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# Convolutions of Certain Analytic Functions

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**Abstract.** Ruscheweyh and Sheil-Small proved the Polya-Schoenberg conjecture that the classes of convex functions, starlike functions and close-to-convex functions are closed under convolution with convex functions. By making use of this result, the radii of starlikeness of order $\alpha$, parabolic starlikeness, and strong starlikeness of order $\gamma$ of the convolution between two starlike functions are determined. Similar convolution results for two classes of analytic functions are also obtained.

**Keywords.** Starlike, parabolic starlike, strongly starlike, radius problem, convolution.

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## 1. Introduction

Let $\mathscr{A}$ be the class of all functions analytic in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = 0 = f'(0) - 1$. Let $\mathscr{S}$ be the subclass of $\mathscr{A}$ consisting of univalent functions. For $0 \leq \alpha < 1$, let $\mathscr{S}^*(\alpha)$ and $\mathscr{C}(\alpha)$ be the subclasses of $\mathscr{S}$ consisting of starlike functions of order $\alpha$ and convex functions of order $\alpha$, respectively. A starlike or convex function of order 0 is respectively called starlike or convex function, and is denoted by $\mathscr{S}^*(0) = \mathscr{S}^*$ and $\mathscr{C}(0) = \mathscr{C}$. The class $\mathscr{S}^*_\gamma$ of strongly starlike functions of order $\gamma$, $0 < \gamma \leq 1$, consists of $f \in \mathscr{S}$ satisfying the inequality

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\gamma\pi}{2}, \quad z \in \mathbb{D}.$$

The function $f \in \mathscr{S}$ is uniformly convex if for every circular arc $\gamma$ contained in $\mathbb{D}$ with center $\zeta \in \mathbb{D}$ the image arc $f(\gamma)$ is convex. The class $\mathscr{UCV}$ of all uniformly convex functions was introduced by Goodman [1]. Rønning [9], as well as Ma and Minda [5], independently proved that

$$f \in \mathscr{UCV} \iff \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right|, \quad z \in \mathbb{D}. \tag{1.1}$$

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Rønning introduced a corresponding class of starlike functions called parabolic starlike functions. A function $f \in \mathscr{A}$ is parabolic starlike if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right|, \quad z \in \mathbb{D}.$$

The class of all such functions is denoted by $\mathscr{S}_{\mathscr{P}}$.

Let $\mathscr{S}_{\mathscr{L}}$ be the class of functions $f \in \mathscr{A}$ satisfying the inequality

$$\left|\left(\frac{zf'(z)}{f(z)}\right)^2 - 1\right| < 1, \quad z \in \mathbb{D}.$$

Thus a function $f$ is in the class $\mathscr{S}_{\mathscr{L}}$ if $zf'(z)/f(z)$ lies in the region bounded by the right-half of the lemniscate of Bernoulli given by $|w^2 - 1| < 1$. This class $\mathscr{S}_{\mathscr{L}}$ was introduced by Sokół and Stankiewicz [15].

For two analytic functions $f, g \in \mathscr{A}$, their convolution or Hadamard product, denoted by $f * g$, is defined by $(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n$. Pólya and Schoenberg [7] conjectured that the class of convex functions $\mathscr{C}$ is preserved under convolution with convex functions: $f, g \in \mathscr{C} \Rightarrow f * g \in \mathscr{C}$. In 1973, Ruscheweyh and Sheil-Small [11] (see also [12]) proved the Polya-Schoenberg conjecture. In fact, they also proved that the classes of starlike functions and close-to-convex functions are closed under convolution with convex functions. The proofs of these facts are evident from the following result which is also needed for our investigation.

