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On a Conjecture of Fournier, Ma, and Ruscheweyh for Bounded Convex Functions

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Abstract. We examine a conjecture of Fournier, Ma, and Ruscheweyh on the maximal value for $|a_4|$ in terms of $|a_2|$ for bounded convex functions. Using the Julia variation, we show that we need only consider maps onto regions with 3 proper sides, and we examine the remaining possibilities to see what geometric conclusions can be made. In particular, we show that the triangle maps conjectured by Fournier, Ma, and Ruscheweyh to be extremal are indeed bounded by their conjectured bound, but the bound is not obtained. It appears only in the limit as the triangles degenerate. We also introduce another candidate for the extremal which also produces the conjectured bound in the limit.

Keywords. Extremal Problems, Convex Functions.

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1. Introduction

From the introduction of the Bieberbach conjecture in 1916, until its proof by Louis de Branges (1984) a great deal of development has been made in the field of geometric function theory. Although the Bieberbach Conjecture was initially the principal driving force behind the field, the field itself grew beyond the initial constraints of the conjecture and, even after the solution to the Bieberbach conjecture, many problems and unanswered questions remain. Some examples of these include finding extremal values for certain functionals restricted to various classes of univalent functions. Ideally, given desired geometric properties that the image of a function might have, we would like to be able to identify precisely what values the coefficients of a series expansion for that function may have, perhaps dependent on the domain or range space for the function, and vice versa.

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We will consider the fourth coefficient for bounded convex functions. Historically, while upper bounds for the fourth coefficient within various classes of functions have been computed, these bounds are notoriously misleading, as they are only realized, generally, for large values of the second coefficient. In addition, the fact that we will be dealing with only bounded convex functions makes this question more difficult still. Among other reasons, this is due to the fact that while the class of convex functions $\mathcal{C}$ is compact, as is the class of unbounded convex functions $\mathcal{C}_u$, the class of bounded convex functions $\mathcal{C}_b$ is not. Hence $\mathcal{C}_b$ may not contain extreme points for some functionals. In addition, the functional we are examining is either upper-semicontinuous, and need not achieve extreme values, even over some compact spaces, or the necessary restrictions upon the domain prevent the domain from being compact.

In 1998, Richard Fournier, Jing Ma and Stephan Ruscheweyh [10] improved upon the results of Martin Chuaqui and Brad Osgood [8] concerning omitted values within the Nehari class. Fournier, Ma, and Ruscheweyh were able to show that a bound exists over the class $\mathcal{C}_b$ for the functional $|a_4|$ in terms of the functional $|a_2|$ and M, the radius of the smallest disk centered at the origin which contains $f(\mathbb{D})$, where $\mathbb{D} = \{z : |z| < 1\}$. In particular, they proved

$$|a_4| \leq \frac{7}{3}|a_2| + \frac{2}{3M} \text{ over } \mathcal{C}_b, \tag{1}$$

where the coefficient $\frac{2}{3}$ in (1) is sharp, while $\frac{7}{3}$ most likely was not. However, considering this same functional, over the class of unbounded functions, they were able show that

$$|a_4| \leq 2|a_2| \text{ over } \mathcal{C}_u. \tag{2}$$

Based on separate results by Ma[13], the three stated that any configuration for a function at which this coefficient of $|a_2|$ in (1) achieves a maximal value must have at most 4 proper sides. They also saw good numerical reasons to believe that the maximal configuration should have fewer sides than this.

The combination of these observations led them to try to reconcile the difference in the appropriate coefficient of $|a_2|$ between (1) and (2) by stating the following conjecture:

Fournier-Ma-Ruscheweyh Conjecture. *For $f \in \mathcal{C}_b$, with f normalized as above,*

$$|a_4| \leq 2|a_2| + \frac{2}{3M},$$

where 2 is sharp for a function of the form

$$f_0(z) = \int_0^z \frac{dt}{(1-t)^{2\lambda}(1+2ct+t^2)^{1-\lambda}}$$

which maps the disk to a triangular region symmetric with respect to $\mathbb{R}$.

After providing the necessary background on the Julia Variation in section 2, we will improve on Fournier, Ma, and Ruscheweyh's results in section 3 to reduce the number of proper sides we must consider from 4 to only 3. In section 4, we will describe the geometry of the remaining possibilities and suggest other candidates for the extremal besides Fournier, Ma, and Ruscheweyh's triangle map.

2. The Julia Variation Technique

This method of variational analysis of a function was introduced in a rudimentary form by Jan Krzyz in 1963 [12], by approximating bounded convex functions with polygons and then using the variational technique on the sides of polygons. Later circa 1975, papers by Roger Barnard and John Lewis [1, 2] presented modifications of the variational techniques which could be used on figures of a much more general type, by showing that their variational formulas worked on corners and sides which curve along the boundary of figures. That is, that any resulting error from utilizing these techniques in this way is absorbed into the $o(\epsilon)$ term and does not affect the procedure of using these techniques significantly.

Excellent formulations of the proofs of these formulas can be found in [1, 2, 3, 4, 5, 6, 7], but briefly, if we have a function $f : \mathbb{D} \to \Omega$ we can outline a distortion in the range, Ω, by defining a new boundary $\partial\Omega_\epsilon$ and then expressing the change from $\partial\Omega$ to the new boundary by functions $p(w)n(w)$ where $w = f\left(e^{i\phi}\right)$, $n(w)$ represents the normal direction to the curve $\partial\Omega$ at w, and $p(w)$ is a real valued, piecewise continuous function defining a scaled translation necessary along the normal direction to go from $w \in \partial\Omega$ to $w + p(w)n(w) \in \partial\Omega_\epsilon$.

In practice, this method is applied to figures whose boundaries can be partitioned into "sides" and it may be necessary to only adjust a few sides of the figure, so $p(w)$ can be set to 0 along all remaining sides.

Then, for any $z \in \mathbb{D}$, the varied function $f_\epsilon : \mathbb{D} \to \Omega_\epsilon$ becomes, by the Julia Variational Formula as extended by Barnard and Lewis [1, 2]:

$$f_\epsilon(z) \;=\; f(z) + \epsilon z f'(z) \int_{\partial\Omega} \frac{1+\zeta z}{1-\zeta z}\, d\Psi \;+ o(\epsilon)$$

where $\zeta = e^{i\phi}$, $o(\epsilon)$ is uniform for z in compact subsets of $\mathbb{D}$ and for $w = f(\zeta)$,

$$d\Psi \;=\; \frac{p(w)n(w)}{2\pi i\left[\zeta f'(\zeta)\right]^2} dw.$$

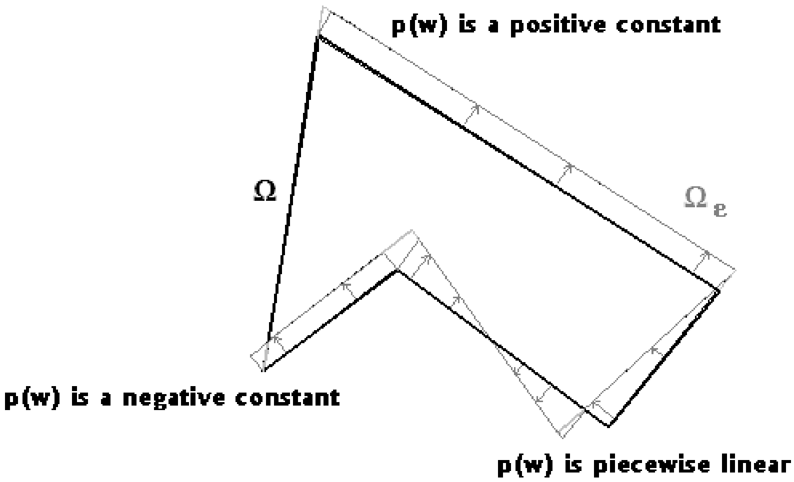

FIGURE 1. Illustration of some common variations.

However, this differential can be expressed through a change in coordinates as

$$d\Psi(\phi) \;=\; \frac{p(w)}{2\pi|f'(\zeta)|}d\phi$$

adjusting the form of the map to

$$f_\epsilon(z) = f(z) + \epsilon z f'(z)\int_{\partial\mathbb{D}} \frac{1+\zeta z}{1-\zeta z}\, d\Psi(\phi) \;+ o(\epsilon). \tag{1}$$

It is important to note here that $d\Psi(\phi)$ is real valued, and its sign is determined entirely by the sign of $p(w)$.

