left|\int_a^b f_n(x)dx - \int_a^b f(x)dx\right| \le \int_a^b |f_n(x) - f(x)|dx.$$

Let $\varepsilon > 0$ be given. Since $f_n \to f$ uniformly on $[a,b]$, there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \ge N$ and for all $x \in [a,b]$. Hence,

$$\left|\int_a^b f_n(x)dx - \int_a^b f(x)dx\right| \le \int_a^b |f_n(x) - f(x)|dx < (b-a)\varepsilon \quad \forall\, n \ge N.$$

2. By the fundamental theorem of Riemann integration,

$$f_n(x) = f_n(x_0) - \int_{x_0}^x f_n'(t)dt.$$

Let $\alpha := \lim_{n\to\infty} f_n(x_0)$. Then we have

$$f_n(x) = f_n(x_0) - \int_{x_0}^x f_n'(t)dt \to \alpha + \int_{x_0}^x g(t)dt.$$

Let

$$f(x) := \alpha + \int_{x_0}^x g(t)dt, \quad x \in [a,b].$$

Then, $f(x_0) = \alpha$ and

$$f_n(x) - f(x) = [f_n(x_0) - \alpha] + \int_{x_0}^x [f_n'(t) - g(t)]dt.$$

Let $\varepsilon > 0$ be given. Since $f_n' \to g$ uniformly on $[a,b]$, $g \in C[a,b]$ and there exists $N_1 \in \mathbb{N}$ such that $|f_n'(x) - g(x)| < \varepsilon$ for all $n \geq N$ and for all $x \in [a,b]$. Also, there exists $N_2 \in \mathbb{N}$ such that $|f_n(x_0) - \alpha| < \varepsilon$ for all $n \geq N$. Hence,

$$|f_n(x) - f(x)| \leq |f_n(x_0) - \alpha| + \int_{x_0}^{x} |f_n'(t) - g(t)|dt < \varepsilon + \varepsilon(b-a) \quad \forall\, n \geq N \,\forall\, x \in [a,b].$$

Hence $f_n \to f$ uniformly on $[a,b]$. Also, since $f(x) := \alpha + \int_{x_0}^{x} g(t)dt$ for all $x \in [a,b]$, f is differentiable and $f' = g$. □

Proof of Theorem 180. Let $f(x) := \sum_{n=0}^{\infty} a_n x^n$ and $f_k(x) := \sum_{n=0}^{k} a_n x^n$ for $x \in (-R,R)$. Let $0 < r < R$ such that $[-r,r] \subseteq (-R,R)$. Then, by Theorem 179, $f_n \to f$ uniformly on $[-r,r]$.

1. We may take r such that $[a,b] \subseteq [-r,r] \subseteq (-R,R)$. By Proposition 181,

$$\int_a^b f(x)dx = \lim_{k\to\infty} \int_a^b \left(\sum_{n=0}^{k} a_n x^n \right) dx = \lim_{k\to\infty} \sum_{n=0}^{k} \int_a^b a_n x^n = \lim_{k\to\infty} \sum_{n=0}^{k} a_n \frac{b^{k+1} - a^{k+1}}{k+1}.$$

2. We have

$$f_k'(x) = \sum_{n=1}^{k} n a_n x^{n-1}, \quad x \in (-R,R).$$

Since

$$\limsup_n |n a_n|^{1/n} = \limsup_n n^{1/n} |a_n|^{1/n} = \limsup_n |a_n|^{1/n},$$

$R := 1/\left(\limsup_n |a_n|^{1/n}\right)$ is the radius of convergence of $\sum_{n=1}^{\infty} n a_n x^{n-1}$ as well. Hence, $\sum_{n=1}^{\infty} n a_n x^{n-1}$ converges for all $|x| < R$ and diverges for $|x| > R$. Let $g(x) := \sum_{n=1}^{\infty} n a_n x^{n-1}$, $|x| < R$. Then, using the arguments as in (1) above, $f_k' \to g$ uniformly on $[-r,r]$, and by Proposition 181, $f' = g$ on $[-r,r]$.

If $x_0 \in (-R,R)$, we may take r such that $x_0 \in (-r,r) \subseteq [-r,r] \subseteq (-R,R)$ so that $f'(x_0) = g(x_0)$. This completes the proof. □

11 Fourier Series

11.1 Trigonometric polynomials

Definition 182. A function of the form

$$f(x) := c_0 + \sum_{n=1}^{k} [a_n \cos nx + b_n \sin nx]$$

is called a **trigonometric polynomial**. ♢

Note that f is a 2π-periodic function, i.e., $f(x + 2\pi) = f(x)$ for all $x \in \mathbb{R}$, and for $m, n \in \mathbb{N}$,

- $\displaystyle \frac{1}{\pi} \int_{-\pi}^{\pi} \cos mx \cos nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin mx \sin nx dx = \begin{cases} 1, & m = n, \\ 0, & m \neq n \end{cases}$
- $\displaystyle \frac{1}{\pi} \int_{-\pi}^{\pi} \cos mx \sin nx dx = \begin{cases} 1, & m = n, \\ 0, & m \neq n \end{cases}$

Hence,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = c_0, \quad \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx dx = a_m, \quad \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx dx = b_m.$$

Since

$$\cos nx = \frac{e^{inx}+e^{-inx}}{2}, \quad \sin nx = \frac{e^{inx}-e^{-inx}}{2i},$$

we obtain

$$\begin{aligned} f(x) &= c_0 + \sum_{n=1}^{k}[a_n\cos nx + b_n \sin nx] \\ &= c_0 + \sum_{n=1}^{k}[a_n\left(\frac{e^{inx}+e^{-inx}}{2}\right) + b_n\left(\frac{e^{inx}-e^{-inx}}{2i}\right)] \\ &= \sum_{n=-k}^{k} c_n e^{inx}, \end{aligned}$$

where, for $n = 1, 2, \ldots, k$,

$$c_n = \frac{a_n}{2}+\frac{b_n}{2i} = \frac{a_n - ib_n}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)[\cos nx - i\sin nx]dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-nx}dx,$$

$$c_{-n} = \frac{a_n}{2}-\frac{b_n}{2i} = \frac{a_n + ib_n}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)[\cos nx + i\sin nx]dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{nx}dx,$$

so that

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-nx}dx \quad \text{for} \quad n = \pm 1, \pm 2, \ldots, \pm k.$$

THEOREM 183. *A function defined on $\mathbb{R}$ is a trigonometric polynomial iff it is of the form $\sum_{n=-k}^{k} c_n e^{inx}$ with $c_n \in \mathbb{C}$.*

Proof. $\Rightarrow$): This part we have already proved.

