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Symmetric Products of the Real Line

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Abstract. We study the symmetric products of the real line from the point of view of geometric function theory. We investigate geodesics, elementary transformations as well as quasiconvexity properties of the symmetric products. We also investigate their biLipschitz embeddability into Euclidean spaces.

Keywords. Symmetric product, biLipschitz maps, quasiconvexity.

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1. Introduction

The notion of symmetric products of topological spaces was introduced by K. Borsuk and S. Ulam ([4]). The n^{th} *symmetric product* $X^{(n)}$ of a topological space X is the quotient space $X^n/\sim$, where $X^n = \prod_1^n X$ is the product space and $(x_1, x_2, \ldots, x_n) \sim (y_1, y_2, \ldots, y_n)$ if and only if $\{x_1, x_2, \ldots, x_n\} = \{y_1, y_2, \ldots, y_n\}$ as subsets of X. We warn the reader that the usage of the term n^{th} symmetric product is not standard; many authors use it for the quotient space X^n/S_n, where S_n is the symmetric group on n symbols. For $n \geq 3$ the spaces $X^{(n)}$ and X^n/S_n are different. As a set the space $X^{(n)}$ can be identified with the set of all nonempty subsets of X of cardinality less than or equal to n. When the topology on X is induced by a metric d, the topology on $X^{(n)}$ is induced by the Hausdorff metric d_H. Recall that for $x = \{x_1, x_2, \ldots, x_n\} \in X^{(n)}$ and $y = \{y_1, y_2, \ldots, y_n\} \in X^{(n)}$ the Hausdorff distance between x and y is defined by

$$d_H(x, y) = \max\big\{ \max_i \min_j d(x_i, y_j),\ \max_i \min_j d(x_j, y_i) \big\}.$$

The notion of symmetric product has been extensively studied in topology (see, for instance, [1],[5],[9]). In contrast, there have been few studies of this notion in geometric function theory, where one can pose the following types of questions in the context of symmetric products of metric spaces. Given that the space X satisfies certain properties such as, *Ahlfors regularity*, *doubling condition*, *Poincaré Inequality*, *quasiconvexity*, does the space $X^{(n)}$ also satisfy the same

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properties? What is the relation between the Hausdorff dimensions of X and $X^{(n)}$? Does the space $X^{(n)}$ support a *biLipschitz embedding* into some standard space? Finally, is the projection $\pi\colon X^n \to X^{(n)}$, given by $\pi(x_1, x_2, \dots, x_n) = \{x_1, x_2, \dots, x_n\}$, *regular* in the sense of S. Semmes? (See ([6, 10]) for more on these concepts).

In ([3]) M. Borovikova and Z. Ibragimov studied the third symmetric product $\mathbb{R}^{(3)}$ of the real line $\mathbb{R}$ and showed that the space $\mathbb{R}^{(3)}$ is biLipschitz equivalent to $\mathbb{R}^3$ ([3, Theorem 6]). Consequently, it possesses all the properties mentioned above. Also, each isometry of $(\mathbb{R}^{(3)}, d_H)$ is induced by an isometry of $\mathbb{R}$ ([3, Theorem 9]). More precisely, for each isometry $F\colon \mathbb{R}^{(3)} \to \mathbb{R}^{(3)}$ there is an isometry $f\colon \mathbb{R} \to \mathbb{R}$ such that $F(x) = \{f(x_1), f(x_2), f(x_3)\}$ for each $x = \{x_1, x_2, x_3\}$. We believe that such a result holds in $\mathbb{R}^{(n)}$ for all $n \geq 4$. Observe that the converse of this statement is true. Namely, if f is an isometry of $\mathbb{R}$, then the map $\widehat{f}$, defined by $\widehat{f}(\{x_1, x_2, \dots, x_n\}) = \{f(x_1), f(x_2), \dots, f(x_n)\}$, is an isometry of $\mathbb{R}^{(n)}$.

The notion of symmetric products of metric spaces has also been used by the second author in his study of hyperbolic fillings of metric spaces ([7, 8]). For example, in ([7]) the natural identification of $\mathbb{R}^{(2)}$ and the upper-half plane $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \colon y > 0\}$ was used to give a positive answer to a weaker version of a problem posed by D. Sullivan. In ([8]) the symmetric products of a metric space were hyperbolized (in the sense of Gromov) to turn them into hyperbolic fillings of the underlying space. More precisely, given a metric space (X, d), the space $\mathcal{X}^{(n)} = \{x \in X^{(n)} \colon \operatorname{card}(x) \geq 2\}$, endowed with the metric

$$d_{\mathcal{H}}(x, y) = 2 \log \frac{d_H(x, y) + \max\{\operatorname{diam}(x),\, \operatorname{diam}(y)\}}{\sqrt{\operatorname{diam}(x) \operatorname{diam}(y)}},$$

is Gromov δ-hyperbolic with $\delta = \log 4$ and that the identity map between $(\mathcal{X}^{(n)}, d_H)$ and $(\mathcal{X}^{(n)}, d_{\mathcal{H}})$ is a homeomorphism ([8, Theorem 4.7]). Moreover, if $f\colon X \to Y$ is a *power quasisymmetry*, then the map $\widehat{f}\colon (\mathcal{X}^{(n)}, d_{\mathcal{H}}) \to (\mathcal{Y}^{(n)}, d_{\mathcal{H}})$, given by $\widehat{f}(\{x_1, x_2, \dots, x_n\}) = \{f(x_1), f(x_2), \dots, f(x_n)\}$, is a *quasiisometry.* ([8, Theorem 6.6]). Recall that a homeomorphism f between metric spaces (X, d_X) and (Y, d_Y) is called a power quasisymmetry if there exist $\lambda \geq 1$ and $\alpha \geq 1$ such that