The new mapping radius resulting from this variation is given by

$$1 + \epsilon\int_{\partial\mathbb{D}} d\Psi(\phi) \;+ o(\epsilon). \tag{2}$$

Ordinarily, if one were attempting to stay within a particular family of functions with a specified mapping radius one would divide the formula (1) by (2) and incorporate the result into a single integral equation to maintain the mapping radius, $|f'(0)|$ which, as in $\mathcal{S}$ is specified in the definition of the class [4]. This is especially important when only changing one side of a figure at a time, as this will almost invariably adjust the mapping radius without it. For our purposes, however, we will be working carefully on more than one side (and sometimes by breaking a side apart) to ensure that $\int_{\partial\mathbb{D}} d\Psi(\phi) = 0$ so that all change to the mapping radius is $o(\epsilon)$ and need not be considered.

3. Narrowing the Field

To begin, we would like to reduce the scope of the types of functions we should consider for extremal cases for this functional. To do this, we apply the Julia Variational techniques and make some observations based on the resultant kernel.

Initially, for any general function from $\mathcal{C}$, we will write f as

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

Define

$$M(f) = \max_{\zeta \in \partial\mathbb{D}} \{|f(\zeta)|\}$$

. Under these assignments, we are looking for the best bound for the functional **A** given in

$$|a_4| \leq \mathbf{A}|a_2| + \frac{2}{3M}. \tag{3}$$

In order to analyze this, assume equality and rewrite the above inequality to find

$$\mathbf{A}(f) = \frac{|a_4| - \frac{2}{3M}}{|a_2|}$$

.

The result of applying the Julia variation given in (1) upon the key functionals a_4 and a_2, as it depends on ϵ is

$$a_{4\epsilon} = a_4 + \epsilon \int_{\Gamma} \left(2\zeta^3 + 4a_2\zeta^2 + 6a_3\zeta + 3a_4\right) d\Psi \ + o(\epsilon)$$

and

$$a_{2\epsilon} = a_2 + \epsilon \int_{\Gamma} (2\zeta + a_2)\, d\Psi \ + o(\epsilon).$$

It will be necessary, since we are dealing with a noncompact space, to localize our search for extremal configurations for f by restricting ourselves to a compact exhaustion of $\mathcal{C}_b$. In particular, we would like to deal with the class $\mathcal{C}_M = \left\{f \in \mathcal{C}_b \;\middle|\; \max_{\xi \in \partial\mathbb{D}} f(\xi) \leq M\right\}$ for some fixed $M \geq 1$. In general, we will assume that $M(f)$ is held constant during our variations, except in one specific instance which will be treated separately.

We will actually focus on a dense family $\mathcal{K}_M$ within $\mathcal{C}_M$ whose images are bounded solely by either straight edges or arcs of $\partial M\mathbb{D}$. In order to determine

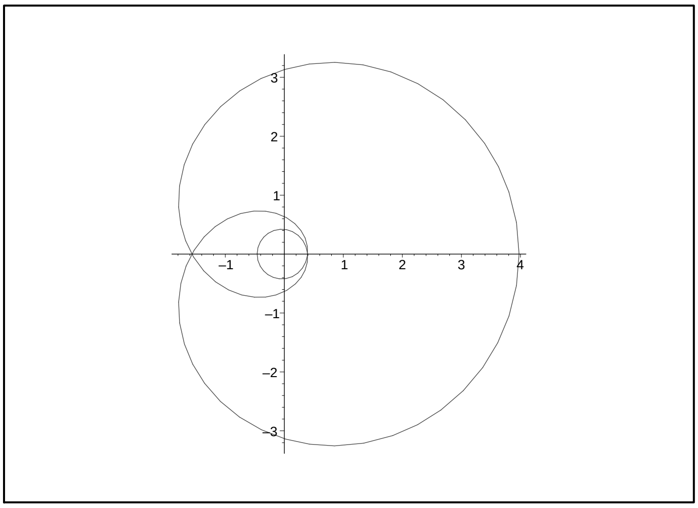

FIGURE 2. The graph of $K(\zeta)$ for $f_{c,\lambda}(z)$ with $c = \frac{-1}{2}$ and $\lambda = \frac{1}{4}$.

the types of functions for which **A** will take on maximal values, we will look at the compact exhaustion of $\mathcal{K}_M$ by the classes

$$\mathcal{K}_{M,n} = \left\{ f \in \mathcal{K}_M \;\middle|\; f(\partial\mathbb{D}) \text{ has at most } n \text{ proper sides} \right\} \tag{4}$$

Since we have restricted our search to functions in $\mathcal{K}_M$, for most of the variations we will perform, the bound M in our functional will be held constant, which will restrict some of the variational movements we will make. There will be only one type of configuration for which we must allow the bound M to decrease, and this will be handled separately, as stated earlier.

For the following discussion, we will refer to the straight edges of the images of functions from $\mathcal{K}_M$ as its proper sides, while any arcs of $\partial M\mathbb{D}$ we will refer to as boundary arcs.

Observe, then, that by (1) we have

$$\begin{aligned} \mathbf{A}_\varepsilon(f) &= \frac{|a_{4\epsilon}| - \frac{2}{3M_\epsilon}}{|a_{2\epsilon}|} \\ &= \frac{\left|a_4 + \epsilon \int_\Gamma (2\zeta^3 + 4a_2\zeta^2 + 6a_3\zeta + 3a_4)\, d\Psi \;\; + o(\epsilon)\right| - \frac{2}{3M_\epsilon}}{\left|a_2 + \epsilon \int_\Gamma (2\zeta + a_2)\, d\Psi \;\; + o(\epsilon)\right|}. \end{aligned}$$

We will now use the fact that a necessary condition for f to be a maximal configuration for $\mathbf{A}$ is that

$$\mathfrak{Re}\left(\frac{d}{d\epsilon}\mathbf{A}_\varepsilon(f)\bigg|_{\epsilon=0}\right) = \frac{d}{d\epsilon}\mathfrak{Re}(\mathbf{A}_\varepsilon(f))\bigg|_{\epsilon=0} = 0.$$

Additionally, we will use the fact that this functional is rotationally invariant, that is, if

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

achieves a particular value under $\mathbf{A}$, then so does

$$e^{-i\theta} f(ze^{i\theta}) = z + e^{i\theta} a_2 z^2 + \sum_{k=3}^{\infty} e^{(k-1)i\theta} a_k z^k$$

For this reason, we may assume, without loss of generality, that an appropriate rotation of f has been chosen so that $a_2 \geq 0$. However, since the value $a_2 = 0$ will eliminate our functional from the inequality $|a_4| \leq \mathbf{A}a_2 + \frac{2}{3M}$, we will need to look only among functions for which $a_2 > 0$, as the case $a_2 = 0$ has a bound which is already fully understood. For this reason, either our function space is no longer compact, or the functional is now only upper semi-continuous.

Utilizing this, we find

$$\frac{d}{d\epsilon}\mathfrak{Re}(\mathbf{A}_\varepsilon(f))\bigg|_{\epsilon=0}$$

$$= \frac{d}{d\epsilon}\left(\frac{|a_4| + \epsilon\int_\Gamma \mathfrak{Re}(2\zeta^3 + 4a_2\zeta^2 + 6a_3\zeta + 3a_4)d\Psi(\phi) + o(\epsilon) - \frac{2}{3M_\epsilon}}{a_2 + \epsilon\int_\Gamma \mathfrak{Re}(2\zeta + a_2)d\Psi(\phi)}\right)\Bigg|_{\epsilon=0}$$

$$= \frac{1}{a_2^2}\Bigg(a_2\int_\Gamma \mathfrak{Re}(2\zeta^3 + 4a_2\zeta^2 + 6a_3\zeta + 3a4)d\Psi(\phi)$$
$$- \left(|a_4| - \frac{2}{3M}\right)\int_\Gamma \mathfrak{Re}(2\zeta + a_2)d\Psi(\phi) + \frac{2a_2}{3M^2}\frac{d}{d\epsilon}M_\epsilon\bigg|_{\epsilon=0}\Bigg)$$

$$(5) \quad = \frac{\int_\Gamma \mathfrak{Re}\left[A_3\zeta^3 + A_2\zeta^2 + A_1\zeta + A_0\right] d\Psi(\phi) + \frac{2a_2}{3M^2}\frac{d}{d\epsilon}M_\epsilon\Big|_{\epsilon=0}}{a_2^2}$$

$$=\frac{\int_\Gamma \mathfrak{Re}(K(\zeta))d\Psi(\phi)+\frac{2a_2}{3M^2}\frac{d}{d\epsilon}M_\epsilon\Big|_{\epsilon=0}}{a_2^2},$$

where

$$\begin{aligned} A_3 &= 2a_2\\ A_2 &= 4a_2^2\\ A_1 &= 6a_2a_3-2|a_4|+\frac{4}{3M}\\ A_0 &= 3a_2a_4-a_2|a_4|+\frac{2a_2}{3M}. \end{aligned}$$

are the coefficients of the kernel of the principal integral in this variation, which we shall call $K(\zeta)$. We notice, first of all, that $K(\zeta)$ is a cubic in ζ. Since, as we stated earlier, we are only interested in functions for which $|a_2|\neq 0$, we may disregard the denominator in the above expression (5), since $a_2^2>0$ and thus determines neither the sign nor the root of the derivative. Notice that since, in most of the variations we will deal with, M is being held constant, the portion of the numerator not accounted for by the integral of $K(\zeta)$ vanishes.