$\Leftarrow$): Exercise. □

Also, observe that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{nx}e^{-mx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(n-m)x}dx = \begin{cases} 1, & m = n, \\ 0, & m \neq n \end{cases}$$

Let

$$\varphi_0 = \frac{1}{\sqrt{2\pi}}, \quad \varphi_{2n-1} = \frac{1}{\sqrt{\pi}}\sin nx, \quad \varphi_{2n}(x) := \frac{1}{\sqrt{\pi}}\cos nx, \quad n \in \mathbb{N},$$

and

$$e_n(x) := \frac{1}{\sqrt{2\pi}}e^{inx}, \quad n \in \mathbb{Z}.$$

Then the sets $\{\varphi_n : n \in \mathbb{N}_0\}$ and $\{e_n : n \in \mathbb{Z}\}$ are orthonormal in the sense that

$$\langle \varphi_n, \varphi_m \rangle := \int_{-\pi}^{\pi} \varphi_n(x)\overline{\varphi_m(x)}dx = \begin{cases} 1, & m = n, \\ 0, & m \neq n \end{cases}$$

$$\langle e_n, e_m \rangle := \int_{-\pi}^{\pi} e_n(x)\overline{e_m(x)}dx = \begin{cases} 1, & m = n, \\ 0, & m \neq n \end{cases}$$

11.2 Trigonometric series

Definition 184. A series of the form $c_0 + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$ with $a_n, b_n \in \mathbb{R}$ is called a trigonometric series. ♢

THEOREM 185. *Every trigonometric series is of the form* $\sum_{n=-\infty}^{\infty} c_n e^{inx}$*. Conversely every series of the form* $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ *is a trigonometric series.*

Suppose the trigonometric series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ converges at some point $x \in \mathbb{R}$. Then it also converges at $x + 2\pi$. Thus, it is enough to discuss its convergence in an interval of length 2π, in particular in $[-\pi, \pi]$.

THEOREM 186. *Suppose* $c_n \in \mathbb{C}$ *such that* $\sum_{n\in\mathbb{Z}} |c_n|$ *converges. Then* $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ *converges uniformly on* $[-\pi, \pi]$*, and in that case* $c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-nx}dx$ *and*

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-nx} \quad \textit{for} \quad n = \pm 1, \pm 2, \ldots, \pm k.$$

Definition 187. The trigonometric series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ is called a Fourier series of a 2π-periodic function f which is integrable on $[-\pi, \pi]$ if $c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-nx}dx$ and

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-nx} \quad \text{for} \quad n = \pm 1, \pm 2, \ldots, \pm k,$$

and in that case we write

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

♢

THEOREM 188. *Suppose* f *is square integrable on* $[-\pi, \pi]$ *and* $f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$*. If* $f_k(x) := \sum_{n=-k}^{k} c_n e^{inx}$*, then*

$$\int_{-\pi}^{\pi} |f(x) - f_k(x)|^2 dx \le \int_{-\pi}^{\pi} |f(x) - g(x)|^2 dx$$

for every g *of the form* $g(x) := \sum_{n=-k}^{k} d_n e^{inx}$ *with* $d_n \in \mathbb{C}$, $n = 0, \pm 1, \cdots, \pm n$*. Further,*

$$\sum_{n=-k}^{k} |c_n|^2 \le \int_{-\pi}^{\pi} |f(x)|^2 dx \quad \forall\, k \in \mathbb{N},$$

and consequently, $c_n \to 0$ *as* $|n| \to \infty$*.*

Proof. For any square integrable functions φ, ψ on $[-\pi, \pi]$, let us denote

$$\langle \varphi, \psi \rangle := \int_{-\pi}^{\pi} \varphi(x)\overline{\psi(x)}dx.$$

Let $\varphi_n(x) = \dfrac{e^{inx}}{\sqrt{2\pi}}$, $n \in \mathbb{Z}$. Then we have $\langle \varphi_n, \varphi_n \rangle = \delta_{nm}$. Let $g(x) := \sum_{n=-k}^{k} d_n e^{inx}$ with complex numbers $d_n\, n = 0, \pm 1, \cdots, \pm n$. Note that

$$\begin{aligned}
\int_{-\pi}^{\pi} |f(x) - g)|^2 dx &= \langle f - g, f - g \rangle \\
&= \langle (f - f_k) + (f_k - g), (f - f_k) + (f_k - g) \rangle \\
&= \langle f - f_k, f - f_k \rangle + \langle f - f_k, f_k - g) \rangle + \langle f_k - g, f - f_k \rangle + \langle f_k - g, f_k - g \rangle.
\end{aligned}$$

Since

$$f_k(x) - g(x) = \sum_{n=-k}^{k} (c - d_n) e^{inx} = \sqrt{2\pi} \sum_{n=-k}^{k} (c - d_n) \varphi_n(x)$$

Hence, using the *orthogonality property* $\langle \varphi_n, \varphi_n \rangle = \delta_{nm}$, we have

$$\langle f - f_k, f_k - g) \rangle = 0 = \langle f_k - g, f - f_k \rangle = 0.$$

Thus,

$$\begin{aligned}
\int_{-\pi}^{\pi} |f(x) - g(x))|^2 dx &= \langle f - f_k, f - f_k \rangle + \langle f_k - g, f_k - g \rangle \\
&= \int_{-\pi}^{\pi} |f(x) - f_k(x))|^2 dx + \int_{-\pi}^{\pi} |f_k(x) - g(x))|^2 dx \\
&\geq \int_{-\pi}^{\pi} |f(x) - f_k(x))|^2 dx.
\end{aligned}$$

Taking $g = 0$ in

$$\int_{-\pi}^{\pi} |f(x) - g(x))|^2 dx = \int_{-\pi}^{\pi} |f(x) - f_k(x))|^2 dx + \int_{-\pi}^{\pi} |f_k(x) - g(x))|^2 dx$$

we obtain

$$\begin{aligned}
\int_{-\pi}^{\pi} |f(x)|^2 dx &= \int_{-\pi}^{\pi} |f(x) - f_k(x))|^2 dx + \int_{-\pi}^{\pi} |f_k(x)|^2 dx \\
&\geq \int_{-\pi}^{\pi} |f_k(x)|^2 dx.
\end{aligned}$$

It can be seen that

$$\int_{-\pi}^{\pi} |f_k(x)|^2 dx = \sum_{n=-k}^{k} |c_n|^2.$$

Thus,

$$\sum_{n=-k}^{k} |c_n|^2 \leq \int_{-\pi}^{\pi} |f(x)|^2 dx \quad \forall\, k \in \mathbb{N},$$

and consequently, $c_n \to 0$ as $|n| \to \infty$. □

Now a theorem on convergence.

THEOREM 189. *Let f be square integrable on $[-\pi,\pi]$ and $f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$. Let $x \in (-\pi,\pi)$. If there exists $\delta > 0$ and $M > 0$ such that*

$$|f(x+t) - f(x) \leq M|t| \quad \forall\, t \in (-\delta,\delta),$$

then $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ converges to $f(x)$.