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \leq \eta\Big(\frac{d_X(x, y)}{d_X(x, z)}\Big) \qquad \text{for all} \quad x, y, z \in X,$$

where $\eta(t) = \lambda \max\{t^{1/\alpha}, t^\alpha\}$. A map $g\colon X \to Y$ is called a *quasiisometry* if there exist constants $\lambda \geq 1$ and $k \geq 0$ such that $\operatorname{dist}(y, g(X)) \leq k$ for each $y \in Y$ and

$$\frac{1}{\lambda} d_X(x, y) - k \leq d_Y(g(x), g(y)) \leq \lambda d_X(x, y) + k \quad \text{for all} \quad x, y \in X.$$

One of the useful features of using the symmetric products as hyperbolic fillings is the fact that the associated extension operator $f \mapsto \widehat{f}$ is compatible with composition. That is, $\widehat{f \circ g} = \widehat{f} \circ \widehat{g}$. This paves the way for a study of groups acting on a metric space (X, d) by extending them to groups acting on $(\mathcal{X}^{(n)}, d_{\mathcal{H}})$ and studying the latter within the theory of Gromov hyperbolic spaces. For example, if X is compact and if G is a group that acts on X by homeomorphisms, then G is *hyperbolic* provided the induced action on the triple space $\mathrm{Tri}(X) = \{(x, y, z)\colon x \neq y \neq z \neq x\}$ is both *properly discontinuous* and *cocompact*. It would be interesting to obtain a similar result with $\mathrm{Tri}(X)$ replaced by $(\mathcal{X}^{(n)}, d_{\mathcal{H}})$ (or even by $(\mathcal{X}^{(3)}, d_{\mathcal{H}})$). Also, finding a sufficient condition on (X, d) so that the space $(X^{(n)}, d_H)$ is locally compact, rectifiably connected and uniform is important in light of the open problem of characterizing metric spaces that can be identified with the boundary at infinity of a CAT(-1) space (see ([7, 8]) for more discussions).

In this paper we study the n^{th} symmetric product $\mathbb{R}^{(n)}$ of $\mathbb{R}$. Section 2 contains basic concepts and some technical results needed in the rest of the paper. Most of these results are direct generalizations of the corresponding results for $\mathbb{R}^{(3)}$ obtained in ([3]), but proofs are provided for completeness. In Section 3 we obtain two sufficient conditions for biLipschitz embeddability of $\mathbb{R}^{(n)}$ into Euclidean spaces (Theorem 3.1 and Corollary 3.3). Even though we could not obtain our desired result that $\mathbb{R}^{(n)}$ can be embedded into some $\mathbb{R}^m$ by a biLipschitz map, as a byproduct of our embedding results, we provide a partial answer to a question posed by K. Borsuk and S. Ulam. The latter asks if the space $[0, 1]^{(n)}$ $(n \geq 4)$ can be topologically embedded into $\mathbb{R}^{n+1}$ ([4, p. 882]). We show that the space $\mathbb{R}^{(4)}$ and hence the space $[0, 1]^{(4)}$ can be topologically embedded into $\mathbb{R}^5$. It was shown by K. Borsuk and S. Ulam that the space $[0, 1]^{(n)}$ is not homeomorphic to any subset of $\mathbb{R}^n$ for $n \geq 4$ ([4, Theorem 7]). Finally, in Section 4 we show that the space $\mathbb{R}^{(n)}$ is quasiconvex and that the projection map is regular (Theorem 4.1 and Theorem 4.2, respectively).

2. Preliminary results

We begin by discussing some basic properties of $\mathbb{R}^{(n)}$, $n \geq 3$. The Euclidean distance in $\mathbb{R}$ as well as in $\mathbb{R}^n$ is denoted by $|-|$. For $p, q \in \mathbb{R}$ we set $p \wedge q = \min\{p, q\}$ and $p \vee q = \max\{p, q\}$. A simple observation shows that $(p \wedge q) \vee (q \wedge r) = q \wedge (p \vee r)$ for all $p, q, r \in \mathbb{R}$. The points in $\mathbb{R}^{(n)}$ can be viewed as subsets of $\mathbb{R}$. For example, when we refer to the *cardinality* $\mathrm{card}(a)$ of $a \in \mathbb{R}^{(n)}$ we mean the cardinality of a as a subset of $\mathbb{R}$. Similarly, for $r \in \mathbb{R}$ and $a \in \mathbb{R}^{(n)}$, we define $\mathrm{dist}(r, a) = \min\{|r - s|\colon\ s \in a\}$. In particular, the Hausdorff distance between

$a, b \in \mathbb{R}^{(n)}$ can be expressed as

$$d_H(a,b) = \max_{s \in a} \operatorname{dist}(s,b) \vee \max_{r \in b} \operatorname{dist}(r,a).$$

The point $0 \in \mathbb{R}$ belongs to each $\mathbb{R}^{(n)}$ and is referred to as the *the origin* and denoted by o.

The space $\mathbb{R}^{(n)}$ can also be represented as a set of n ordered real numbers $(a_1, a_2, \dots, a_n)$, $-\infty < a_1 \le a_2 \le \cdots \le a_n < +\infty$, modulo some identifications, which for $n = 4$ reduce to $(a,a,a,b) \sim (a,a,b,b) \sim (a,b,b,b)$ and $(a,a,b,c) \sim (a,b,b,c) \sim (a,b,c,c)$. For $n = 3$ the identification is $(a,a,b) \sim (a,b,b)$ (see also [5]). In general, there are $\binom{n-1}{k-1}$ different ways of expressing each point of cardinality $k \le n$ as n-tuples. In what follows we use this representation of $\mathbb{R}^{(n)}$. The numbers a_i are called the coordinates of the point $(a_1, a_2, \dots, a_n)$. If we want to emphasize the coordinates of a point we denote it by $(a_i)_1^n$ or just by (a_i) if there is no danger of confusion. If $\operatorname{card}(a) = 1$, we say that a is a *singleton.* In general, we say that a is a k-tuple $(1 \le k \le n)$ if $\operatorname{card}(a) = k$. By a *norm* of a point $a \in \mathbb{R}^{(n)}$ we mean $d_H(o,a)$ and denote it by $|a|$. It is easy to see that if $a = (a_i)_1^n$ then $|a| = |a_1| \vee |a_n|$.