We can also express $\zeta=e^{i\phi}$ and then, recalling that for general complex z and w,

$$\mathfrak{Re}(zw)=\mathfrak{Re}(z)\mathfrak{Re}(w)-\mathfrak{Im}(z)\mathfrak{Im}(w),$$

we rewrite the above expression from (5) in terms of trigonometric functions as

$$\begin{aligned} \int_\Gamma \Bigg[2a_2\cos(3\phi)+4a_2^2\cos(2\phi)+\left(6a_2\mathfrak{Re}(a_3)-2|a_4|+\frac{4}{3M}\right)\cos\phi\\ -6a_2\mathfrak{Im}(a_3)\sin\phi+3a_2\mathfrak{Re}(a_4)-a_2|a_4|+\frac{2a_2}{3M}\Bigg]d\Psi=0 \end{aligned} \tag{6}$$

There are three key facts to notice about the expressions (5) and (6). First, the kernel in (6), being a third order trigonometric polynomial, has at most six roots. Second, the image of $\mathbb{D}$ under the kernel of (5) will take $\partial\mathbb{D}$ to a closed loop crossing the imaginary axis, and in fact any vertical line, the same number of times as the kernel in (6) has roots, which, as we noted, is at most six (see figures 3 and 2 for examples). Finally, notice that the derivative of $\mathfrak{Re}(K(\zeta))$ will also be a third order trigonometric polynomial, indicating that this loop can change directions (with respect to the real axis) at most 6 times as well, at most 3 of which are either minima or maxima with respect to the real axis.

We should also note the allowable movements for our variations on this problem. On the straight edges we are allowed to push these in or out using a constant

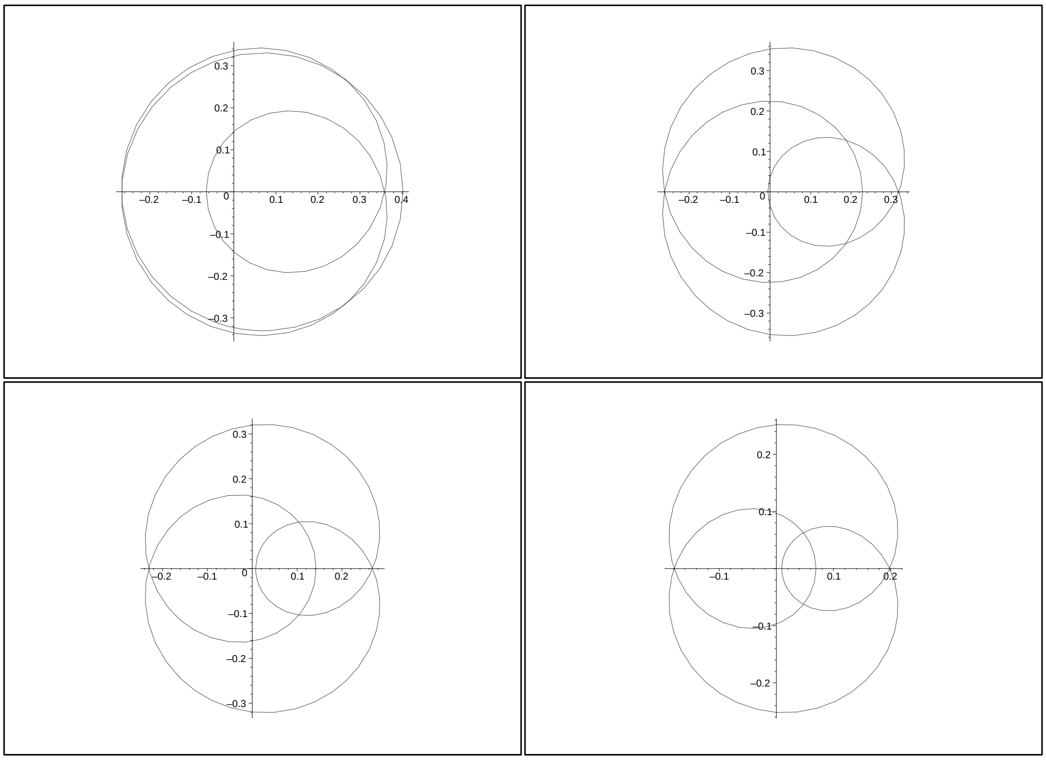

FIGURE 3. A selection of possible $K(\partial\mathbb{D})$ graphs.

$p(w)$, or we can pivot the side on a fixed point in the side by using a linear $p(w)$ which is equal to 0 at a predetermined fixed point. In some cases, if necessary, we will be able to push in on a subarc of a boundary arc in order to create a new proper side. We may also create piecewise linear variations, but we must be careful to maintain convexity; so while one part of a straight edge may be pivoted inwards from an interior point (provided this segment extends to one of the endpoints), we cannot pivot such a segment outward as this would violate convexity. In addition, there is one possible scenario in which we cannot move certain sides inward without altering the bound M. The only situation in which this can happen is if the image Ω has exactly one vertex $f(\xi)$ which lies on the bounding circle $M\mathbb{D}$ and $\mathfrak{Re}(K(\xi)) < 0$. In this situation the allowed movements will not change, but we must be careful to observe that moving either of the sides connecting to $f(\xi)$ inward will result in a decrease in M_ϵ, whose effect on $\frac{d}{d\epsilon}\mathbf{A}_\varepsilon$ must be checked in a different manner than the rest.

With these observations in mind, we can state and prove our main theorem.

Theorem 3.1. *For the functional*

$$\mathbf{A} = \frac{|a_4| - \frac{2}{3M}}{|a_2|},$$

where a_k are the normalized coefficients from above, considered over the class $\mathcal{K}_M$, an extremal value for the functional, if it exists, will be obtained by a function that takes $\mathbb{D} = \{z\ : |z| < 1\}$ onto a region with at most 3 proper sides.

Proof. First, we shall assume a function $f \in \mathcal{K}_{M,n}$ is a local extremum for $\mathbf{A}$ at which $\mathbf{A}$ achieves a maximal value, and we shall assume that the boundary of $f(\mathbb{D}) = \Omega$ has some finite number of proper sides $0 < s \leq n$ and some finite number of bounding arcs $r \geq 0$ which lie along $\partial M\mathbb{D}$. We will not be considering "false" sides, that is if any two sides, as have been explained, meet in such a way that $f'(\zeta)$ is continuous at that point, then either these two sides are collinear, or they are connected arcs along $\partial M\mathbb{D}$. In either case, we will simply consider the combination of these as a single side. For this reason, we can state that $r \leq s$, if $s \neq 0$. Note that the case $s = 0$ would force f to be the identity map, and need not be considered here.

We will label these sides $\Gamma_1,\Gamma_2,\ \ldots,\ \Gamma_s,\Gamma_{s+1},\ldots,\Gamma_{s+r}$ in such a way that $j \leq s$ implies that Γ_j is a proper side and $j > s$ implies that Γ_j is a bounding arc. For each Γ_j, there is an arc along $\partial\mathbb{D}$, call it γ_j for which $f(\gamma_j) = \Gamma_j$. Furthermore, when considering the kernel $K(\zeta)$ given in (6), $K(\gamma_j) = \sigma_j$ is an arc along the closed loop $K(\partial\mathbb{D})$ (see Figure 4).

Now, by the pigeon hole principle, at most 6 of $\{\sigma_j\}$ may cross any vertical line, because (6) has at most 6 roots, as discussed earlier. This fact will be utilized in the following Lemma.