Proof. First we observe that

$$\begin{aligned} f_k(x) &:= \sum_{n=-k}^{k} c_n e^{inx} \\ &= \sum_{n=-k}^{k} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-iny}dy\right)e^{inx} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)D_k(x-y)dy \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)D_k(y)dy, \end{aligned}$$

where

$$D_k(y) := \sum_{n=-k}^{k} e^{iny},$$

called the *Dirichlet kernel.* Clearly,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_k(y)dt = 1.$$

Further,

$$D_k(y) := \sum_{n=-k}^{k} e^{iny} = \frac{\sin{(k+\frac{1}{2})y}}{\sin\frac{y}{2}},$$

which can be seen by observing that

$$(e^{ix} - 1)D_k(t) = e^{i(k+1)t} - e^{-ikt}.$$

Thus,

$$\begin{aligned} f_k(x) - f(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x-y) - f(y)]D_k(y)dy = \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x-y)-f(y)]\frac{\sin{(k+\frac{1}{2})y}}{\sin\frac{y}{2}}dy \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(x-y)-f(y)}{\sin\frac{y}{2}}\sin{(k+\frac{1}{2})}ydy. \end{aligned}$$

Let $g(y) := \frac{f(x-y)-f(y)}{\sin\frac{y}{2}}$. Then,

$$\begin{aligned} f_k(x) - f(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(y)\sin{(k+\frac{1}{2})}ydy = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(y)[\sin ky\cos\frac{y}{2} + \cos ky\sin\frac{y}{2}]dy \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{[g(y)\cos\frac{y}{2}]\sin ky + [g(y)\sin\frac{y}{2}]\cos ky\right\}dy. \end{aligned}$$

Since $g(y)\cos\frac{y}{2}$ and $g(y)\sin\frac{y}{2}$ are bounded sqyuare integrable functions, by Theorem **??**,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[g(y)\cos\frac{y}{2}]\sin kydy \to 0, \quad \frac{1}{2\pi}\int_{-\pi}^{\pi}[g(y)\sin\frac{y}{2}]\cos kydy \to 0$$

as $k \to \infty$. □

12 Assignments

12.1 Assignment - I

Prove/verify/Justify the following statements:

1. Let $(S, \preceq)$ be a partially ordered set and $A \subseteq S$.
   (a) Least upper bound of A, if exists, is unique.
   (b) Greatest lower bound of A, if exists, is unique.
   (c) If $\prec$ is an order relation on S, then $\preceq$ defined by $x \preceq y$ for either $x \prec y$ or $x = y$, is a partial order.

2. (a) Let $S = \mathbb{N} \times \mathbb{N}$ and for (m, n), (p, q) in S, define $(m, n) \sim (p, q) \iff mq = np$. Then $\sim$ is an equivalence relation on S. In this case the equivalence class of (m, n) is denoted by $\frac{m}{n}$. The set of all equivalence classes is th set of all *positive rational numbers*.
   (b) Let $n \in \mathbb{Z}$. For $a, b \in \mathbb{N}$, define $a \sim b \iff a-b$ is a multiple of n. Then $\sim$ is an equivalence relation on $\mathbb{Z}$.
   (c) Let G be a group and H be a subgroup of G. For $a, b \in G$, define $a \sim b \iff a - b \in H$. Then $\sim$ is an equivalence relation on S.
   (d) Let V be a vector space and W be a subspace of V. For $x, y \in V$, define $x \sim y \iff x - y \in W$. Then $\sim$ is an equivalence relation on V.
   (e) For a nonempty set X let $S = 2^X$. For A, B in S, define $A \preceq B \iff A \subseteq B$. Then $\preceq$ is a partial order on S.
   (f) Let $S = re^{i\theta} : 0 \leq \theta < 2\pi, 0 \leq r \leq 1\}$. For $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$ in S, define $z_1 \preceq z_2 \iff \theta_1 = \theta_2$ and $r_1 \leq r_2$. Then $\preceq$ is a partial order on S. In this case,
      i. for each θ, the set $A_\theta := \{re^{i\theta} : 0 \leq r \leq 1\}$ is a totally ordered subset for which $e^{i\theta}$ is an upper bounded.
      ii. Every $e^{i\theta}$ is a maximal element of S.

3. Let $x \in C[a, b]$. Then there exists $t_0 \in [a, b]$ such that $\sup\limits_{a \leq t \leq b} |x(t)| = |x(t_0)|$.

4. For $x \in C[a, b]$, show that $\int_a^b |x(t)|\, dt = 0 \Rightarrow x(t) = 0$ for all $t \in [a, b]$.

5. For a non-empty subset, let $B(S)$ be the set of all real valued bounded functions defined on S. For x, y in $B(S)$, let $d(x, y) := \sup\{|x(s) - y(s)| : s \in S\}$. Then d is a metric on $B(S)$. What is $B(\mathbb{N})$?

6. Let $1 < p < \infty$ and let $q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. For x, y in $\mathbb{R}^n$,
   (a) $\sum_{k=1}^n |x_k y_k| \leq (\sum_{k=1}^n |x_k|^p)^{\frac{1}{p}} (\sum_{k=1}^n |y_k|^q)^{\frac{1}{q}}$. *(Hölder's inequality)*
   [You may assume the relation $\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}$ for every $\alpha \geq 0$, $\beta \geq 0$.]
   (b) $(\sum_{k=1}^n |x_k + y_k|^p)^{\frac{1}{p}} \leq (\sum_{k=1}^n |x_k|^p)^{\frac{1}{p}} + (\sum_{k=1}^n |y_k|^p)^{\frac{1}{p}}$. *(Minkowski's inequality)*
   (c) $d_p(x, y) := (\sum_{k=1}^n |x_k - y_k|^p)^{\frac{1}{p}}$ defines a metric on $\mathbb{R}^n$.

7. Let $1 < p < \infty$ and let $q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. For x, y in $C[a, b]$,

(a) $\int_a^b |x(t)y(t)|\,dt \le \left(\int_a^b |x(t)|^p dt\right)^{\frac{1}{p}} \left(\int_a^b |y(t)|^q dt\right)^{\frac{1}{q}}$. *(Hölder's inequality)*

[You may assume the relation $\alpha\beta \le \frac{\alpha^p}{p} + \frac{\beta^q}{q}$ for every $\alpha \ge 0$, $\beta \ge 0$.]

(b) $\left(\int_a^b |x(t)+y(t)|^p\,dt\right)^{\frac{1}{p}} \le \left(\int_a^b |x(t)|^p dt\right)^{\frac{1}{p}} + \left(\int_a^b |y(t)|^p dt\right)^{\frac{1}{p}}$. *(Minkowski's inequality)*

(c) $d_p(x,y) := \left(\int_a^b |x(t)-y(t)|^p dt\right)^{\frac{1}{p}}$ defines a metric on $C[a,b]$.

8. Two metrics d and ρ are said to be equivalent if there exist constants $C_1 > 0, C_2 > 0$ such that $C_1 d(x,y) \le \rho(x,y) \le C_2 d(x,y) \quad \forall\, x,y \in X$.

   (a) For any two metrics d and ρ on X, define $d \sim \rho$ iff d and ρ are equivalent. Then $\sim$ is an equivalence relation on the set of all metrics on X.

   (b) For $x,y \in \mathbb{R}^k$, let $d_\infty(x,y) := \max\{|x_k - y_k| : k = 1,\ldots,k\}$. Then for every p with $1 \le p < \infty$, d_p and d_∞ are equivalent metrics on $\mathbb{R}^k$. Further, for any p, r with $1 \le p \le \infty$ and $1 \le r \le \infty$, the metrics d_p and d_ρ on $\mathbb{R}^n$ are equivalent.