**Theorem 1.2.** [12, Theorem 2.4] *If $f \in \mathscr{S}^*$ and $\varphi \in \mathscr{C}$, then*

$$\frac{\varphi * fF}{\varphi * f}(\mathbb{D}) \subset \overline{co}(F(\mathbb{D}))$$

*for any function $F$ analytic in $\mathbb{D}$, where $\overline{co}(F(\mathbb{D}))$ denotes the closed convex hull of $F(\mathbb{D})$.*

The radius of a property $\mathscr{P}$ of functions in a set $\mathscr{M}$ is the largest number $R$ such that every function in the set $\mathscr{M}$ has the property $\mathscr{P}$ in each disk $\mathbb{D}_r = \{z \in \mathbb{D} : |z| < r\}$ for every $r < R$. The convolution of the Koebe function $k(z) = z/(1-z)^2$ with itself is not univalent. Thus, the convolution of two univalent (or starlike) functions need not be univalent. Since the radius of convexity of functions in the class $\mathscr{S}^*$ is $2 - \sqrt{3}$, a result of Ruscheweyh and Sheil-Small showed that the radius of starlikeness of the convolution between two starlike functions is $2 - \sqrt{3}$ (see [4]). Silverman [14] has determined the radius of univalence for the convolution of a normalized univalent function with a close-to-convex function. He also found a lower bound for the radius of univalence of convolution between two univalent functions.

By making use of Theorem 1.2, the $\mathscr{S}_{\mathscr{P}}$-radius (and the $\mathscr{S}^*(\alpha)$, $\mathscr{S}_{\mathscr{L}}$, and $\mathscr{S}^*_{\gamma}$ radii) is determined for the convolution of two starlike functions. Certain classes of analytic functions are also proved to be closed under convolution with convex functions.

## 2. Convolution of two starlike functions

Rønning [10] proved that the class $\mathcal{S}_{\mathcal{P}}$ is closed under convolution with a starlike function of order 1/2. However, the convolution of two starlike functions need not be in the class $\mathcal{S}_{\mathcal{P}}$. Therefore it is natural to determine the $\mathcal{S}_{\mathcal{P}}$, $\mathcal{S}^*(\alpha)$, $\mathcal{S}_{\mathcal{L}}$, and $\mathcal{S}^*_\gamma$ radii of the convolution $f*g$ between two starlike functions $f$ and $g$. We do this in Theorem 2.1.

**Theorem 2.1.** *Let $f,g\in\mathcal{S}^*$ and $h_\rho(z):=(f*g)(\rho z)/\rho$. Then*

*(a) $h_\rho\in\mathcal{S}_{\mathcal{P}}$ for $0\le\rho\le(4-\sqrt{13})/3\approx 0.13148$,*
*(b) $h_\rho\in\mathcal{S}^*(\alpha)$ for $0\le\rho\le(2-\sqrt{3+\alpha^2})/(1+\alpha)$,*
*(c) $h_\rho\in\mathcal{S}_{\mathcal{L}}$ for $0\le\rho\le(\sqrt{5}-2)(\sqrt{2}-1)\approx 0.09778$,*
*(d) $h_\rho\in\mathcal{S}^*_\gamma$ for $0\le\rho\le(2-\sqrt{4-b^2})/b$ where $b=\sin(\pi\gamma/2)$.*

*The upper bound for $\rho$ in each case is sharp.*

**Proof.** Let $H(z)=z+\sum_{n=2}^{\infty}n^2z^n$ and consider the disk containing the values $zH'(z)/H(z)$. The function $H$ in closed form is

$$H(z)=\frac{z(1+z)}{(1-z)^3}.$$

It is easy to see that

$$\left|\frac{zH'(z)}{H(z)}-\frac{1+r^2}{1-r^2}\right|\le\frac{4r}{1-r^2},\quad |z|=r<1. \tag{2.2}$$

Let $a>1/2$. It is known that [13] the disk $\{w:|w-a|<R_a\}$ is contained in the parabolic region $\{w:|w-1|<\operatorname{Re}w\}$ if the number $R_a$ satisfies

$$R_a=\begin{cases}a-\frac{1}{2} & (\frac{1}{2}<a\le\frac{3}{2})\\ \sqrt{2a-2} & (a\ge\frac{3}{2}).\end{cases}$$