Lemma 3.2. *For the functional $\mathbf{A}$ considered over the class $\mathcal{K}_M$, if f is assumed to be a function at which $\mathbf{A}$ attains a maximal value,*

1. *$f(\partial\mathbb{D})$ must be made up of at most 6 proper sides, i.e. $f \in \mathcal{K}_{M,6}$, and either*
2. *all proper sides of $f(\partial\mathbb{D})$ must be associated to arcs whose mean points (under the measure $d\Psi(\phi)$ with a piecewise constant $p(w)$) lie on the same vertical line, or*
3. *all proper sides of $f(\partial\mathbb{D})$ except two must be associated to arcs whose mean points under the measure $d\Psi(\phi)$ with a piecewise constant $p(w)$ lie on the same vertical line along with the mean points of the remaining two arcs considered under a pivoting definition for $p(w)$.*

Proof of Lemma 3.2. Assume that $f \in \mathcal{K}_M$ is a maximal configuration for the functional $\mathbf{A}$. Let us assume that $f(\partial\mathbb{D})$ has more than 6 proper sides $(s > 6)$,

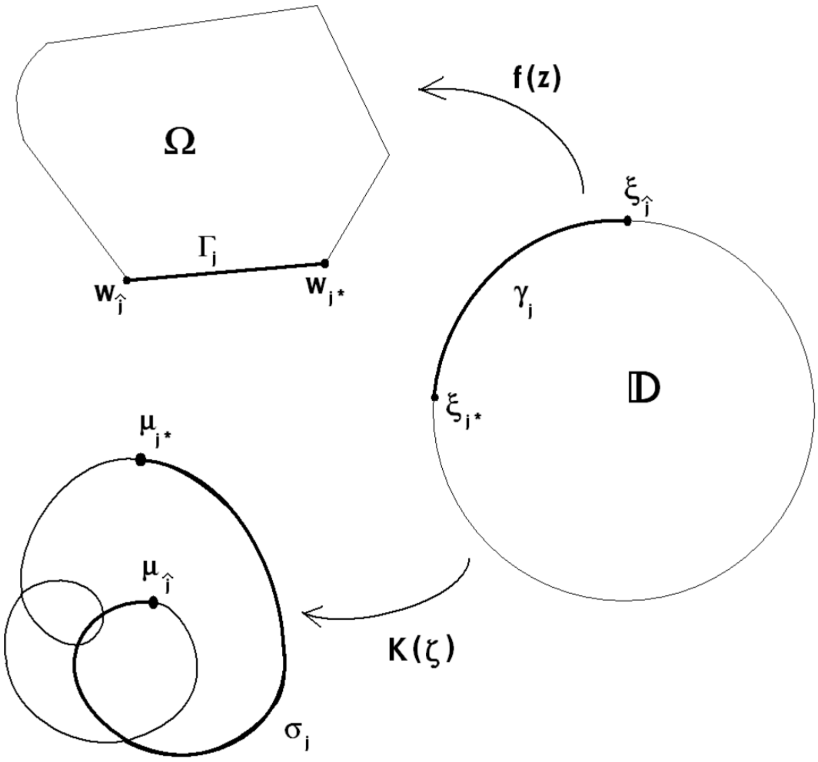

FIGURE 4. An illustration of the notation to be used.

$\Gamma_1,\ \Gamma_2,\ \cdots,\Gamma_s$ and we label by $\xi_j \in \gamma_j$ those points which satisfy the mean value theorem for the respective integrals

$$\mathfrak{Re}(K(\xi_j))\int_{\gamma_j}\frac{1}{2\pi|f'(\zeta)|}d\phi=\int_{\gamma_j}\mathfrak{Re}(K(\zeta))\frac{1}{2\pi|f'(\zeta)|}d\phi, \tag{7}$$

for $j=1,2,\ldots,s$. Then by the pigeon hole principle, at least one of the points $K(\xi_j)$ must fail to lie on the vertical line given by $\mathfrak{Re}(z)=\mathfrak{Re}(K(\xi_1))$. If $n\le 6$ denotes the number of ξ_j for which $K(\xi_j)$ do lie on the vertical line $\mathfrak{Re}(z)=\mathfrak{Re}(K(\xi_1))$, let us renumber these remaining points so that only the lines $\Gamma_1,\ \Gamma_2,\ \cdots\Gamma_n$ have the property that $\mathfrak{Re}(K(\xi_j))=\mathfrak{Re}(K(\xi_j))$ for $j=1..n$. Now, assume that $\mathfrak{Re}(K(\xi_1))<\mathfrak{Re}(K(\xi_s))$, and for some $\alpha>0$ assign to $p(w)$ the values

$$\frac{-\alpha}{\int_{\gamma_1}\frac{1}{2\pi|f'(\zeta)|}d\phi}\ ,\ \text{for } w\in\Gamma_1, \tag{8}$$

the value

$$\frac{\alpha}{\int_{\gamma_s} \frac{1}{2\pi|f'(\zeta)|} d\phi} \text{ , for } w \in \Gamma_s, \tag{9}$$

and 0 otherwise. Notice that using these values,

$$\int_{\partial\mathbb{D}} d\Psi(\phi) = \alpha - \alpha = 0. \tag{10}$$

Thus, since our variation has not adjusted M, and neither has our renormalization, we know that $f_\epsilon \in \mathcal{K}_M$.

Now notice

$$\int_{\partial\mathbb{D}} \mathfrak{Re}(K(\zeta)) \frac{p(w)}{2\pi|f'(\zeta)|} d\phi$$

$$= \int_{\gamma_1} \mathfrak{Re}(K(\zeta)) \frac{p(w)}{2\pi|f'(\zeta)|} d\phi + \int_{\gamma_s} \mathfrak{Re}(K(\zeta)) \frac{p(w)}{2\pi|f'(\zeta)|} d\phi \tag{11}$$

$$= \int_{\gamma_1} \frac{-\alpha \mathfrak{Re}(K(\zeta))}{\int_{\gamma_1} \frac{1}{|f'(\zeta)|} d\phi |f'(\zeta)|} d\phi + \int_{\gamma_s} \frac{\alpha \mathfrak{Re}(K(\zeta))}{\int_{\gamma_s} \frac{1}{|f'(\zeta)|} d\phi |f'(\zeta)|} d\phi$$

$$= \frac{-\alpha}{\int_{\gamma_1} \frac{1}{|f'(\zeta)|} d\phi} \int_{\gamma_1} \frac{\mathfrak{Re}(K(\zeta))}{|f'(\zeta)|} d\phi + \frac{\alpha}{\int_{\gamma_s} \frac{1}{|f'(\zeta)|} d\phi} \int_{\gamma_s} \frac{\mathfrak{Re}(K(\zeta))}{|f'(\zeta)|} d\phi$$

$$= \frac{-\alpha \mathfrak{Re}(K(\xi_1))}{\int_{\gamma_1} \frac{1}{|f'(\zeta)|} d\phi} \int_{\gamma_1} \frac{1}{|f'(\zeta)|} d\phi + \frac{\alpha \mathfrak{Re}(K(\xi_s))}{\int_{\gamma_s} \frac{1}{|f'(\zeta)|} d\phi} \int_{\gamma_s} \frac{1}{|f'(\zeta)|} d\phi$$

$$= \alpha \Big(\mathfrak{Re}(K(\xi_s)) - \mathfrak{Re}(K(\xi_1) \Big), \tag{12}$$

which by construction is positive.

If $\mathfrak{Re}(K(\xi_s)) < \mathfrak{Re}(K(\xi_1))$, this entire procedure can be completed using $\alpha < 0$ instead, and will yield the same results. Thus, we have increased the value of $\mathbf{A}$, contradicting the maximality of f. Since this procedure can be implemented for any $f \in \mathcal{K}_M \setminus \mathcal{K}_{M,6}$, we see that maximal configurations for $\mathbf{A}$ must lie in $\mathcal{K}_{M,6}$.

Furthermore, we can show that there is only one possible configuration for f under which more than one arc of $K(\partial\mathbb{D})$ which associates to a proper side may lie entirely on one side of any vertical line. Suppose we have some vertical line

$\mathfrak{Re}(z) = \rho$ such that at least two proper sides, Γ_1 and Γ_2 which associate to arcs σ_1 and σ_2 which have no interior intersection points with the vertical line. Then we have two possible situations, either these two arcs lie on the same side of the vertical line, or they lie on opposite sides.

If they lie on opposite sides of the the vertical line, then, without loss of generality, assume $\mathfrak{Re}(\mu) < \rho$ for all $\mu \in \sigma_1$ and $\mathfrak{Re}(\mu) > \rho$ for all $\mu \in \sigma_2$. Then, for some real number α we may assign to $p(w)$ the values given in (8) and (9) substituting $s = 2$.