   (c) For $x,y \in C[a,b]$, let $d_\infty(x,y) := \max\{|x(t)-y(t)| : a \le t \le b\}$. Then for every p with $1 \le p < \infty$, d_p and d_∞ are not equivalent metrics on $C[a,b]$.

9. Suppose X is a set. A function $\rho \times X \to [0,\infty)$ is called a *pseudo metric* on X if

   (i) $\rho(x,x) = 0 \quad \forall\, x \in X$,

   (ii) $\rho(x,y) = \rho(y,x) \quad \forall\, x,y \in X$,

   (iii) $\rho(x,y) \le \rho(x,z) + \rho(z,y) \quad \forall\, x,y,z \in X$.

   (a) Let $\mathcal{R}[a,b]$ be the set of all real valued Riemann integrable functions defined on $[a,b]$. For $x,y \in C[a,b]$, let $d(x,y) := \int_a^b |x(t)-y(t)|dt$. Then d is a pseudo metric, but not a metric on $\mathcal{R}[a,b]$.

   (b) The function ρ defined by $\rho((x_1,x_2),(y_1,y_2)) = |x_1 - y_1|, \quad (x_1,x_2),(y_1,y_2) \in \mathbb{R}^2$ is a pseudo metric on $\mathbb{R}^2$, but not a metric.

   (c) Let $t_0 \in [a,b]$. Then ρ defined by $\rho(x,y) = |x(t_0) - y(t_0)|, \quad x,y \in C[a,b]$ is a pseudo metric on $C[a,b]$, but not a metric.

10. Let ρ be a pseudo metric on X. For $x,y \in X$, define $x \sim y \iff \rho(x,y) = 0$.

    (a) $\sim$ is an equivalence relation on X.

    (b) $\tilde{X}$ be the set of all equivalence classes. On $\tilde{X}$, define $d([x],[y]) := \rho(x,y)$. Then d is a metric on $\tilde{X}$, called the *metric induced by the pseudo metric* ρ.

    (c) Taking $X = \mathcal{R}[a,b]$ and ρ as in Problem 4(a), what are the equivalence classes of an $x \in \mathcal{R}[a,b]$?

    (d) Taking $X = C[a,b]$ and ρ as in Problem 4(c), what are the equivalence classes of an $x \in C[a,b]$?

11. Let (X,d) be a metric space and $A \subseteq X$. The following are equivalent:

    (i) A is closed.

    (ii) A contains all its closure points.

    (iii) A contains all its limit points.

    (iv) A contains all its boundary points.

    (v) A^c is open.

    (vi) For every (x_n) in A, if $x_n \to x$ for some $x \in X$, then $x \in A$.

12. Let (X, d) be a metric space, $A \subseteq X$.

    (a) If x_0 is a limit point of A, then every open ball containing x_0 contains infinitely many points from A.

    (b) If A is a finite set, then A does not have any limit point.

13. Let (X, d) be a metric space and $A \subseteq X$.

    (i) $\bar{A} = A \cup \partial A$

    (ii) $\bar{A} = A \cup A'$, where A' is the set of all limit points of A.

    (iii) $\bar{A} = \cap\{F : F \text{ is closed in } X\}$.

    (iv) A is open iff $A = A^o$.

    (v) A is closed iff $A = \bar{A}$.

14. If A is a subset of $\mathbb{R}$ and A is bounded above, then $\sup(A) \in \bar{A}$.

15. Let (X, d) be a metric space.

    (i) Every open ball in X is an open set.

    (ii) Union of every arbitrary collection of open sets in X is open in X.

    (iii) Finite intersection of opens sets in X is open in X.

    (iv) Intersection of every arbitrary collection of closed sets in X is closed in X.

    (v) Finite union of of closed sets in X is closed in X.

16. Let (X, d) be a metric space and (x_n) be a sequence in X. Then $x_n \to x$ iff for every $r > 0$, there exists $N \in \mathbb{N}$ such that $x_n \in B(x, r)$ for all $n \geq N$.

17. Let X be with discrete metric.

    (i) Every subset of X is open and closed.

    (ii) A sequence in X is convergent iff it is eventually constant.

    (iii) A sequence in X is a Cauchy sequence iff it is eventually constant.

18. Let X be with discrete metric, (x_n) be a sequence in X and $x \in X$.

    (i) $x_n \to x$ iff every subsequence of (x_n) converges to x.

    (ii) If (x_n) is a sequence, then $x_n \to x$ iff (x_n) has a subsequence which converges to x.

19. (a) Every closed subset of a complete metric space is complete.

    (b) Every complete subset of a metric space is closed.

20. Suppose X_1, X_2 are metric spaces with metrics d_1, d_2, respectively, and $X_1 \subseteq X_2$ and $d_{2|_{X_1 \times X_1}} = d_1$. If X_1 is not closed in X_2, then X_1 is not complete.

21. (a) For a nonempty set S, the metric space $B(S)$ w.r.t. the sup-metric is complete.

    (b) $C[a, b]$ is a closed subset of $B[a, b]$.

    (c) $C[a, b]$ with sup-metric is complete.

22. If d and ρ are equivalent metrics on a set X, and if d is complete, then ρ is also complete.

23. For $1 \leq p < \infty$, $(C[a, b], d_p)$ is not complete.

24. Let (X, d) be a metric space and $Y \subseteq X$. let $\rho := d_{|_{Y \times Y}}$. Then a set $A \subseteq Y$ is open iff there exists an open set G in X such that $A = G \cap Y$.

12.2 Assignment - II

Prove/verify/Justify the following statements:

1. Let (X, d) be a metric space and $A \subseteq X$.
   (i) $\bar{A} = A \cup \partial A = A \cup A'$
   (ii) A is open iff $A = A^o$.
   (iii) A is closed iff $A = \bar{A}$.
2. Let (X, d) be a metric space and $A \subseteq X$.
   (i) A^o is an open set.
   (ii) $\bar{A}$ is a closed set.
3. Let (X, d) be a metric space and $A \subseteq X$.
   (i) A^o is the largest open set (w.r.t. the partial order $\subseteq$) contained in A, i.e., if G is an open set such that $G \subseteq A$, then $G \subseteq A^o$.
   (ii) $\bar{A}$ is the smallest closed set (w.r.t. the partial order $\subseteq$) containing A, i.e., if F is a closed set such that $A \subseteq F$, then $\bar{A} \subseteq F$.
4. Let (X, d) be a metric space and $A \subseteq X$.
   (i) A^o is the union of all open sets contained in A, that is,
   $$A^o = \bigcup\{G \subseteq X : G \text{ open and } G \subseteq A\}.$$
   (ii) $\bar{A}$ is the intersection of all open sets containing A, that is,
   $$\bar{A} = \bigcap\{G \subseteq X : G \text{ open and } G \subseteq A\}.$$
5. Let (X, d) be a metric space and A and B are subsets of X.
   (i) $A \subseteq B \Rightarrow \bar{A} \subseteq \bar{B}$.
   (ii) $A \subseteq B \Rightarrow A^0 \subseteq \bar{B}^o$.
   (iii) If $A \subseteq B$, then it is not necessary that $\partial A \subseteq \partial B$.
6. Let (X, d) be a metric space and $A \subseteq X$.
   (i) $x \in \bar{A} \iff \exists (x_n)$ in A such that $x_n \to x$.
   (ii) $x \in A' \iff \forall\, r > 0$, $B(x, r) \cap A$ is an infinite set.
7. Let $A \subseteq \mathbb{R}$.
   (i) If A is bounded above, then $\sup(A) \in \bar{A}$.
   (ii) If A is bounded below, then $\inf(A) \in \bar{A}$.
8. Every finite subset of a metric space is closed, and it does not have any limit point.
9. Let $X = \mathbb{R}$ with usual metric $d(x, y) := |x - y|$, $x, y \in \mathbb{R}$. Let $A = (0, 1]$. Then
   (i) $A^o = (0, 1)$ and $\bar{A} = [0, 1]$.
   (ii) A is neither open nor closed.
   (iii) $\partial A = \{0, 1\}$.