For each $\lambda \in \mathbb{R}$, $\mu \ge 0$ and $a = (a_1, a_2, \dots, a_n) \in \mathbb{R}^{(n)}$, we define

$$\lambda + a = (\lambda + a_i)_1^n = (\lambda + a_1, \lambda + a_2, \dots, \lambda + a_n),$$

$$\mu a = (\mu a_i)_1^n = (\mu a_1, \mu a_2, \dots, \mu a_n)$$

and

$$\overline{a} = (-a_{n+1-i})_1^n = (-a_n, -a_{n-1}, \dots, -a_2, -a_1).$$

The point $\overline{a}$ is called *the conjugate* of a. There is a natural lift of each linear transformation h of $\mathbb{R}$ to a transformation $\tilde{h}$ of $\mathbb{R}^{(n)}$ given by $\tilde{h}((a_1, a_2, \dots, a_n)) = (h(a_1), h(a_2), \dots, h(a_n))$ if h is orientation-preserving, and $\tilde{h}((a_1, a_2, \dots, a_n)) = (h(a_n), h(a_{n-1}), \dots, h(a_2), h(a_1))$ if h is orientation-reversing. In particular, the following elementary transformations of $\mathbb{R}$, namely translations $x \mapsto x + \lambda$, dilations $x \mapsto \mu x$, $\mu \in (0, +\infty)$ and reflections $x \mapsto 2\lambda - x$, $\lambda \in \mathbb{R}$, induce the corresponding transformations of $\mathbb{R}^{(n)}$: *vertical translations:* $T_\lambda(a) = \lambda + a$, *dilations:* $D_\mu(a) = \mu a$ and *reflections:* $R_\lambda(a) = 2\lambda + \overline{a}$. Observe that the vertical translations and reflections of $\mathbb{R}^{(n)}$ are easily seen to be isometries as they are induced from the isometries of $\mathbb{R}$. (By an isometry we mean a distance preserving transformation of a space). Clearly, the dilations $x \mapsto \mu x$ act on $\mathbb{R}^{(n)}$ as dilatations, namely $d_H(\mu a, \mu b) = \mu d_H(a,b)$ for all $a, b \in \mathbb{R}^{(n)}$.

Next, for each $a \in \mathbb{R}^{(n)}$, we define

$$\Gamma(a) = \{T_\lambda(a) \colon \lambda \in \mathbb{R}\} \quad \text{and} \quad \Delta(a) = \{D_\mu(a) \colon \mu \in (0, +\infty)\}.$$

Also, for each $\lambda \in \mathbb{R}$, we define $\Pi_\lambda = T_\lambda(\Pi)$, where

$$\Pi = \{(-r, a_2, a_3, \dots, a_{n-1}, r) \in \mathbb{R}^{(n)} \colon\ r \in [0, +\infty) \text{ and } -r \le a_i \le r\}.$$

We shall refer to the sets $\Gamma(a)$, $\Delta(a)$ and Π_λ as *vertical lines*, *rays* and *horizontal planes*, respectively. Clearly, for each $a, b \in \mathbb{R}^{(n)}$, the sets $\Gamma(a)$ and $\Gamma(b)$ are either disjoint or the same. Similarly, the sets $\Delta(a)$ and $\Delta(b)$ as well as the sets Π_{λ_1} and Π_{λ_2} are either disjoint or the same. Moreover,

$$\mathbb{R}^{(n)} = \bigcup_{a \in \mathbb{R}^{(n)}} \Gamma(a), \qquad \mathbb{R}^{(n)} = \bigcup_{a \in \mathbb{R}^{(n)}} \Delta(a) \qquad \text{and} \qquad \mathbb{R}^{(n)} = \bigcup_{\lambda \in \mathbb{R}} \Pi_\lambda.$$

Finally, for each $\mu > 0$, we define $\mathcal{S}_\mu = D_\mu(\mathcal{S})$, where

$$\mathcal{S} = \{(-1, a_2, a_3, \dots, a_{n-1}, 1) \in \mathbb{R}^{(n)} \colon\ -1 \le a_2 \le a_3 \le \dots \le a_{n-1} \le 1\}.$$

The next four lemmas are direct generalizations of corresponding results for $n = 3$ obtained in ([3]). They give estimates for distances between two sets of type Π_λ, $\Gamma(a)$, $\mathcal{S}_r$ and $\Delta(a)$, respectively. For completeness, we provide the proofs.

Lemma 2.1. *Given $a \in \Pi_{\lambda_1}$, we have*

$$d_H(a, b) \ge |\lambda_1 - \lambda_2| \qquad \textit{for all} \quad b \in \Pi_{\lambda_2}.$$

The equality holds if $b \in \Gamma(a)$.

Proof. Without loss of generality we can assume that $\lambda_1 \le \lambda_2$. Let $a = (\lambda_1 - r_1, a_2, \dots, a_{n-1}, \lambda_1 + r_1)$ and $b = (\lambda_2 - r_2, b_2, \dots, b_{n-1}, \lambda_2 + r_2)$. If $r_2 \ge r_1$ we have

$$d_H(a, b) \ge \operatorname{dist}(\lambda_2 + r_2, a) = \lambda_2 + r_2 - \lambda_1 - r_1 \ge \lambda_2 - \lambda_1.$$

If $r_2 \le r_1$ then $\lambda_1 - r_1 \le \lambda_2 - r_2$ and hence

$$d_H(a, b) \ge \operatorname{dist}(\lambda_1 - r_1, b) = \lambda_2 - r_2 - \lambda_1 + r_1 \ge \lambda_2 - \lambda_1.$$

Hence $d_H(a, b) \ge \lambda_2 - \lambda_1$. Now if $b \in \Pi_{\lambda_2} \cap \Gamma(a)$ then $b = (\lambda_2 - r_1, b_2, \dots, b_{n-1}, \lambda_2 + r_1)$, where $b_k = a_k + \lambda_2 - \lambda_1$ for each $k \in \{2, 3, \dots, n-1\}$. For each k we have

$$\operatorname{dist}(a_k, b) \vee \operatorname{dist}(b_k, a) \le |a_k - b_k| = \lambda_2 - \lambda_1.$$

Also,

$$\operatorname{dist}(\lambda_2 - r_1, a) \le (\lambda_2 - r_1) - (\lambda_1 - r_1) = \lambda_2 - \lambda_1$$

and

$$\operatorname{dist}(\lambda_1 + r_1, b) \le (\lambda_2 + r_1) - (\lambda_1 + r_1) = \lambda_2 - \lambda_1.$$

Finally, since

$$\operatorname{dist}(\lambda_1 - r_1, b) = \operatorname{dist}(\lambda_2 + r_1, a) = \lambda_2 - \lambda_1,$$

we obtain $d_H(a, b) = \lambda_2 - \lambda_1$, as required.