Under this assignment (10) holds, and by construction $\mathfrak{Re}(K(\xi_1)) < \mathfrak{Re}(K(\xi_2))$ so (12) also holds and is positive. Notice that if $\mathfrak{Re}(\mu) > \rho$ for $\mu \in \sigma_1$ and $\mathfrak{Re}(\mu) < \rho$ for $\mu \in \sigma_2$, then choose $\alpha < 0$ and both of these results still hold. Thus, in this situation, we have found a manipulation of these sides which will increase the value of $\mathbf{A}$, which contradicts the assumed maximality of f.

Notice that (10) still holds so $f_\epsilon \in \mathcal{K}_M$, and the integral in (6) will take on the same value,

$$\alpha\Bigg(\mathfrak{Re}\big(K(\xi_2)\big) - \mathfrak{Re}\big(K(\xi_1)\big) \Bigg) > 0$$

again contradicting the assumed maximality of f.

Thus, in order for more than one side to be associated to arcs of $K(\partial\mathbb{D})$ which lie entirely on the same side of some vertical line, the mean value points for those arcs must lie on the same vertical line.

Since $K(\partial\mathbb{D})$ is a compact loop, we can take some $B > 0$ such that $\mathfrak{Re}(K(\zeta)) < B$ for all $\zeta \in \partial\mathbb{D}$. In this way, we can show that all the mean value points ξ_j for arcs associated to proper sides must lie on the same vertical line.

Note now, that we have only to consider the possibility that, as mentioned before, there is exactly a single point of $\partial\Omega$ on the bounding circle $M\mathbb{D}$ and the two sides, say Γ_1 and Γ_2, connected to that point have associated arcs whose mean value points, as explained in (7), might lie to the left of all other associated arcs. In this situation, we cannot simply push in the side, as this would change M. Instead, let w_0 be the single point of $\partial\Omega$ on $M\mathbb{D}$, and define by ξ_j the points such that

$$\mathfrak{Re}(K(\xi_j)) \int_{\gamma_j} \frac{|w - w_0|}{2\pi|f'(\zeta)|}\, d\phi = \int_{\gamma_j} \mathfrak{Re}(K(\zeta)) \frac{|w - w_0|}{2\pi|f'(\zeta)|}\, d\phi. \tag{13}$$

for $j = 1, 2$.

Now, if any of this new collection of mean value points (2 modified, the rest the same as before) is such that $\mathfrak{Re}(K(\xi_j)) < \mathfrak{Re}(K(\xi_k))$ for any $k \neq j$, we may again perform a variation to raise the value of the functional, but if one of the sides Γ_1 or Γ_2 must be moved, we define $p(w)$ to be

$$\frac{-\alpha|w-w_0|}{\int_{\gamma_j}\frac{|w-w_0|}{2\pi|f'(\zeta)|}\,d\phi}\text{, for } w\in\Gamma_j, \tag{14}$$

where again, α may be taken to be either positive or negative, depending on whether $\mathfrak{Re}(K(\xi_j)) < \mathfrak{Re}(K(\xi_k))$ or the opposite. ■

Now in order to formulate a step down lemma which will allow us to further reduce the number of sides possible, we shall prove several other lemmas which we will use in its proof.

For each of the following, again suppose $f(z)$ is a local maximum for **A** over $\mathcal{K}_M$. Then, by Lemma 3.2 there can be at most 6 straight sides belonging to the image $f(\partial\mathbb{D})$.

Lemma 3.3. *Provided that we consider only the situations where M may be held constant, no proper side in this maximal configuration may associate to an arc which crosses any vertical line exactly once (Note that for a line to cross a vertical line exactly once, it must have exactly one interior point which lies on the vertical line, and at at least one interior point must lie on each side of the same line.)*

Lemma 3.4. *Provided that we consider only the situations where M may be held constant, because of Lemma 3.3, each arc σ_j which associates to a proper side must contain at least one point $\mu_j = K(\zeta_j)$ within its interior for which $\frac{d}{d\phi}\mathfrak{Re}(K(\zeta_j)) = 0$.*

Lemma 3.5. *Provided that we consider only the situations where M may be held constant, each arc which associates to a proper side of $\partial\Omega$ which contains only one direction change point within it's interior must be situated so that both of its endpoints have equal real parts.*

Lemma 3.6. *Provided that we consider only the situations where M may be held constant, no arc which associates to a proper side can be situated in such a way that a vertical line can separate the endpoints of that segment from every interior direction change point of the arc (with respect to the real axis), unless the endpoints would lie to the right of such a vertical line.*

And finally, we state the actual step down lemma.

Lemma 3.7. *Provided that we consider only the situations where M may be held constant, because of Lemmas 3.3-3.6 and because both $\mathfrak{Re}(K(\zeta))$ and $\frac{d}{d\phi}\mathfrak{Re}(K(\zeta))$ each have at most 6 roots, the maximum number of proper sides of $f(\partial\mathbb{D})$ is 3.*

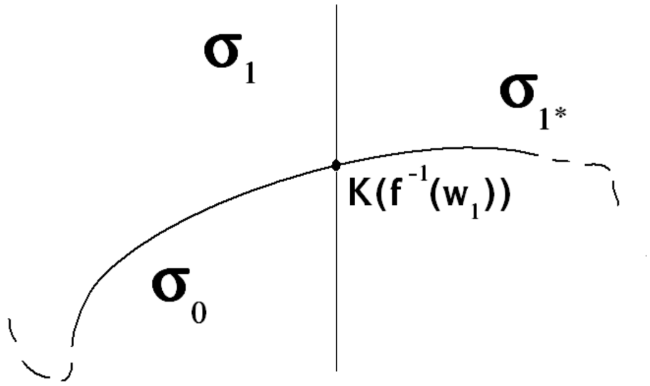

FIGURE 5. An arc σ_1 which crosses a vertical line exactly once.

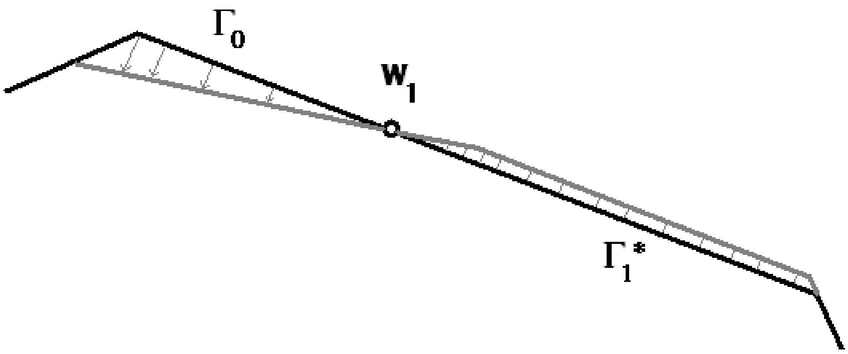

FIGURE 6. Illustration of piecewise $p(w)$ used to prove Lemma 3.3.

Proof of Lemma 3.3. Suppose that one side Γ_1 has an arc σ_1 which the kernel $K(\zeta)$ associates to it, and σ_1 contains exactly one interior point $K(f^{-1}(w_1))$ which lies on some vertical line $\mathfrak{Re}(z) = \rho$, and that, furthermore, at least one point in σ_1 lies on both sides of the same vertical line. Label the portion of side Γ_1 which is mapped to the left of the vertical line Γ_0 and label the remainder of the side Γ_{1*}. Now, for some $\alpha > 0$ assign to $p(w)$ the values

$$\frac{-\alpha|w - w_1|}{\int_{\gamma_0} \frac{|w-w_1|}{2\pi|f'(\zeta)|} d\phi} \text{ , for } w \in \Gamma_0, \tag{15}$$

the value

$$\frac{\alpha}{\int_{\gamma_{1*}} \frac{1}{2\pi|f'(\zeta)|} d\phi} \text{ , for } w \in \Gamma_{1*}, \tag{16}$$

and 0 otherwise.

Notice that the movements indicated by these variations correspond to a pivoting inward of one part of the line, upon the fixed point w_1, and then the second part of the line is pushed outward constantly, as shown in Figure 6. The corner of this variation can be thought of in the same way which Barnard and Lewis showed that a side may be pushed out against a boundary, which it cannot cross, and the resulting error is absorbed into the $o(\epsilon)$ term shown in (1) and thus has no appreciable effect on either the change in the mapping radius or the resultant effect on the kernel given in (6).

Define by ξ_{1*} the point guaranteed by the Mean Value Theorem so that (7) applies to Γ_{1*} and by ξ_0 the point guaranteed by the Mean Value Theorem so that

$$\mathfrak{Re}(K(\xi_0))\int_{\gamma_0}\frac{|w-w_1|}{2\pi|f'(\zeta)|}d\phi=\int_{\gamma_0}\mathfrak{Re}(K(\zeta))\frac{|w-w_1|}{2\pi|f'(\zeta)|}d\phi. \tag{17}$$

Unlike some of our previous variations, this particular procedure cannot be reversed, since using $\alpha < 0$ would immediately violate convexity.