10. Let $X = \mathbb{R}$ with usual metric $d(x, y) := |x - y|$, $x, y \in \mathbb{R}$. Let $A = \mathbb{Z}$. Then
   (i) $A^o = \varnothing$ and $\bar{A} = \mathbb{Z}$.
   (ii) A is a closed set, not open.
   (iii) $\partial A = \mathbb{Z}$.
11. Let $X = \mathbb{R}$ with usual metric $d(x, y) := |x - y|$, $x, y \in \mathbb{R}$. Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. Then
   (i) $A^o = \varnothing$ and $\bar{A} = A \cup \{0\}$.
   (ii) A is neither closed nor open.
   (iii) $\partial A = \{0\}$.
12. Let X be with discrete metric and $A \subseteq X$. Then we have the following (verify):
   (i) $A^o = A = \bar{A}$.
   (ii) A is open and closed.
   (iii) $\partial A = \varnothing$.
13. Let $X = \mathbb{R}$ be with usual (Euclidian) metric and $A = (0, 1) \times \{0\}$. Then we have the following (verify):
   (i) $A^o = \varnothing, \quad \bar{A} = [0, 1] \times \{0\}$.
   (ii) A is closed but not open.
   (iii) $\partial A = [0, 1] \times \{0\}$.
14. Let $X = (0, 1]$ with usual metric $d(x, y) := |x - y|$, $x, y \in (0, 1]$. Let $A = (0, \frac{1}{2}]$ and $B = (\frac{1}{2}, 1]$. Then
   (i) A is closed, but not open in X
   (ii) B is open, but not closed, in X.
   (iii) $\partial A = \{\frac{1}{2}\} = \partial B$.
   (iv) $A^o = (0, \frac{1}{2})$ and $B^0 = (\frac{1}{2}, 1]$
   (v) $\bar{A} = (0, \frac{1}{2}]$ and $\bar{B} = [\frac{1}{2}, 1]$
15. Let (X, d) be a metric space and $Y \subseteq X$. Let $A \subseteq Y$. Then A is open in Y iff there exists an open set G in X such that $A = G \cap Y$. Let $X = \mathbb{R}$ with usual metric.
   (i) If $Y = \mathbb{Z}$, what are the open subsets of Y?
   (ii) If $Y = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$, then what are the open subsets of Y?
   (iii) If $Y = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$, then what are the open subsets of Y? Is $\{0\}$ open in Y?
16. $C[a, b]$ is not complete w.r.t the metric $d_p(x, y) := \left(\int_a^b |x(t) - y(t)|^p dt\right)^{\frac{1}{p}}$ for any $p \in [1, \infty)$.

12.3 Assignment - III

Prove/verify/Justify the following statements:

1. Every closed and bounded subset of $\mathbb{R}^k$ has a limit point.
2. If Ω is a closed and bounded subset of $\mathbb{R}^k$, then $C(\Omega) \subseteq B(\Omega)$.

3. Let Ω is a metric space, and $C_b(\Omega)$ be the set of all continuous bounded functions defined on Ω. Then
   (a) $C_b(\Omega)$ is a closed subset of $B(\Omega)$,
   (b) $C_b(\Omega)$ is complete w.r.t. the sup-metric.
4. The set $\mathbb{R} \setminus \mathbb{Q}$ of rational numbers is separable w.r.t.the usual metric.
5. A closed set is nowhere[4] dense iff its compliment is dense.
6. Every infinite subset of a compact set K has a limit point in K.
7. If $E \subseteq \mathbb{R}^k$ and if every infinite subset of E has a limit point in E, then E is compact.
8. If Every infinite subset of a metric space has a limit point, then the metric space separable.
9. Every compact subset of a metric space is separable.
10. Every bounded infinite subset of $\mathbb{R}^k$ has a limit point. (This result is known as *Bolzano-Weierstrass theorem*)
11. Let E be a subset of a metric space X. Then E is disconnected iff there exist disjoint open sets A and B in X such that $E \subseteq A \cup B, \quad E \cap A \neq \varnothing, \quad E \cap B \neq \varnothing$.
12. If E_1 and E_2 are connected subsets of a metric space such that $E_1 \cap E_2 \neq \varnothing$, then $E_1 \cup E_2$ is also connected.
13. Closure of a connected set is connected.
14. Interior of a connected set need not be connected.
15. Every nonempty open subset of $\mathbb{R}$ is a countable disjoint union of open intervals.
16. If E is a nonempty connected subset of a metric space, and if E is not singleton, then E is an uncountable set.
17. Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that for every $\alpha \in \mathbb{R}$, either $f^{-1}(\alpha)$ is singleton or $\#(f^{-1}(\alpha)) \leq 2$. Then, there exists $\beta \in \mathbb{R}$ such that $f^{-1}(\beta)$ is singleton. [Hint: β is either the maximum value or the minimum value of f.]
18. Let X and Y be metric spaces. If $f : X \to Y$ is uniformly continuous, then for every Cauchy sequence (x_n) in X, the sequence $(f(x_n))$ is Cauchy in Y. Give an example to show that the assumption of uniform continuity of f cannot be dropped.

12.4 Assignment - IV

Prove/verify/Justify the following statements:

1. Let (f_n) be a sequence of functions defined on an interval I and f is also a function defined on I. Then the following are true:
   (a) If there exists a sequence (α_n) of positive real numbers such that $\alpha_n \to 0$ as $n \to \infty$ and $|f_n(x) - f(x)| \leq \alpha_n \, \forall \, n \in \mathbb{N}, \forall \, x \in I$, then $f_n \to f$ uniformly.

[4] A subset of a metric space is *nowhere dense* if interior of its closure is empty.

(b) If $f_n \to f$ uniformly, then for every sequence (x_n) in I, $|f_n(x_n) - f(x_n)| \to 0$ as $n \to \infty$.

2. For $n \in \mathbb{N}$, let $f_n(x) = x^n$, $x \in I := (0,1)$. Then

   (a) (f_n) converges pointwise to the zero function on I, but not uniformly,

   (b) (f_n) converges uniformly on $(0,a)$ for any a with $0 < a < 1$.

3. For $n \in \mathbb{N}$, let $f_n(x) = \frac{nx}{1+n^2x^2}$, $x \in I := [0,1]$. Then (f_n) converges pointwise to the zero function on I, but not uniformly.