■

Lemma 2.2. *Let $a = (-r, a_2, \dots, a_{n-1}, r) \in \Pi$ and $b = (-s, b_2, \dots, b_{n-1}, s) \in \Pi$ be arbitrary points. Then for all $a' \in \Gamma(a)$ and $b' \in \Gamma(b)$ we have*

$$2d_H(a', b') \geq d_H(a, b).$$

Equality holds for $a' = (-r, r, r, \dots, r)$ and $b' = (-r/2, r/2, 3r/2, \dots, 3r/2)$.

Proof. Using a vertical translation, if necessary, we may assume that $a' = a$. Let $b' = T_\lambda(b)$ for some $\lambda \in \mathbb{R}$. By Lemma 2.1, $d_H(b', b) = |\lambda|$ and $d_H(a, b') \geq |\lambda|$. By the triangle inequality,

$$d_H(a, b) \leq d_H(a, b') + d_H(b', b) = d_H(a, b') + |\lambda| \leq 2d_H(a, b'),$$

as claimed. ■

Lemma 2.3. *Given $a = (-r, a_2, \dots, a_{n-1}, r) \in \mathcal{S}_r$, we have*

$$d_H(a, b) \geq |r - t| \quad \textit{for all} \quad b = (-t, b_2, \dots, b_{n-1}, t) \in \mathcal{S}_t.$$

Equality holds if $\operatorname{dist}(a_k, b) \vee \operatorname{dist}(b_k, a) \leq |r - t|$ for each $k = \{2, 3, \dots, n-1\}$.

Proof. We have

$$d_H(a, b) = \operatorname{dist}(-r, b) \vee \operatorname{dist}(r, b) \vee \operatorname{dist}(a_k, b) \vee \operatorname{dist}(b_k, a) \vee \operatorname{dist}(-t, a) \vee \operatorname{dist}(t, a).$$

Hence

$$d_H(a, b) \geq \operatorname{dist}(r, b) \vee \operatorname{dist}(t, a) = |r - t|.$$

If $\operatorname{dist}(a_k, b) \vee \operatorname{dist}(b_k, a) \leq |r - t|$ for each k, then

$$d_H(a, b) \leq |r - t| \vee \operatorname{dist}(r, b) \vee \operatorname{dist}(t, a) = |r - t|, \quad \text{i.e.,} \quad d_H(a, b) = |r - t|.$$

■

Lemma 2.4. *Let $a = (-r, a_2, \dots, a_{n-1}, r)$ and $b = (-s, b_2, \dots, b_{n-1}, s)$ be arbitrary points with $s > 0$ and $r \leq s$. Put $c = (r/s)b$. Then*

$$d_H(a, c) \leq 2d_H(a, b).$$

Proof. When b is re-scaled by the factor r/s, none of its points move by more than $s - r$ from their original positions. Hence $d_H(b, c) \leq s - r$. On the other hand, $s - r = \operatorname{dist}(s, a) \leq d_H(a, b)$. By the triangle inequality, we have $d_H(a, c) \leq d_H(a, b) + d_H(b, c) \leq 2d_H(a, b)$. ■

We end this section with a discussion of geodesics and isometries in $\mathbb{R}^{(n)}$. By an open arc γ in $\mathbb{R}^{(n)}$ we mean a homeomorphic embedding $\gamma\colon (s, t) \to \mathbb{R}^{(n)}$, where (s, t) is an open interval $(s, t) \subset \mathbb{R}$. We also identify γ with the image set $\gamma((s, t)) \subset \mathbb{R}^{(n)}$. We say that an open arc $\gamma \subset \mathbb{R}^{(n)}$ is geodesic if $d_H(a, b) = d_H(a, c) + d_H(c, b)$ for each ordered triple of points $a, c, b \in \gamma$. Observe that both rays and vertical lines can be regarded as isometric embeddings of $(0,+\infty)$ and $\mathbb{R}$ into $\mathbb{R}^{(n)}$, respectively. Indeed, if Δ is a ray, then $\Delta = \Delta(a)$ for some a with $|a| = 1$. Hence $\Delta = h\big((0,+\infty)\big)$, where $h(\mu) = \mu a$ and $d_H(h(\mu_1), h(\mu_2)) = |\mu_1 - \mu_2|$. Similarly, if $\Gamma(a)$ is a vertical line, then $\Gamma(a) = g(\mathbb{R})$, where $g(\lambda) = \lambda + a$ and $d_H(g(\lambda_1), g(\lambda_2)) = |\lambda_1 - \lambda_2|$. Thus, both rays and vertical lines are geodesics.

Our immediate goal is to show that the vertical line $\Gamma(o)$ has a property not shared by other vertical lines. This property played a crucial role in proving the isometry result for $n = 3$ ([3, Theorem 9]). For simplicity we denote $\Gamma(o)$ by Γ_0.