Now, since σ_0 lies to the left of the chosen vertical line, and σ_{1*} lies to the right, we know that

$$\mathfrak{Re}(K(\xi_0)) < \mathfrak{Re}(K(\xi_{1*})).$$

Now, as before, we see that

$$\int_{\partial\mathbb{D}}\mathfrak{Re}(K(\zeta))\frac{p(w)}{2\pi|f'(\zeta)|}d\phi$$

$$= \int_{\gamma_0}\mathfrak{Re}(K(\zeta))\frac{p(w)}{2\pi|f'(\zeta)|}d\phi+\int_{\gamma_{1*}}\mathfrak{Re}(K(\zeta))\frac{p(w)}{2\pi|f'(\zeta)|}d\phi \tag{18}$$

$$= \int_{\gamma_0}\frac{-\alpha\mathfrak{Re}(K(\zeta))|w-w_1|}{\int_{\gamma_0}\frac{|w-w_1|}{|f'(\zeta)|}d\phi|f'(\zeta)|}d\phi+\int_{\gamma_{1*}}\frac{\alpha\mathfrak{Re}(K(\zeta))}{\int_{\gamma_{1*}}\frac{1}{|f'(\zeta)|}d\phi|f'(\zeta)|}d\phi$$

$$= \frac{-\alpha}{\int_{\gamma_0}\frac{|w-w_1|}{|f'(\zeta)|}d\phi}\int_{\gamma_0}\frac{\mathfrak{Re}(K(\zeta))|w-w_1|}{|f'(\zeta)|}d\phi+\frac{\alpha}{\int_{\gamma_{1*}}\frac{1}{|f'(\zeta)|}d\phi}\int_{\gamma_{1*}}\frac{\mathfrak{Re}(K(\zeta))}{|f'(\zeta)|}d\phi$$

$$= \frac{-\alpha\mathfrak{Re}(K(\xi_0))}{\int_{\gamma_0}\frac{|w-w_1|}{|f'(\zeta)|}d\phi}\int_{\gamma_0}\frac{|w-w_1|}{|f'(\zeta)|}d\phi+\frac{\alpha\mathfrak{Re}(K(\xi_{1*}))}{\int_{\gamma_{1*}}\frac{1}{|f'(\zeta)|}d\phi}\int_{\gamma_{1*}}\frac{1}{|f'(\zeta)|}d\phi$$

$$= \alpha\Big(\mathfrak{Re}(K(\xi_{1*})) - \mathfrak{Re}(K(\xi_0)\Big), \tag{19}$$

which is once again positive.

So, applying this variation yields a larger value for the functional, and in addition it increases the number of straight sides, since we've just bent part of a side inward and pushed the remaining portion outward. Furthermore, of the arcs associated to these two new sides, one is located entirely to the right of the other one, and thus, the newly created figure also cannot be a maximal configuration. Since **A** evaluated on $f(z)$ is less than **A** evaluated on $f_\epsilon(z)$, and the new configuration is also not maximal, $f(z)$ also could not have been maximal. ■

Proof of Lemma 3.4. Assume that some proper side, which we can renumber to call Γ_1, does not contain a point within its interior at which the direction of the loop $K(\partial\mathbb{D})$ changes direction (with respect to the real axis). Then it can cross any vertical line at most once, and hence, by Lemma 3.3 f cannot be a maximal configuration. Thus, by contradiction, we may assume that every arc of $K(\partial\mathbb{D})$ which associates to a proper side of $f(\mathbb{D})$ changes direction at least once within its interior. ■

Proof of Lemma 3.5. Suppose that Γ_1 associates to an arc σ_1 which changes direction exactly once. Now, suppose that the two endpoints of σ_1, μ_0 and μ_1 are situated so that $\mathfrak{Re}(\mu_0) \neq \mathfrak{Re}(\mu_1)$.

Then, since the curve σ_1 changes direction only once on its interior, σ_1 can cross any vertical line at most twice. Furthermore, it crosses only one of $\mathfrak{Re}(z) = \mathfrak{Re}(\mu_1)$ or $\mathfrak{Re}(z) = \mathfrak{Re}(\mu_0)$ exactly twice, and the other it only touches once. Without loss of generality, assume that the line $\mathfrak{Re}(z) = \mathfrak{Re}(\mu_0)$ lies to the left of the line $\mathfrak{Re}(z) = \mathfrak{Re}(\mu_1)$.

Then, let ρ be any real number between $\mathfrak{Re}(\mu_0)$ and $\mathfrak{Re}(\mu_1)$, then the vertical line $\mathfrak{Re}(z) = \rho$ crosses the line σ_1 only once, and σ_1 properly crosses the line, thus by Lemma 3.3, this contradicts the maximality of f. ■

Proof of Lemma 3.6. If we assume that there is some proper side Γ_1 which associates to an arc σ_1 which is configured in such a way that its two endpoints can be separated from all interior points of σ_1 at which σ_1 changes direction with respect to the real axis in such a way that σ_1 crosses the vertical line exactly twice, and if the two endpoints of σ_1 lie on the left of this vertical line, then we may move the vertical line to the left as far as liked, provided that the endpoints of σ_1 remain to the left of the new vertical line.

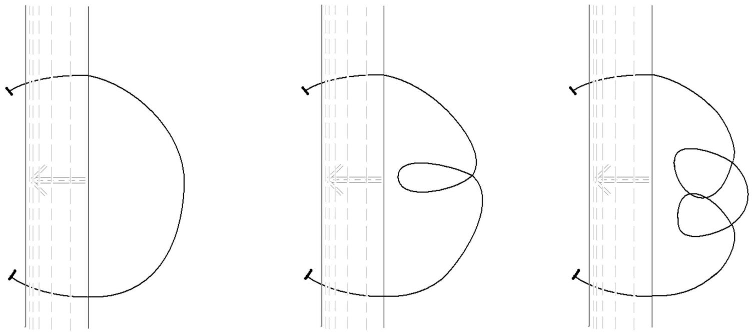

Figure 7. An illustration of the types of curves being ruled out under Lemma 3.6.

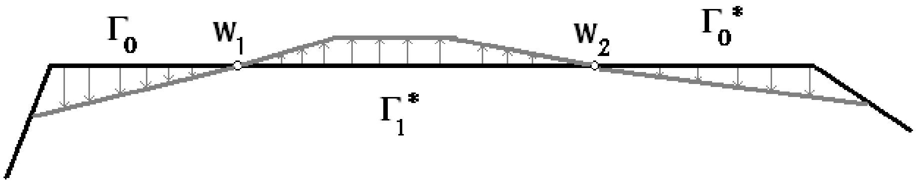

Figure 8. Illustration of piecewise $p(w)$ used in the proof of Lemma 3.6.

In this way, we may prevent the map $p(w)$, which we will formulate, from causing the two pivoted boundaries for the central side to meet in such a way as to eliminate the remaining side all together. Once such a vertical line has been chosen and we have relabeled the points of intersection between σ_1 and this new line to be w_1 and w_2, we can label the portion of Γ_1 which lies entirely on the left of the line, and which connects to w_1 Γ_0, label by Γ_{1*} the portion of the side entirely on the right side of the line between w_1 and w_2, and by Γ_{0*} the remaining portion of the side, which also lies on the left side of the chosen vertical line and which is bordered by w_2. Now, assign to $p(w)$ the values

$$\frac{-\alpha|w-w_1|}{\int_{\gamma_0} \frac{|w-w_1|}{2\pi|f'(\zeta)|}d\phi}\text{ , for } w \in \Gamma_0, \tag{20}$$

the value

$$\frac{\alpha+\beta}{\int_{\gamma_{1*}} \frac{1}{2\pi|f'(\zeta)|}d\phi}\text{ , for } w \in \Gamma_{1*}, \tag{21}$$

the value

$$-\frac{\beta|w-w_2|}{\int_{\gamma_{0*}} \frac{|w-w_2|}{2\pi|f'(\zeta)|}d\phi}\ , \text{ for } w \in \Gamma_{0*}, \tag{22}$$

and 0 otherwise. Here we must require that both α and β be positive in order to maintain convexity.