4. For $n \in \mathbb{N}$, let $f_n(x) = \frac{2nx}{1+n^4x^2}$, $x \in I := [0,1]$. Then (f_n) converges to the zero function uniformly.

5. Give an example of a sequence (f_n) such that

   (a) each f_n has continuous derivative on an interval I and

   (b) (f_n) converges uniformly to a differentiable function f on I,

   but $f_n' \not\to f$ pointwise.

6. The series $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ converges uniformly on $\mathbb{R}$.

7. Let $R > 0$ be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ and let $f(x) := \sum_{n=0}^{\infty} a_n x^n$, $-R < x < R$. Then

   (a) $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-r, r]$ for any r with $0 < r < R$,

   (b) for any $k \in \mathbb{N}$, $\sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_k x^{n-k}$ converges uniformly on $[-r, r]$ for any r with $0 < r < R$,

   (c) for any $k \in \mathbb{N}$, $f^{(k)}(0)$ exists and $a_k = \frac{f^{(k)}(0)}{k!}$.

8. Let $R_1 > 0$ and R_2 be the radii of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$, respectively. If $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$ for all x in a neibourhood of 0, then $a_n = b_n$ for all $n \in \mathbb{N} \cup \{0\}$.

9. From the relation $\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$ for $|x| < 1$, derive $\log(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}$ for $|x| < 1$.

10. From the relation $\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$ for $|x| < 1$, derive $\frac{\pi}{4} = \sum_{n=0}^{\infty}(-1)^n \frac{1}{2n+1}$.

12.5 Assignment - Final

Prove/verify/Justify the following statements:

1. Let $(S, \preceq)$ be a partially ordered set and $A \subseteq S$. Then

   (a) Least upper bound of A, if exists, is unique.

   (b) Greatest lower bound of A, if exists, is unique.

   (c) If $\prec$ is an order relation on S, then $\preceq$ defined by $x \preceq y$ for either $x \prec y$ or $x = y$, is a partial order.

2. (a) Let $n \in \mathbb{Z}$. For $a, b \in \mathbb{N}$, define $a \sim b \iff a - b$ is a multiple of n. Then $\sim$ is an equivalence relation on $\mathbb{Z}$.

(b) Let G be a group and H be a subgroup of G. For $a, b \in G$, define $a \sim b \iff a - b \in H$. Then $\sim$ is an equivalence relation on S.

(c) Let V be a vector space and W be a subspace of V. For $x, y \in V$, define $x \sim y \iff x - y \in W$. Then $\sim$ is an equivalence relation on V.

(d) For a nonempty set X let $S = 2^X$. For A, B in S, define $A \preceq B \iff A \subseteq B$. Then $\preceq$ is a partial order on S.

(e) Let $S = re^{i\theta} : 0 \le \theta < 2\pi, 0 \le r \le 1\}$. For $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$ in S, define $z_1 \preceq z_2 \iff \theta_1 = \theta_2$ and $r_1 \le r_2$. Then $\preceq$ is a partial order on S. In this case,

   i. for each θ, the set $A_\theta := \{re^{i\theta} : 0 \le r \le 1\}$ is a totally ordered subset for which $e^{i\theta}$ is an upper bounded.

   ii. Every $e^{i\theta}$ is a maximal element of S.

3. (a) For $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, let

$$\begin{aligned} d_\infty(x, y) &:= \max\{|x_k - y_k| : k = 1, \ldots, n\}, \\ d_1(x, y) &= \sum_{k=1}^{n} |x_k - y_k|, \\ d_2(x, y) &= \Big(\sum_{k=1}^{n} |x_k - y_k|^2\Big)^{1/2}. \end{aligned}$$

Then d_∞, d_1, d_2 are metrics on $\mathbb{R}^n$; and any two of them are equivalent.

(b) For $x, y \in C[a, b]$, let

$$\begin{aligned} d_\infty(x, y) &:= \sup\{|x(t) - y(t)| : t \in [a, b]\}, \\ d_1(x, y) &= \int_a^b |x(t) - y(t)| dt, \\ d_2(x, y) &= \Big(\int_a^b |x(t) - y(t)|^2 dt)^{1/2}. \end{aligned}$$

Then d_∞, d_1, d_2 are metrics on $C[a, b]$, and they are not equivalent.

(c) For a non-empty subset, let $B(S)$ be the set of all real valued bounded functions defined on S. For x, y in $B(S)$, let $d(x, y) := \sup\{|x(s) - y(s)| : s \in S\}$. Then d is a metric on $B(S)$. What is $B(\mathbb{N})$?

4. Suppose X is a set. A function $\rho : X \times X \to [0, \infty)$ is called a *pseudo metric* on X if

   (i) $\rho(x, x) = 0 \quad \forall x \in X$,

   (ii) $\rho(x, y) = \rho(y, x) \quad \forall x, y \in X$,

   (iii) $\rho(x, y) \le \rho(x, z) + \rho(z, y) \quad \forall x, y, z \in X$.

(a) Let $\mathcal{R}[a, b]$ be the set of all real valued Riemann integrable functions defined on $[a, b]$. For $x, y \in C[a, b]$, let $\rho(x, y) := \int_a^b |x(t) - y(t)| dt$. Then ρ is a pseudo metric, but not a metric on $\mathcal{R}[a, b]$.

(b) The function ρ defined by $\rho((x_1, x_2), (y_1, y_2)) = |x_1 - y_1|, \quad (x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$ is a pseudo metric on $\mathbb{R}^2$, but not a metric.

(c) Let $t_0 \in [a, b]$. Then ρ defined by $\rho(x, y) = |x(t_0) - y(t_0)|, \quad x, y \in C[a, b]$ is a pseudo metric on $C[a, b]$, but not a metric.

5. Let ρ be a pseudo metric on X. For $x, y \in X$, define $x \sim y \iff \rho(x,y) = 0$.

   (a) $\sim$ is an equivalence relation on X.

   (b) Let $\tilde{X}$ be the set of all equivalence classes. On $\tilde{X}$, define $d([x],[y]) := \rho(x,y)$. Then d is a metric on $\tilde{X}$, called the *metric induced by the pseudo metric ρ.*

   (c) Taking $X = \mathcal{R}[a,b]$ and ρ as in Problem 4(a), what are the equivalence classes of an $x \in \mathcal{R}[a,b]$?

   (d) Taking $X = C[a,b]$ and ρ as in Problem 4(c), what are the equivalence classes of an $x \in C[a,b]$?

6. Let (X,d) be a metric space and $A \subseteq X$. The following are equivalent:

   (i) A is closed.

   (ii) A contains all its closure points.

   (iii) A contains all its limit points.

   (iv) A contains all its boundary points.

   (v) A^c is open.

   (vi) For every (x_n) in A, if $x_n \to x$ for some $x \in X$, then $x \in A$.

7. Let (X,d) be a metric space and $A \subseteq X$.

   (i) $\bar{A} = A \cup \partial A$

   (ii) $\bar{A} = A \cup A'$, where A' is the set of all limit points of A.

   (iii) $\bar{A} = \cap\{F : F \text{ is closed in } X\}$.

   (iv) A is open iff $A = A^o$.

   (v) A is closed iff $A = \bar{A}$.