Let $0\le r\le(4-\sqrt{13})/3=:\rho_0$. Then $a:=(1+r^2)/(1-r^2)\le 3/2$ for $r\le 1/\sqrt{5}\approx 0.4472$. In particular, $a\le 3/2$ for $0\le r\le\rho_0\approx 0.13148$. Consequently the inequality

$$\left|\frac{zH'(z)}{H(z)}-1\right|<\operatorname{Re}\left(\frac{zH'(z)}{H(z)}\right),\quad |z|=r<1,$$

holds if

$$\frac{1+r^2-4r}{1-r^2}\ge\frac{1}{2},$$

or if $3r^2-8r+1=(\rho_0-r)(1-3\rho_0 r)/\rho_0\geq 0$. As $\rho_0<1/3$, this inequality is clearly satisfied for $0\leq r\leq\rho_0$. Also, with $z=-\rho_0$, then

$$\left|\frac{zH'(z)}{H(z)}-1\right|=\left|\frac{1+4z+z^2}{1-z^2}-1\right|=\frac{4\rho_0-2\rho_0^2}{1-\rho_0^2}=\frac{\rho_0^2-4\rho_0+1}{1-\rho_0^2}=\operatorname{Re}\left(\frac{zH'(z)}{H(z)}\right).$$

This shows that the number $\rho_0$ is sharp.

Define the function $h:\mathbb{D}\to\mathbb{C}$ by $h(z)=f(z)*g(z)$. Then $h(z)=F(z)*G(z)*H(z)$ where $F$ and $G$ are respectively defined by $zF'(z)=f(z)$ and $zG'(z)=g(z)$. Since $f,g$ are starlike, it follows that $F$ and $G$ are convex. Since the convolution of two convex functions is convex, $F*G$ is convex. Also the function $H(\rho_0 z)/\rho_0$ is a function in $\mathscr{S}_\mathscr{P}$ and hence $F(z)*G(z)*H(\rho_0 z)/\rho_0$ is again in the class $\mathscr{S}_\mathscr{P}$. Equivalently, $h_{\rho_0}(z)=(F*G*H)(\rho_0 z)/\rho_0$ is in $\mathscr{S}_\mathscr{P}$. Thus the $\mathscr{S}_\mathscr{P}$-radius of the function $h$ is at least $\rho_0$.

Consider the Koebe function $k(z)=z/(1-z)^2$; it is starlike and the $\mathscr{S}_\mathscr{P}$-radius of $k(z)*g(z)=(z/(1-z)^2)*g(z)=zg'(z)$ is the same as the radius of uniform convexity of $g$. Since $\rho_0$ is the radius of uniform convexity of starlike functions, the radius is sharp. This proves the result in part (a).

We shall now prove the result in part (b). Let $\rho_1=(2-\sqrt{3+\alpha^2})/(1+\alpha)$. From the inequality (2.2), it follows that

$$\operatorname{Re}\frac{zH'(z)}{H(z)}\geq\frac{1+r^2-4r}{1-r^2}\geq\alpha$$

for $0\leq|z|=r\leq\rho_1$. For $z=-\rho_1$, then

$$\operatorname{Re}\frac{zH'(z)}{H(z)}=\frac{1+\rho_1^2-4\rho_1}{1-\rho_1^2}=\alpha.$$

Since the class of starlike functions of order $\alpha$ is also closed under convolution with convex functions, the function

$$h(\rho_1 z)/\rho_1=(F*G*H)(\rho_1 z)/\rho_1=F(z)*G(z)*H(\rho_1 z)/\rho_1$$

is again a starlike function of order $\alpha$. This proves (b).

To prove (c), note that for $0<a<\sqrt{2}$, $\{w:|w-a|<r_a\}\subseteq\{w:|w^2-1|<1\}$ if $r_a$ is given by

$$r_a=\begin{cases}\left(\sqrt{1-a^2}-(1-a^2)\right)^{1/2} & (0<a\leq 2\sqrt{2}/3)\\ \sqrt{2}-a & (2\sqrt{2}/3\leq a<\sqrt{2}).\end{cases}$$