First, we discuss special types of geodesics. Let $a' = (a_1, a_2, \ldots, a_n) \notin \Gamma_0$ be any point with $a_1 + a_n \geq 0$. Let $\gamma_1 = \Delta(a')$ and $\gamma_2 = \{(u, u, \ldots, u) \colon u \leq 0\}$. Then $\gamma = \gamma_1 \cup \gamma_2$ is a geodesic. Indeed, since γ_1 and γ_2 are geodesics, it is enough to show that

$$d_H(a, b) = d_H(a, o) + d_H(o, b) \qquad \text{for all} \quad a \in \gamma_1 \ \text{ and } \ b \in \gamma_2.$$

Let $a = \mu a' \in \gamma_1$ and $b = (u, u, \ldots, u) \in \gamma_2$ be arbitrary points. Since $a_1 + a_n \geq 0$ and $u \leq 0$, we have

$$d_H(a, b) = \mu a_n - u = \mu a_n + (-u) = d_H(a, o) + d_H(o, b), \quad \text{as required.}$$

Due to invariance under vertical translations, these types of geodesics can be constructed starting with an arbitrary singleton. Next lemma shows that all geodesics containing a singleton are of this type.

Lemma 2.5. *Let γ be a geodesic and let $c \in \gamma$ be a singleton. Put $\gamma = \gamma_1 \cup \{c\} \cup \gamma_2$. Then either $\gamma_1 \subset \Gamma_0$ or $\gamma_2 \subset \Gamma_0$.*

Proof. We will prove the following stronger result which, in particular, contains the lemma. Let $a, b \in \mathbb{R}^{(n)}$ be arbitrary points. If there exists a singleton c such that $d_H(a, b) = d_H(a, c) + d_H(c, b)$, then either a or b is a singleton.

There is nothing to prove if either $c = a$ or $c = b$, so we assume that $c \neq a, b$. Due to invariance under vertical translations we can further assume that $c = o$. Let $a = (r, a_2, a_3, \ldots, a_{n-1}, t)$ and $b = (u, b_2, b_3, \ldots b_{n-1}, w)$. Then

$$d_H(a, o) = |r| \vee |t| \qquad \text{and} \qquad d_H(b, o) = |u| \vee |w|.$$

Using a conjugation, if necessary, we can assume that $|r| \vee |t| \vee |u| \leq w$. Assume that neither a nor b is a singleton, or equivalently, $r \neq t$ and $u \neq w$.

Case 1: $|r| \leq |t|$. Since $r < t$, we have $t > 0$. Then $d_H(a, o) + d_H(o, b) = w + t$. A simple observation shows that for each $i = 2, 3, \ldots, n-1$ the quantities $\operatorname{dist}(u, a)$, $\operatorname{dist}(b_i, a)$, $\operatorname{dist}(v, a)$, $\operatorname{dist}(r, b)$, $\operatorname{dist}(a_i, b)$ and $\operatorname{dist}(t, b)$ are strictly less than $w + t$. Hence $d_H(a, b) < w + t$, a contradiction.

Case 2: $|r| \geq |t|$. Since $r < t$, we have $r < 0$. Then $d_H(a, o) + d_H(o, b) = w - r$. Then for each $i = 2, 3, \ldots, n-1$ the quantities $\operatorname{dist}(u, a)$, $\operatorname{dist}(b_i, a)$, $\operatorname{dist}(v, a)$,

$\text{dist}(r,b)$, $\text{dist}(a_i,b)$ and $\text{dist}(t,b)$ are strictly less than $w-r$. Hence $d(a,b) < w-r$, a contradiction.

■

Lemma 2.5 implies that every geodesic containing a singleton shares a common arc with Γ_0. In particular, given a singleton c and any three geodesics $\gamma_1, \gamma_2, \gamma_3$ containing c, at least one of the intersections $\gamma_1 \cap \gamma_2 \cap \Gamma_0$, $\gamma_1 \cap \gamma_3 \cap \Gamma_0$ and $\gamma_2 \cap \gamma_3 \cap \Gamma_0$ contains an open arc. In contrast, next lemma shows that each k-tuple with $k > 1$ is contained in at least $k+2$ geodesics having only this point in common. Recall that by a k-tuple, $1 \le k \le n$, we mean a point $a \in \mathbb{R}^{(n)}$ of cardinality k when a is viewed as a subset of $\mathbb{R}$.

Lemma 2.6. *Given $k \ge 2$, for each k-tuple $a = (a_1, a_2, \dots, a_k) \in \mathbb{R}^{(n)}$ there exist $k+2$ geodesics $\gamma_1, \gamma_2, \dots, \gamma_k$ such that $\gamma_m \cap \gamma_l = \{a\}$ for each $m \neq l$.*

Proof. Put
$$\epsilon = \frac{1}{4}\big[(a_2 - a_1) \wedge (a_3 - a_2) \wedge \cdots \wedge (a_k - a_{k-1})\big]$$
and for each $m = 1, 2, \dots, k$ define $\gamma_m \colon (-\epsilon, \epsilon) \to \mathbb{R}^{(n)}$ by
$$\gamma_m(x) = (a_1, a_2, \dots, a_{m-1}, a_m + x, a_{m+1}, \dots, a_k).$$
Also, define $\gamma_{k+1}(a) = \Gamma(a)$ and $\gamma_{k+2}(a) = \Delta(a)$. Then the arcs $\gamma_1, \gamma_2, \dots, \gamma_{k+2}$ are the required geodesics.

■

As we pointed out at the beginning, the vertical translations and reflections are isometries of $\mathbb{R}^{(n)}$. These isometries are induced from isometries of $\mathbb{R}$. Observe that each vertical translation is a composition of two reflections. For example, the vertical translation $a \mapsto \lambda + a$ is a composition of $a \mapsto \overline{a}$ followed by $a \mapsto \lambda + \overline{a}$. Recall that each Euclidean isometry of $\mathbb{R}^n$ is a composition of at most $n+1$ reflections in planes (see, for example, [2, Theorem 3.1.3]). We believe that isometries of $\mathbb{R}^{(n)}$ are more rigid.

Conjecture 2.1. Each isometry of $\mathbb{R}^{(n)}$ is a composition of at most three reflections.