8. Let $X = \mathbb{R}$ with usual metric $d(x,y) := |x - y|$, $x, y \in \mathbb{R}$.

   (a) Let $A = (0,1]$. Then

      (i) $A^o = (0,1)$ and $\bar{A} = [0,1]$.

      (ii) A is neither open nor closed.

      (iii) $\partial A = \{0,1\}$.

   (b) Let $A = \mathbb{Z}$. Then

      (i) $A^o = \varnothing$ and $\bar{A} = \mathbb{Z}$.

      (ii) A is a closed set, not open.

      (iii) $\partial A = \mathbb{Z}$.

   (c) Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. Then

      (i) $A^o = \varnothing$ and $\bar{A} = A \cup \{0\}$.

      (ii) A is neither closed nor open.

      (iii) $\partial A = \{0\}$.

9. Let X be with discrete metric and $A \subseteq X$. Then we have the following (verify):

   (i) $A^o = A = \bar{A}$.

   (ii) A is open and closed.

   (iii) $\partial A = \varnothing$.

10. Let $X = \mathbb{R}$ be with usual (Euclidian) metric and $A = (0,1) \times \{0\}$. Then we have the following (verify):

    (i) $A^o = \varnothing, \quad \bar{A} = [0,1] \times \{0\}$.

    (ii) A is closed but not open.

    (iii) $\partial A = [0,1] \times \{0\}$.

11. Let $X = (0,1]$ with usual metric $d(x,y) := |x-y|$, $x,y \in (0,1]$. Let $A = (0, \frac{1}{2}]$ and $B = (\frac{1}{2}, 1]$. Then

    (i) A is closed, but not open in X

    (ii) B is open, but not closed, in X.

    (iii) $\partial A = \{\frac{1}{2}\} = \partial B$.

    (iv) $A^o = (0, \frac{1}{2})$ and $B^0 = (\frac{1}{2}, 1]$

    (v) $\bar{A} = (0, \frac{1}{2}]$ and $\bar{B} = [\frac{1}{2}, 1]$

12. Let (X,d) be a metric space, $A \subseteq X$.

    (a) If x_0 is a limit point of A, then every open ball containing x_0 contains infinitely many points from A.

    (b) If A is a finite set, then A does not have any limit point.

13. If A is a subset of $\mathbb{R}$ and A is bounded above, then $\sup(A) \in \bar{A}$.

14. Let (X,d) be a metric space.

    (i) Every open ball in X is an open set.

    (ii) Union of every arbitrary collection of open sets in X is open in X.

    (iii) Finite intersection of opens sets in X is open in X.

    (iv) Inersection of every arbitrary collection of closed sets in X is closed in X.

    (v) Finite union of of closed sets in X is closed in X.

15. Let (X,d) be a metric space and (x_n) be a sequene in X. Then $x_n \to x$ iff for every $r > 0$, there exists $N \in \mathbb{N}$ such that $x_n \in B(x,r)$ for all $n \geq N$.

16. Let X be with discrete metric.

    (i) Every subset of X is open and closed.

    (ii) A sequence in X is convergent iff it is eventually constant.

    (iii) A sequence in X is a Cauchy sequence iff it is eventually constant.

17. Let X be a metric space, (x_n) be a sequence in X and $x \in X$.

    (i) $x_n \to x$ iff every subsequenece of (x_n) converges to x.

    (ii) $x_n \to x$ iff every subsequence of (x_n) has a subsequence which converges to x.

18. Let (X,d) be a metric space and $Y \subseteq X$. Then a set $A \subseteq Y$ is open in Y (w.r.t. the induced metric on Y) iff there exists an open set G in X such that $A = G \cap Y$.

19. Let X be a metric space and A and Y be such that $A \subseteq Y \subseteq X$. Then

    (a) Suppose Y is closed in X. Then A is closed in Y iff A is closed in X.

(b) Suppose Y is open in X. Then A is open in Y iff A is open in X.

(c) Result in (19a) need not hold if Y is not closed in X.

(d) Result in (19b) need not hold if Y is not open in X.

20. Let (X, d) be a metric space and $A \subseteq X$.

    (i) A^o is an open set.

    (ii) $\bar{A}$ is a closed set.

21. Let (X, d) be a metric space and $A \subseteq X$.

    (i) A^o is the largest open set (w.r.t. the partial order $\subseteq$) contained in A, i.e., if G is an open set such that $G \subseteq A$, then $G \subseteq A^o$.

    (ii) $\bar{A}$ is the smallest closed set (w.r.t. the partial order $\subseteq$) containing A, i.e., if F is a closed set such that $A \subseteq F$, then $\bar{A} \subseteq F$.

22. Let (X, d) be a metric space and $A \subseteq X$.

    (i) A^o is the union of all open sets contained in A, i.e., $A^o = \bigcup\{G \subseteq X : G \text{ open and } G \subseteq A\}$.

    (ii) $\bar{A}$ is the intersection of all open sets containing A, i.e., $\bar{A} = \bigcap\{G \subseteq X : G \text{ open and } G \subseteq A\}$.

23. Let (X, d) be a metric space and A and B are subsets of X.

    (i) $A \subseteq B \Rightarrow \bar{A} \subseteq \bar{B}$.

    (ii) $A \subseteq B \Rightarrow A^0 \subseteq B^o$.

    (iii) If $A \subseteq B$, then it is not necessary that $\partial A \subseteq \partial B$.

24. Let (X, d) be a metric space and $A \subseteq X$.

    (i) $x \in \bar{A} \iff \exists (x_n)$ in A such that $x_n \to x$.

    (ii) $x \in A' \iff \forall r > 0$, $B(x, r) \cap A$ is an infinite set.

25. (a) Every closed subset of a complete metric space is complete.

    (b) Every complete subset of a metric space is closed.

26. Let X be a metric space and $Y \subseteq X$. If Y is not closed in X, then Y is not complete (w.r.t. the induced metric).

27. (a) For a nonempty set S, the metric space $B(S)$ w.r.t. the sup-metric is complete.

    (b) $C[a, b]$ is a closed subset of $B[a, b]$.

    (c) $C[a, b]$ with sup-metric is complete.

28. If d and ρ are equivalent metrics on a set X, and if d is complete, then ρ is also complete.

29. Then metric space $(C[a, b], d_p)$ for $p \in \{1, 2\}$ is not complete.

30. Let X be a complete metric space and (F_n) is a sequence of closed nonempty subsets of X such that $F_n \supseteq F_{n+1}$ for all $n \in \mathbb{N}$ and $\text{diam}(F_n) \to 0$ as $n \to \infty$. Then $\bigcap_{n=1}^{\infty} F_n$ is nonempty and singleton.

31. Let $A \subseteq \mathbb{R}$.

    (i) If A is bounded above, then $\sup(A) \in \bar{A}$.

    (ii) If A is bounded below, then $\inf(A) \in \bar{A}$.

32. Every finite subset of a metric space is closed, and it does not have any limit point.

33. Every closed and bounded subset of $\mathbb{R}^k$ has a limit point.

34. If Ω is a closed and bounded subset of $\mathbb{R}^k$, then $C(\Omega) \subseteq B(\Omega)$.

35. Let Ω be a metric space, and $C_b(\Omega)$ be the set of all continuous bounded functions defined on Ω. Then

    (a) $C_b(\Omega)$ is a closed subset of $B(\Omega)$,

    (b) $C_b(\Omega)$ is complete w.r.t. the sup-metric.