Of some concern here, is whether or not the constant motion on side Γ_{1*} would need to move the side further than the two pivoting maps would allow in order to prevent the change to the mapping radius. It can be seen, however, that provided we have chosen α and β large enough so that the inequality

$$\frac{\alpha\beta|w_1-w_2|}{\alpha+\beta} > \frac{\alpha\int_{\gamma_{0*}} \frac{|w-w_2|}{2\pi|f'(\zeta)|}d\phi + \beta\int_{\gamma_0} \frac{|w-w_1|}{2\pi|f'(\zeta)|}d\phi}{\int_{\gamma_{1*}} \frac{1}{|f'(\zeta)|}d\phi}$$

holds, then the angle at which the variation pushes in the corners will make any possible intersection between the created diagonal lines further out than the middle side is moved. Notice that because we had some control initially over the size of all three arcs, we can reduce the value of the two integrals $\int_{\gamma_{0*}} \frac{|w-w_2|}{2\pi|f'(\zeta)|}d\phi$ and $\int_{\gamma_0} \frac{|w-w_1|}{2\pi|f'(\zeta)|}d\phi$ by choosing a line closer to the two endpoints, which at the same time will increase the values of $|w_1-w_2|$ and $\int_{\gamma_{1*}} \frac{1}{2\pi|f'(\zeta)|}d\phi$.

Now, having performed this variance as described we see that we achieve a resulting positive variance in the functional value in a manner similar to (19), and at the same time we have introduced more sides whose associated arcs lie such that one of them lies entirely to the right of the the other two. Thus, this new mapping which yields a larger functional value also cannot be a maximal configuration, and neither could the original configuration. ■

Proof of Lemma 3.7. Suppose that f is a maximal configuration for the functional $\mathbf{A}$. We know already by Lemma 3.2 that the maximum number of proper sides must be 6 and by the second part of Lemma 3.2, the line $\mathfrak{Re}(z) = \mathfrak{Re}(K(\xi_1))$ contains at least one interior point of each arc which associates to a proper side. These interior points must either be places at which the arc σ_j crosses this vertical line, or they must be points of tangency to the line, and hence be an interior point at which the arc changes direction (with respect to the real axis).

If the arc σ_j crosses the line, then by Lemma 3.3, there must also exist a second interior point at which the arc σ_j crosses this vertical line. Assume these are the only two points at which this arc crosses the vertical line. Also, by Lemma 3.4 and by noting that $K(\zeta)$ is continuous and continuously differentiable, we see that at least one interior point which falls strictly between these two crossings must be a point of direction change. Now by Lemma 3.6, if these are the only

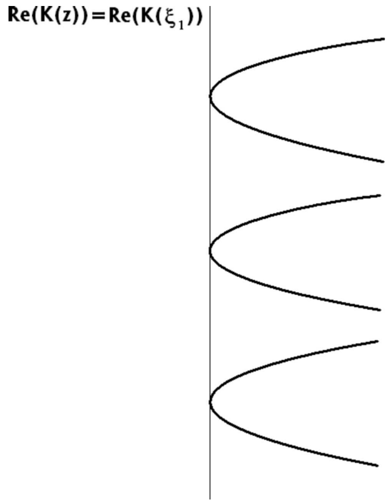

Figure 9. Illustration of three maximal tangencies.

direction changes along σ_j, then the endpoints of σ_j must point to the right, and the subarc between the two crossing points must lie to the left or be tangent to the vertical line.

Similarly, if the arc σ_j is tangent to the line at an interior point at which the direction changes, then if that is the only interior point of the arc where the direction changes, then since we could choose a vertical line far enough towards the endpoints that it would separate this point from the endpoints, according to Lemma 3.6, σ_j must be situated so that it lies entirely to the right of the vertical line, except for the point of tangency.

If every arc which associates to a proper side crosses the vertical line, then as shown, they must each cross at least twice, hence by counting the number of crossings possible, $s \leq 3$.

If every arc which associates to a proper side does not cross the vertical line, but rather is tangent to it, then each of these tangencies must also be points of direction change. If these are the only interior changes of direction for these arcs, then because a vertical line can be found which separates this point from the two endpoints, by Lemma 3.6, the remainder of the arc must lie to the right of the vertical line $\mathfrak{Re}(z) = \mathfrak{Re}(K(\xi_1))$ except for the point at which it is tangent. In this case, we have a situation similar to figure 9 in which three direction changes are tangent to the vertical line, and the endpoints of those sides point to the right of the line.

Notice that in this case, we must have only three additional points at which the curve $K(\zeta)$ can change direction, and each of these must occur within the connections between the six endpoints associated to the arcs which are currently accounted for. Further, any arc which touches the vertical line containing the averaging value points would create a seventh solution of $\mathfrak{Re}(K(\zeta)) = \rho$ for some real number ρ, which is impossible, as has been noted.

Furthermore, in the event that either some arc σ_j crosses the vertical line $\mathfrak{Re}(z) = \mathfrak{Re}(K(\xi_1))$ more than twice, or if the interior of an arc which is tangent to this line contains additional direction change points, then counting points of direction change and/or points where $K(\zeta)$ crosses this vertical line yields that there can be at most two additional sides. Thus, our assertion is verified. ∎

Now, suppose that the configuration for $f(\partial\mathbb{D})$ is as in Lemma 3.3 except that the only side which crosses some vertical line exactly once is one of the two lines which meet at the only point w_0 which lies on the bounding circle $M\mathbb{D}$. Suppose also that both of the arcs σ_1 and σ_2, whose associated sides connect to w_0, contain one direction change within their interiors (and another one somewhere between these). Then this leaves only 3 remaining direction change points, and since Lemmas 3.3-3.6 apply to all other sides of $\partial\Omega$, and since we know that at most one further direction change can happen to be a minimum (with respect to the real axis), then there can be only one proper side remaining, and possibly two bounding arcs which lie entirely to the right of the rest of the figure and the assertion of Theorem 3.1 is satisfied.

Now, if one of the two arcs does not contain a direction change within its interior, say this side is Γ_1, then any vertical line which crosses σ_1 crosses it only once. Unlike the situation in Lemma 3.3, note that rather than showing that the integral of the kernel is positive, it must also be larger than the remaining term in the numerator given in (6), which will be negative.

In order to have

$$\frac{\int_{\partial\mathbb{D}} \mathfrak{Re}(K(\zeta))d\Psi(\phi) + \frac{2a_2}{3M^2}\frac{d}{d\epsilon}M_\epsilon\Big|_{\epsilon=0}}{a_2^2} \geq 0$$

it is sufficient to have

$$\frac{1}{a_2}\int_{\partial\mathbb{D}} \mathfrak{Re}(K(\zeta))d\Psi(\phi) \geq -\frac{2}{3M^2}\frac{d}{d\epsilon}M_\epsilon\Big|_{\epsilon=0}.$$

Further, notice that

$$\frac{1}{a_2}\int_{\partial\mathbb{D}} \mathfrak{Re}(K(\zeta))d\Psi(\phi) \geq \int_{\partial\mathbb{D}} \mathfrak{Re}(K(\zeta))d\Psi(\phi)$$

and that

$$-\frac{d}{d\epsilon}M_\epsilon\Big|_{\epsilon=0} \geq -\frac{2}{3}\frac{d}{d\epsilon}M_\epsilon\Big|_{\epsilon=0} \geq -\frac{2}{3M^2}\frac{d}{d\epsilon}M_\epsilon\Big|_{\epsilon=0},$$

since $a_2 \leq 1$ and $M \geq 1$.

Since any vertical line crosses Γ_1 only once, we may choose a vertical line l far enough to the left that the distance between the point w_0 and w_l, the point associated to the intersection between l and σ_1, is as small as necessary. Further, note that geometrically we may state that $\left|\frac{d}{d\epsilon}M_\epsilon\right| \leq |w_0 - w_l|$. Then, we apply the same variational assignment to $p(w)$ found in (15) and (16), and observe that we then need to assert the inequality

$$\alpha\Bigg(\mathfrak{Re}(K(\xi_1)) - \mathfrak{Re}(K(\xi_0))\Bigg) \geq |w_0 - w_l|$$

in order to verify that the resultant variation, indeed increases the functional.

Since the term on the right is independent of α, however, we simply must choose α sufficiently large to make this inequality true.

Thus, this contradicts the assumed maximality of f also, just as in Lemma 3.3. Now, adding this new information to the remaining Lemmas 3.4-3.7 proves the Theorem for this remaining exception.

Moreover, this gives us a list of the possible configurations for which a maximal function could take $\mathbb{D}$ onto a figure with 3 proper sides.

■

In Figure 10 we see a listing of the possibilities for figures which have at most three proper sides. Notice the configuration which Fournier, Ma, and Ruscheweyh initially proposed falls into the category of possible maximal configurations.