We will give the following arguments towards the validity of the conjecture. Let $F \colon \mathbb{R}^{(n)} \to \mathbb{R}^{(n)}$ be an isometry. First, Lemma 2.5 and Lemma 2.6 imply that singletons are mapped to singletons. By means of a preliminary vertical translation we can assume that $F(o) = o$. Since $|c| = d(o,c) = d(o,F(c)) = |F(c)|$ for each singleton c, we have either $F(c) = \overline{c}$ or $F(c) = c$. Using a conjugation, if necessary, we can assume that $F(c) = c$ for all singletons c. Since we have already used up to three reflections, we must now show that F is the identity map.

Next, we show that all 2-tuples are fixed. For $r > 0$, we put $\mathcal{C}(r) = \{a \in \mathbb{R}^{(n)} : |a| = r\}$. Observe that given any k-tuple $a = (a_1, a_2, \ldots, a_{k-1}, a_k) \in \mathbb{R}^{(n)}$ with $2 < k \leq n$, all the geodesics $\gamma_2, \gamma_3, \ldots, \gamma_{k-1}$ given in Lemma 2.6 contain a and lie in $\Pi_s \cap \mathcal{C}(|a|)$, where $s = (a_1 + a_k)/2$. Note that $|a| = d_H(o, a) = |a_1| \vee |a_k|$. On the other hand, there are no geodesics containing a 2-tuple $b = (r, t)$ and lying in $\Pi_{(r+t)/2} \cap \mathcal{C}(|b|)$. Indeed, if γ is any arc containing b and lying in $\Pi_{(r+t)/2} \cap \mathcal{C}(|b|)$ then there exists $\epsilon > 0$ such that the points $b' = (r, t - \epsilon/2, t)$, $b'' = (r, t - \epsilon/3, t)$ and $b''' = (r, r + \epsilon, t)$ lie on γ in this order. Since $d_H(b', b'') = \epsilon/6$, $d_H(b'', b''') = \epsilon$ and $d_H(b', b'') = \epsilon$, γ can not be a geodesic. We conclude that all 2-tuples are fixed by F.

Finally, let $a = (r, a_2, \ldots, a_{k-1}, t)$ be arbitrary k-tuple with $k \geq 3$. Then $F(a) = (r', b_2, \ldots, b_{m-1}, t')$ for some $m \geq 3$. Choose arbitrary $\lambda_1 \in \mathbb{R}$ and $\lambda_2 \in \mathbb{R}$ with $\lambda_1 < r \wedge r'$ and $\lambda_2 > t \vee t'$. Let $c' = (\lambda_1)_1^n$ and $c'' = (\lambda_2)_1^n$ be singletons. Then

$$t - \lambda_1 = d_H(c', a) = d_H(c', F(a)) = t' - \lambda_1$$

and

$$\lambda_2 - r = d_H(c'', a) = d_H(c'', F(a)) = \lambda_2 - r'$$

Hence $r' = r$ and $t' = t$ and we conclude that F preserves each horizontal plane Π_λ as well as the sets $\mathcal{C}(\rho) \cap \Pi_\lambda$ for each $\rho > 0$.

3. Embeddings of $\mathbb{R}^{(n)}$ into Euclidean spaces

Suppose that (X, d_X) and (Y, d_Y) are metric spaces. A map $\phi \colon X \to Y$ is an *embedding* if it is a homeomorphism onto its image. An embedding ϕ is *biLipschitz* if there exists $L \geq 1$ such that

$$\frac{1}{L} d_X(a, b) \leq d_Y(\phi(a), \phi(b)) \leq L d_X(a, b)$$

whenever $a, b \in X$. Let $\mathbb{S}$ be the unit circle in the complex plane. Let $\mathbb{S}^n$ and $\mathbb{S}^{(n)}$ denote the unit n-dimensional sphere in $\mathbb{R}^{n+1}$ and the n^{th} symmetric product of $\mathbb{S}$, respectively.

Theorem 3.1. *Let $n \geq 4$ and $m \geq n$. Suppose that there is a biLipschitz embedding of $\mathbb{S}^{(n-1)}$ into $\mathbb{S}^{m-1}$ in $\mathbb{R}^m$. Then there is a biLipschitz embedding of $\mathbb{R}^{(n)}$ into $\mathbb{R}^{m+1}$.*

Proof. We begin by showing that the set $\mathcal{S}$ can be embedded into $\mathbb{S}^{(n-1)}$ by a biLipschitz map. Recall that

$$\mathcal{S} = \big\{(-1, a_2, a_3, \dots, a_{n-1}, 1) \in \mathbb{R}^{(n)}\colon\ -1 \le a_2 \le a_3 \le \dots \le a_{n-1} \le 1\big\}.$$

Define a map $h\colon \mathcal{S} \to \mathbb{S}^{(n-1)}$ by

$$h\big((-1, a_2, a_3, \dots, a_{n-1}, 1)\big) = \big\{-1, e^{i\pi a_2}, e^{i\pi a_3}, \dots, e^{i\pi a_{n-1}}\big\}.$$

We will show that

$$2d_H(a,b) \le d_H\big(h(a), h(b)\big) \le \pi d_H(a,b)$$

for all $a, b \in \mathcal{S}$. Indeed, let $a, b \in \mathcal{S}$ be arbitrary distinct points. Without loss of generality we can assume that $d_H(a,b) = \operatorname{dist}(s,b) = |s-t|$ for some $s \in a$ and $t \in b$. Note that $s \notin \{-1,1\}$. Then $\operatorname{dist}(e^{i\pi s}, h(b)) = |e^{i\pi s} - e^{i\pi t}|$. Hence

$$d_H\big(h(a), h(b)\big) \ge \operatorname{dist}(e^{i\pi s}, h(b)) = 2\sin\frac{\pi|s-t|}{2} \ge 2|s-t| = 2d_H(a,b).$$

On the other hand, we have either $d_H\big(h(a), h(b)\big) = \operatorname{dist}(e^{i\pi s}, h(b))$ for some $s \in a$ or $d_H\big(h(a), h(b)\big) = \operatorname{dist}(e^{i\pi t}, h(a))$ for some $t \in b$. Let us assume that the latter is the case. Then $d_H\big(h(a), h(b)\big) = \operatorname{dist}(e^{i\pi t}, h(a)) = |e^{i\pi t} - e^{i\pi u}|$ for some $u \in a$. Note that $t \notin \{-1,1\}$ and, in particular, $\operatorname{dist}(t,a) = |t-u|$. Hence

$$d_H\big(h(a), h(b)\big) = |e^{i\pi t} - e^{i\pi u}| = 2\sin\frac{\pi|t-u|}{2} \le \pi|t-u| = \pi \operatorname{dist}(t,a) \le \pi d_H(a,b),$$

as required.