    (c) Ω compact implies $C(\Omega) = C_b(\Omega)$.

36. If Ω is a compact metric space, then for every $f \in C(\Omega)$, there exists $u, v \in \Omega$ such that $f(u) = \sup_{x\in\Omega} f(x)$ and $f(v) = \inf_{x\in\Omega} f(x)$.

37. Let X and Y metric spaces and $f : X \to Y$ is a continuous bijective function. If X is compact, then $f^{-1} : Y \to X$ is continuous.

38. Let E be a bounded subset of $\mathbb{R}$ (w.r.t. the usual metric). If E is not closed, then

    (a) there exists a continuous function f on E such that $f(E)$ is not bounded.

    (b) there exists a continuous bounded function f on E such that f does not attain maximum on E.
    [Hint: Take $x_0 \in \bar{E} \setminus E$ and $f_1(x) := 1/(x - x_0)$, $f_2(x) = 1/[1 + (x - x_0)^2]$.]

39. The set $\mathbb{R} \setminus \mathbb{Q}$ of rational numbers is separable[5] w.r.t.the usual metric.

40. A closed set is nowhere[6] dense iff its compliment is dense.

41. Every infinite subset of a compact set K has a limit point in K.

42. If $E \subseteq \mathbb{R}^k$ and if every infinite subset of E has a limit point in E, then E is compact.

43. If Every infinite subset of a metric space has a limit point, then the metric space separable.

44. Every compact subset of a metric space is separable.

45. Every bounded infinite subset of $\mathbb{R}^k$ has a limit point. (This result is known as *Bolzano-Weierstrass theorem*)

46. Let X be a metric space and $E \subseteq X$. Then the following are equivalent:

    (a) There exist disjoint open sets A and B in X such that $E \subseteq A \cup B$, $E \cap A \neq \varnothing$, $E \cap B \neq \varnothing$.

    (b) E is disconnected w.r.t. the induced metric on E.

    (c) There exist nonempty sets A and B such that $E = A \cup B$, $\bar{A} \cap B = \varnothing$, $A \cap \bar{B} = \varnothing$.

47. Let E be a subset of a metric space X. Then E is disconnected iff there exist disjoint open sets A and B in X such that $E \subseteq A \cup B$, $E \cap A \neq \varnothing$, $E \cap B \neq \varnothing$.

48. If E_1 and E_2 are connected subsets of a metric space such that $E_1 \cap E_2 \neq \varnothing$, then $E_1 \cup E_2$ is also connected.

49. Closure of a connected set is connected.

[5] A metric space is separable if its closure is the whole space.

[6] A subset of a metric space is *nowhere dense* if interior of its closure is empty.

50. Interior of a connected set need not be connected.

51. Every nonempty open subset of $\mathbb{R}$ is a countable disjoint union of open intervals.

52. If E is a nonempty connected subset of a metric space, and if E is not singleton, then E is an uncountable set.

53. Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that for every $\alpha \in \mathbb{R}$, either $f^{-1}(\alpha)$ is singleton or $\#(f^{-1}(\alpha)) \leq 2$. Then, there exists $\beta \in \mathbb{R}$ such that $f^{-1}(\beta)$ is singleton. [Hint: β is either the maximum value or the minimum value of f.]

54. Let X and Y be metric spaces. If $f : X \to Y$ is uniformly continuous, then for every Cauchy sequence (x_n) in X, the sequence $(f(x_n))$ is Cauchy in Y. Give an example to show that the assumption of uniform continuity of f cannot be dropped.

55. Let (f_n) be a sequence of functions defined on an interval I and f is also a function defined on I. Then the following are true:

    (a) If there exists a sequence (α_n) of positive real numbers such that $\alpha_n \to 0$ as $n \to \infty$ and $|f_n(x) - f(x)| \leq \alpha_n \,\forall\, n \in \mathbb{N}, \forall\, x \in I$, then $f_n \to f$ uniformly.

    (b) If $f_n \to f$ uniformly, then for every sequence (x_n) in I, $|f_n(x_n) - f(x_n)| \to 0$ as $n \to \infty$.

56. For $n \in \mathbb{N}$, let $f_n(x) = x^n$, $x \in I := (0, 1)$. Then

    (a) (f_n) converges pointwise to the zero function on I, but not uniformly,

    (b) (f_n) converges uniformly on $(0, a)$ for any a with $0 < a < 1$.

57. For $n \in \mathbb{N}$, let $f_n(x) = \frac{nx}{1+n^2x^2}$, $x \in I := [0, 1]$. Then (f_n) converges pointwise to the zero function on I, but not uniformly.

58. For $n \in \mathbb{N}$, let $f_n(x) = \frac{2nx}{1+n^4x^2}$, $x \in I := [0, 1]$. Then (f_n) converges to the zero function uniformly.

59. (Dini's theorem): Let (f_n) be a sequence of continuous (real valued) functions defined on a compact metric space Ω such that $f_n \geq f_{n+1}$ for all $n \in \mathbb{N}$ and $f_n \to f$ pointwise on Ω. Then $f_n \to f$ uniformly on Ω.

    (Hint: Take $g_n = f_n - f$ and for $\varepsilon > 0$, define $E_n := \{x \in \Omega : g_n(x) \geq \varepsilon\}$; observe that E_ns are compact and $En \supseteq E_{n+1}$ for all $n \in \mathbb{N}$; use nested theorem.)

60. Let (f_n) be a sequence of continuous (real valued) functions defined on a compact metric space Ω and $f : \Omega \to \mathbb{R}$. Suppose $f_n \to f$ pointwise on Ω. If (f_n) is equicontinuous, then $f_n \to f$ uniformly on Ω.

61. Give an example of a sequence (f_n) such that

    (a) each f_n has continuous derivative on an interval I and

    (b) (f_n) converges uniformly to a differentiable function f on I,

    but $f_n' \not\to f$ pointwise.

62. The series $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ converges uniformly on $\mathbb{R}$.

63. Let $R > 0$ be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ and let $f(x) := \sum_{n=0}^{\infty} a_n x^n$, $-R < x < R$. Then

    (a) $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-r, r]$ for any r with $0 < r < R$,

(b) for any $k \in \mathbb{N}$, $\sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_k x^{n-k}$ converges uniformly on $[-r, r]$ for any r with $0 < r < R$,

(c) for any $k \in \mathbb{N}$, $f^{(k)}(0)$ exists and $a_k = \frac{f^{(k)}(0)}{k!}$.

64. Let $R_1 > 0$ and R_2 be the radii of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$, respectively. If $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$ for all x in a neibourhood of 0, then $a_n = b_n$ for all $n \in \mathbb{N} \cup \{0\}$.

65. From the relation $\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$ for $|x| < 1$, derive $\log(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}$ for $|x| < 1$.

66. From the relation $\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$ for $|x| < 1$, derive $\frac{\pi}{4} = \sum_{n=0}^{\infty}(-1)^n \frac{1}{2n+1}$.