Since the disk in (2.2) is centered at the point $a:=(1+r^2)/(1-r^2)\geq 1$, the inequality

$$\left|\left(\frac{zH'(z)}{H(z)}\right)^2-1\right|<1$$

is satisfied for $|z| = r < 1$ if

$$\frac{4r}{1-r^2} < \sqrt{2} - \frac{1+r^2}{1-r^2},$$

or $(\sqrt{2}+1)r^2 + 4r - (\sqrt{2}-1) < 0$. This yields

$$0 \le r \le (-2+\sqrt{5})(\sqrt{2}-1) =: \rho_2.$$

For $z = \rho_2$, then

$$\left| \left( \frac{zH'(z)}{H(z)} \right)^2 - 1 \right| = 1.$$

The remaining part of the proof is similar to the other two parts notwithstanding the fact that the class $\mathcal{S}_{\mathcal{L}}$ is closed under convolution with convex functions. However this is known even more generally for any analytic function $f$ for which $zf'(z)/f(z)$ lies in a convex domain [6]; in case of functions in the class $\mathcal{S}_{\mathcal{L}}$, the convex domain is the right half of the lemniscate of Bernoulli.

The proof of part (d) is similar, with the observation that the disk $|w-a| \le R_a$ is contained in the sector $|\arg w| \le \frac{\pi\gamma}{2}$, $0 < \gamma \le 1$ whenever $R_a \le a \sin\left(\frac{\pi\gamma}{2}\right)$. ■

**Remark 2.3.** The proof that the $\mathcal{S}_{\mathcal{P}}$-radius of the function $H$ is $\rho_0$ shows that the $\mathcal{S}_{\mathcal{P}}$-radius of the (ordinary) product of a starlike function and a function with positive real part is also $\rho_0$.

**Corollary 2.4.** [4] *The radius of starlikeness of convolution of two starlike functions is* $2-\sqrt{3}$.

## 3. Classes closed under convolution with convex functions

**Definition 3.1.** For $\alpha \ge 0$, let $\mathcal{L}(\alpha)$ be the class of analytic functions $f \in \mathcal{A}$ satisfying the condition

$$\operatorname{Re}\left\{ \frac{zf'(z)}{f(z)} \left( \alpha \frac{zf''(z)}{f'(z)} + 1 \right) \right\} > -\frac{\alpha}{2}, \quad z \in \mathbb{D}.$$

The following result is obtained.

**Theorem 3.2.** *If* $f \in \mathcal{L}(\alpha)$ *and* $\varphi \in \mathcal{C}$, *then* $f * \varphi \in \mathcal{L}(\alpha)$.

**Proof.** Let $h : \mathbb{D} \to \mathbb{C}$ be the function defined by

$$h(z) := \frac{z + (2\alpha-1)z^2}{(1-z)^3}.$$

Then

$$\frac{zf'(z)}{f(z)} \left( \alpha \frac{zf''(z)}{f'(z)} + 1 \right) = \frac{\alpha z^2 f''(z) + zf'(z)}{f(z)} = \frac{f(z) * h(z)}{f(z)}.$$

Let $F : \mathbb{D} \to \mathbb{C}$ be defined by

$$F(z) := \frac{f(z) * h(z)}{f(z)}.$$

The function $F$ is clearly well-defined and analytic in $\mathbb{D}$, with $F(\mathbb{D}) \subseteq \{w : \Re w > -\alpha/2\}$. Li and Owa [3] proved that every function in the class $\mathscr{L}(\alpha)$ is starlike. Therefore, the function $f * \varphi$ is starlike univalent and hence the function $\frac{\varphi * fF}{\varphi * f}$ is well-defined and analytic in $\mathbb{D}$. Also,

$$\frac{\varphi * fF}{\varphi * f}(\mathbb{D}) \subset \overline{co}(F(\mathbb{D})),$$

or equivalently

$$\operatorname{Re} \frac{\varphi * fF}{\varphi * f} \geq -\frac{\alpha}{2}.$$

Since

$$\frac{(\varphi * fF)(z)}{(\varphi * f)(z)} = \frac{(\varphi * (f * h))(z)}{(f * \varphi)(z)} = \frac{((\varphi * f) * h)(z)}{(f * \varphi)(z)},$$

by using the minimum principle for harmonic functions, it follows that

$$\operatorname{Re} \left( \frac{((\varphi * f) * h)(z)}{(f * \varphi)(z)} \right) > -\frac{\alpha}{2},$$

or equivalently

$$\operatorname{Re} \left( \frac{z(\varphi * f)'(z)}{(\varphi * f)(z)} \left( \alpha \frac{z(\varphi * f)''(z)}{(\varphi * f)'(z)} + 1 \right) \right) > -\frac{\alpha}{2}, \quad z \in \mathbb{D}.$$