Suppose now that there is a biLipschitz embedding of $\mathbb{S}^{(n-1)}$ into $\mathbb{S}^{m-1}$. Using the map h we see that there is a biLipschitz embedding of $\mathcal{S}$ into $\mathbb{S}^{m-1}$, say f. Let $M = f(\mathcal{S}) \subset \mathbb{S}^{m-1}$. Define a map $g\colon \Pi \to \mathbb{R}^m$ by $g(a) = |a| f\big((1/|a|)a\big)$ if $a \ne o$, and $g(o) = (0, 0, \dots, 0)$. Observe that if $a \in \mathcal{S}$, then $g(a) = f(a)$.

Assume now that f is L-biLipschitz, i.e., $(1/L)d_H(a,b) \le |f(a) - f(b)| \le L d_H(a,b)$ for all $a, b \in \mathcal{S}$. We will show that g is $(2L+1)$-biLipschitz. Note that

$$|g(\mu a) - g(\mu b)| = \mu|g(a) - g(b)| \qquad \text{and} \qquad |g(\mu a) - g(a)| = d_H(\mu a, a)$$

for all $a, b \in \Pi$ and $\mu \in [0, \infty)$. Let $a, b \in \Pi$ be arbitrary points. Without loss of generality we can assume that $|a| \ge |b|$. Let $c = (|b|/|a|)a$. Then $a, c \in \Delta(a)$ and $b, c \in \mathcal{S}_{|b|}$. In particular, $|g(a) - g(c)| = d_H(a,c)$ and $|g(b) - g(c)| = |b||f(\tilde{b}) - f(\tilde{c})|$, where $\tilde{b} = (1/|b|)b$ and $\tilde{c} = (1/|c|)c$. Hence $(1/L)d_H(b,c) \le |g(b) - g(c)| \le L d_H(b,c)$. Thus,

$$d_H(a,b) \le d_H(a,c) + d_H(b,c) \le |g(a) - g(c)| + L|g(b) - g(c)| \le (L+1)|g(a) - g(b)|$$

and using Lemma 2.3 and Lemma 2.4 we obtain

$$|g(a) - g(b)| \le |g(a) - g(c)| + |g(b) - g(c)| \le d_H(a,c) + L d_H(b,c) \le (2L+1) d_H(a,b),$$

as required.

Next, we define a map $F\colon \mathbb{R}^{(n)} \to \mathbb{R}^{m+1}$ by

$$F\big((x_1, x_2, \ldots, x_n)\big) = \Big(g\big((x_1 - t, x_2 - t, \ldots, x_n - t)\big), t\Big), \tag{3.2}$$

where $t = (x_1 + x_n)/2$. Note that $F(a) = (g(a), 0)$ for all $a \in \Pi$. It remains to show that F is biLipschitz. Let now $a, b \in \mathbb{R}^{(n)}$ be arbitrary points. Since

$$d_H(\lambda + a, \lambda + b) = d_H(a, b) \quad \text{and} \quad |F(\lambda + a) - F(\lambda + b)| = |F(a) - F(b)|$$

for all $\lambda \in \mathbb{R}$, we may assume that $b \in \Pi$. Denote the intersection point of $\Gamma(a)$ with Π by c. Hence $a, c \in \Gamma(a)$ and $b, c \in \Pi$. Note that

$$|F(a) - F(b)|^2 = |F(a) - F(c)|^2 + |F(c) - F(b)|^2.$$

It is easy to check that $|F(a) - F(c)| = d(a, c)$. Since $F(b) = (g(b), 0)$ and $F(c) = (g(c), 0)$, using our assumption we obtain

$$\frac{1}{2L+1} d_H(b, c) \leq |F(b) - F(c)| \leq (2L + 1) d_H(b, c).$$

On the other hand, Lemma 2.1 and Lemma 2.2 imply that $d_H(a, c) \leq d_H(a, b)$ and $d_H(b, c) \leq 2d_H(a, b)$. Hence

$$\begin{aligned} d_H(a, b) &\leq d_H(a, c) + d_H(c, b) \\ &\leq |F(a) - F(c)| + (2L + 1)|F(c) - F(b)| \leq (2L + 2)|F(a) - F(b)| \end{aligned}$$

and

$$\begin{aligned} |F(a) - F(b)| &\leq |F(a) - F(c)| + |F(c) - F(b)| \\ &\leq d_H(a, c) + (2L + 1) d_H(c, b) \leq (4L + 3) d_H(a, b). \end{aligned}$$

Thus, F is $(4L + 3)$-biLipschitz, completing the proof.

■

The following corollary is an immediate consequence of the proof of Theorem 3.1.

Corollary 3.3. *If there is a biLipschitz embedding of* Π *into* $\mathbb{R}^m$ *for some* m*, then there is a biLipschitz embedding of the space* $\mathbb{R}^{(n)}$ *into* $\mathbb{R}^{(m+1)}$.

Observe that if the map f in the proof of Theorem 3.1 is a homeomorphism, then so are the maps g and F. Hence if there is a topological embedding of $\mathbb{S}^{(n-1)}$ into an $(m-1)$ - dimensional sphere $\mathbb{S}^{m-1}$ in $\mathbb{R}^m$, then there is also a topological embedding of $\mathbb{R}^{(n)}$ into $\mathbb{R}^{m+1}$. Since $\mathbb{S}^{(3)}$ is homeomorphic to a three-dimensional sphere $\mathbb{S}^3$ in $\mathbb{R}^4$ ([5]), we conclude that there is a topological embedding of $\mathbb{R}^{(4)}$ into $\mathbb{R}^5$. In particular, the space $\mathbb{I}^{(4)}$, where $\mathbb{I} = [0, 1]$, can be topologically embedded into $\mathbb{R}^5$, giving a positive answer to the question of Borsuk and Ulam for $n = 4$.