In the next section, we will analyze this family of functions proposed by Fournier, Ma, and Ruscheweyh to show that the best possible bound for the functional **A** over the class of symmetric isosceles triangle maps must be at least 2. We have found a great deal of numerical evidence to suggest that 2 is actually the best bound over $\mathcal{C}_b$; however, there are other extremal candidates for which this is also true.

4. Analysis of the Extremal Candidates

4.1. Fournier, Ma, and Ruscheweyh's Candidate. In their paper [10], R. Fournier, J. Ma and S. Ruscheweyh conjectured that the extremal value for the functional **A** would be 2 and that this value would be attained by a function of the form:

$$f_{c,\lambda}(z) = \int_0^z \frac{dt}{(1-t)^{2\lambda}(1+2ct+t^2)^{1-\lambda}}$$

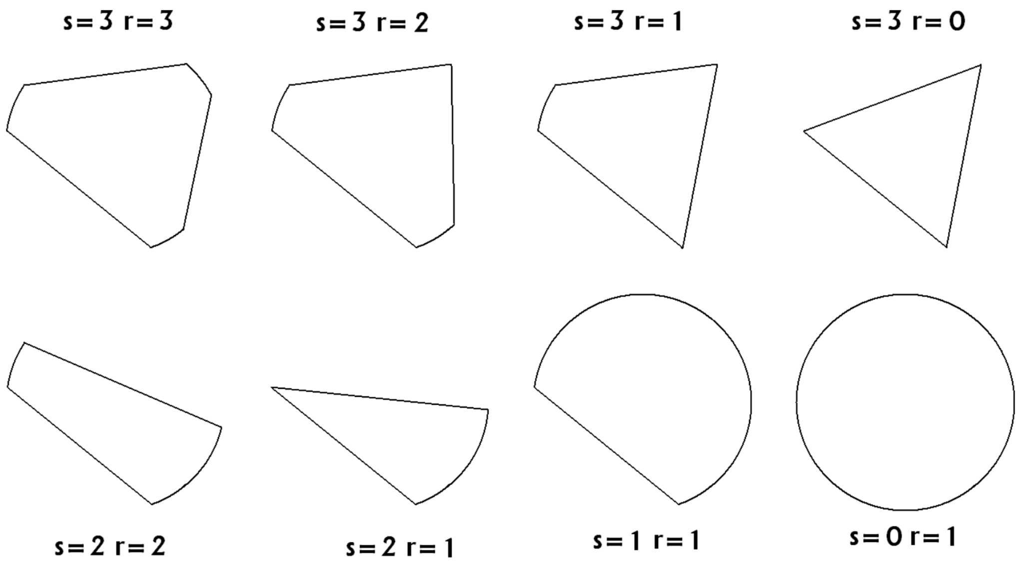

FIGURE 10. The candidates for maximizing $\mathbf{A}$ over K_M.

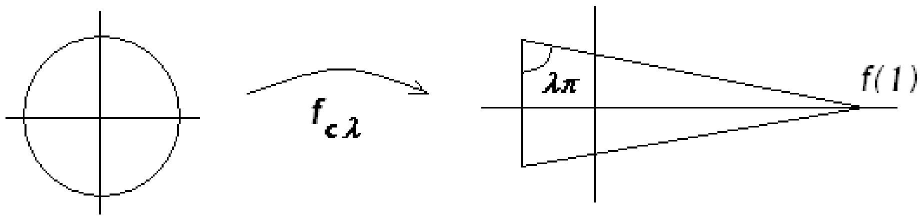

FIGURE 11. The functions $f_{c,\lambda}$ suggested by Fournier, Ma, and Ruscheweyh.

which takes the unit disk to a symmetrically aligned isosceles triangle whose isosceles angles measure $\lambda\pi$ and the remaining angle, which is directly on the real axis, measures $(1-2\lambda)\pi$. They were unable, however, to find a function within this class which would yield the desired value, and they were unable to verify that the functional could not attain higher values over this family. They were able to ascertain that the constant $2/3$ which appears beside the $1/M$ in (1) is the best possible over all bounded convex functions.

With very lengthy computation [11], it is possible to show that the upper bound for our functional for the candidates suggested by Fournier, Ma, and Ruscheweyh must be at least 2.

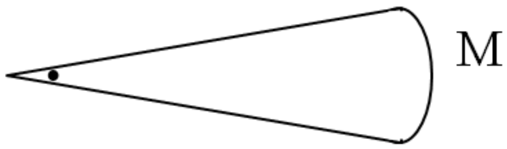

FIGURE 12. A shifted symmetric circular sector.

Proposition 4.1. *Over the family* $\left\{f_{c,\lambda}\middle| -1 < c < 1,\ 0 < \lambda < \frac{1}{2}\right\}$, *an upper bound for the functional* $\mathbf{A}\left(f_{c,\lambda}\right) = \frac{|a_4(f_{c,\lambda})| - \frac{2}{3M(f_{c,\lambda})}}{|a_2(f_{c,\lambda})|}$ *must be at least 2.*

There is actually quite a lot of evidence to suggest that 2 is the best upper bound, but as some of the sections of the parameter space are difficult to deal with analytically, this is difficult to show.

4.2. Shifted Symmetric Circular Sector Maps. There is another set of candidates whose upper bound is also 2, maps onto symmetric curvilinear triangles with two proper sides and the third on a circle of radius M. See figure 12.

We can observe a similar behavior to the case of the Euclidean triangles by observing the behavior of this functional over another family of functions which are derived from the Euclidean triangles by an outward variation along the side perpendicular to the real axis.

Such functions can be thought of as compositions of the following elementary functions:

$$\begin{aligned} w_1 &= \tfrac{1-z}{1+z} \\ w_2 &= kz \\ w_3 &= -z + \sqrt{z^2+4} \\ w_4 &= z^\lambda \\ w_5 &= z - \left(-k + \sqrt{k^2+4}\right)^\lambda \end{aligned}$$

Then define, using a rotation for notational convenience,

$$w_{k,\lambda}(z) = -w_5(w_4(w_3(w_2(w_1(-z)))))$$

we can normalize these functions to be in $\mathcal{N}$ by simply using

$$\begin{aligned} f_{k,\lambda}(z) &= \frac{w_{k,\lambda}(z)}{w'_{k,\lambda}(0)} \\ &= \frac{\sqrt{k^2+4}\left(1 \;-\; (-k+\sqrt{k^2+4})^{-\lambda}\left(\frac{(kz+k)-\sqrt{k^2(z+1)^2+4(z-1)^2}}{(z-1)}\right)^{\lambda}\right)}{2\lambda k} \end{aligned}$$

Using this family of functions, by construction, $0 < k < \infty$ and $0 < \lambda < 1$, where the parameter λ will once more control the interior angle of the sector, while the parameter k will control how far left or right the origin is shifted. The larger the value for k, the closer $f_{k,\lambda}(0)$ shifts towards the curved arc, and the smaller the value of k the closer it is to the vertex lying on the real axis.

Then, we can calculate, for the range of λ and k corresponding to the situation with the Euclidean triangles (large k values and very small λ values),

$$\begin{aligned} |a_2| &= \left|\frac{4-\lambda k\sqrt{k^2+4}}{k^2+4}\right| \\ |a_4| &= \frac{1}{3(k^2+4)}\Big|192+24k^4\lambda^2 \\ &\quad -96k^2(1-\lambda^2)-k\lambda\sqrt{k^2+4}\left(2k^4-16k^2+k^4\lambda^2+4k^2\lambda^2+144\right)\Big| \\ M &= \max\left(\frac{\sqrt{k^2+4}}{\lambda k}\ ,\ \frac{\sqrt{k^2+4}\left(2^{\lambda}\left(-k+\sqrt{k^2+4}\right)^{-\lambda}-1\right)}{2\lambda k}\right) \\ &= \frac{\sqrt{k^2+4}}{\lambda k}. \end{aligned}$$

So, using these values, we have

$$\begin{aligned} \mathbf{A}f_{k,\lambda} = &\frac{1}{3\lambda k(k^2+4)^3-12(k^2+4)^{5/2}}(k^7+8k^5+16k^3)\lambda^3 \\ &-\sqrt{k^2+4}(24k^4+96k^2)\lambda^2+(6k^7+40k^5+272k^3+832k)\lambda \\ &-\sqrt{k^2+4}(24k^4-96k^2+192). \end{aligned}$$

As in the case of the triangle map conjectured to be extremal by Fournier, Ma, and Ruscheweyh, if shrink the central angle toward 0, by letting $\lambda \to 0$ and

$k \to \infty$ at the appropriate relative rates, we again achieve the same maximal limiting value of 2. As is seen in the case of the conjectured triangle, this bound is not attained within the class of bounded functions, but rather only appears in a limiting case.

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