This proves our result. ∎

**Corollary 3.3.** *If $f \in \mathscr{L}(\alpha)$, then the integral transforms $F$ and $G$ given by*

$$F(f(z)) = \frac{\gamma+1}{z^\gamma} \int_0^z \zeta^{\gamma-1} f(\zeta) d\zeta, \quad \operatorname{Re} \gamma > 0 \tag{3.4}$$

*and*

$$G(f(z)) = \int_0^z \frac{f(\zeta) - f(\eta\zeta)}{\zeta - \eta\zeta} d\zeta, \ |\eta| \leq 1, \quad \eta \neq 1 \tag{3.5}$$

*are again in $\mathscr{L}(\alpha)$.*

**Proof.** The results follow since $F = f * \varphi_1$ and $G = f * \varphi_2$ where

$$\varphi_1(z) = \sum_{n=1}^{\infty} \frac{\gamma+1}{\gamma+n} z^n, \quad \varphi_2(z) = \sum_{n=1}^{\infty} \frac{1-\eta^n}{(1-\eta)n} z^n$$

are convex univalent functions in $\mathbb{D}$. ∎

**Definition 3.6.** For $g \in \mathscr{A}$ and $\beta < 1$, let $\mathscr{R}_g(\beta)$ be the class of all analytic functions $f \in \mathscr{A}$ satisfying the condition

$$\operatorname{Re}\left(\frac{(f*g)(z)}{z}\right) > \beta.$$

We show that the class $\mathscr{R}_g(\beta)$ is closed under convolution with convex functions.

**Theorem 3.7.** *If $f \in \mathscr{R}_g(\beta)$ and $\varphi \in \mathscr{C}$, then $f*\varphi \in \mathscr{R}_g(\beta)$.*

**Proof.** Define the function $F$ by

$$F(z) = \frac{(f*g)(z)}{z}.$$

Then

$$\frac{(\varphi * f * g)(z)}{z} = \frac{(\varphi * (zF))(z)}{(\varphi * z)(z)} \in \overline{co}(F(\mathbb{D})).$$

This shows that $\operatorname{Re}(\varphi * f * g)(z)/z) > \beta$ and hence $f*\varphi \in \mathscr{R}_g(\beta)$. ∎

**Corollary 3.8.** *If $f \in \mathscr{R}_g(\beta)$ and the integral transforms $F$ and $G$ are given by* (3.4) *and* (3.5), *then $F, G \in \mathscr{R}_g(\beta)$.*

With $g(z) = (z + (2\alpha - 1)z^2)/(1-z)^3$, the class $\mathscr{R}_g(\beta)$ reduces to the class $\mathscr{R}(\alpha,\beta)$ defined below.

**Definition 3.9.** For $\alpha \in \mathbb{R}$ and $\beta < 1$, let $\mathscr{R}(\alpha,\beta)$ be the class of all analytic functions $f \in \mathscr{A}$ satisfying the condition

$$\operatorname{Re}\left(f'(z) + \alpha z f''(z)\right) > \beta.$$

Several related convolution results on $\mathscr{R}(\alpha,\beta)$ can be found in Ponnusamy and Singh [8]. For this class, the following results are obtained.

**Corollary 3.10.** *If $f \in \mathscr{R}(\alpha,\beta)$ and $\varphi \in \mathscr{C}$, then $f*\varphi \in \mathscr{R}(\alpha,\beta)$.*

**Corollary 3.11.** *If $f \in \mathscr{R}(\alpha,\beta)$ and the integral transforms $F$ and $G$ are respectively given by* (3.4) *and* (3.5), *then $F, G \in \mathscr{R}(\alpha,\beta)$.*

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