That $\mathbb{I}^{(4)}$ can be topologically embedded into $\mathbb{R}^5$ also follows from the works of R. M. Schori ([9]) and R. N. Andersen, M. M. Marjanovic and R. M. Schori ([1]). More precisely, the space $\mathbb{I}^{(4)}$ is homeomorphic to cone$(\mathbb{I}^1_0) \times \mathbb{I}$ (Theorem 6 and Theorem 4 in ([9])), where $\mathbb{I}^1_0 = \{A \in \mathbb{I}^{(4)} \colon 0, 1 \in A\}$. In ([1]) it was shown that $\mathbb{I}^1_0$ is the "Dunce Hat". The latter can be constructed in $\mathbb{R}^3$ and hence cone$(\mathbb{I}^1_0)$ can be embedded into $\mathbb{R}^4$. Therefore cone$(\mathbb{I}^1_0) \times \mathbb{I}$ and hence $\mathbb{I}^{(4)}$ can be embedded into $\mathbb{R}^5$. The authors thank Alejandro Illanes for these observations.

4. Metric properties of $\mathbb{R}^{(n)}$

Suppose that (X, d_X) is a metric space. By a *curve* in X we mean a continuous mapping $\gamma \colon [a, b] \to X$, where $[a, b] \subset \mathbb{R}$ is an interval. We also identify γ with the image set $\gamma([a, b])$. The *length* of a curve γ is defined by

$$L(\gamma) = \sup \sum_{i=0}^{n} d_X(\gamma(t_{i+1}), \gamma(t_i)),$$

where the supremum is taken over all sequences $a = t_0 \le t_1 \le \cdots \le t_{n+1} = b$. A curve γ is said to be *rectifiable* if $L(\gamma) < \infty$. The space X is said to be C-*quasiconvex* if there exists a constant $C \ge 1$ with the property that every pair of points $x, y \in X$ can be joined by a curve whose length is no more than $C \cdot d_X(x, y)$.

Our immediate goal is to show that the space $\mathbb{R}^{(n)}$ is quaisconvex. For $n = 3$ the quasiconvexity of $\mathbb{R}^{(3)}$ follows since $\mathbb{R}^{(3)}$ is biLipschitz equivalent to $\mathbb{R}^3$. More precisely, the map $F \colon \mathbb{R}^{(3)} \to \mathbb{R}^3$, defined by

$$F\big((r, s, t)\big) = \Big(\frac{r-t}{2} \cos \frac{2\pi(s-r)}{t-r},\ \frac{r-t}{2} \sin \frac{2\pi(s-r)}{t-r},\ \frac{t+r}{2}\Big),$$

satisfies

$$d_H(a, b)/2 \le |F(a) - F(b)| \le (3 + 4\pi) d_H(a, b)$$

for all $a, b \in \mathbb{R}^{(n)}$ (see [3, (9) and (10)]). It follows that the space $\mathbb{R}^{(3)}$ is $2(3+4\pi)$-quasiconvex.

Theorem 4.1. *For $n \ge 3$ the space $\mathbb{R}^{(n)}$ is 4^n-quasiconvex.*

Proof. We will show that for every two points a and b in $\mathbb{R}^{(n)}$ there exists a curve joining a and b whose length is less than $4^n d_H(a, b)$. We prove it by induction on n. The claim is true for $n = 3$ as mentioned above. We assume now that it is true for $n - 1 \ge 3$ and we will prove it for n. Suppose $a = (a_1, ..., a_n)$ and $b = (b_1, ..., b_n)$ are two given points in $\mathbb{R}^{(n)}$ and let $D = d_H(a, b)$. If both a and b

are in $\mathbb{R}^{(n-1)}$, then there is nothing to prove. Otherwise, we can assume without loss of generality that $b_1 < b_2 < \cdots < b_n$ and $a_1 \le a_2 \le \cdots \le a_n$. We break the proof into the following two cases.

Case 1: $b_{i+1} - b_i > 2D$ or $a_{i+1} - a_i > 2D$ for every $i = 1, 2 \ldots, n-1$. Without loss of generality we can assume that $b_{i+1} - b_i > 2D$ for every $i = 1, 2 \ldots, n-1$. Note that $d_H(a, b) = D$ implies that every a_i is in a D neighborhood of some b_j and each b_i is in a D neighborhood of some a_j. Since b_i's are at least $2D$ units apart, then each a_i has to be within D neighborhood of b_i, and thus $|a_i - b_i| \le D$ for each $i = 1, 2, \ldots, n$. Therefore

$$D = d_H(a, b) = \max\{|a_k - b_k|\colon 1 \le k \le n\}.$$

We define a map $\gamma\colon [0, 1] \to \mathbb{R}^{(n)}$ by $\gamma(t) = \{a_1 + t(b_1 - a_1), \ldots, a_n + t(b_n - a_n)\}$. We will show that the length of this curve is equal to D. Recall that the length of γ is defined by

$$L(\gamma) = \sup_{0=t_1<t_2\cdots<t_m=1} \sum_{i=1}^{m} d_H(\gamma(t_i), \gamma(t_{i-1}))$$

where the supremum is taken over all finite partitions of the interval $[0, 1]$. Suppose that $\mathcal{P}_1 = \{0 = s_0, s_1, \ldots, s_l = 1\}$ is arbitrary partition of $[0, 1]$. We can choose a partition $\mathcal{P}_2 = \{0 = t_0, t_1, \ldots, t_m = 1\}$ so that $\mathcal{P}_1 \subseteq \mathcal{P}_2$ and that $(t_i - t_{i-1}) < \min\{|a_{j+1} - a_j|\colon 1 \le j \le n-1\}$. Then for each $i = 0, 1, \ldots, m$ and for each $k = 1, 2, \ldots, n$ we have

$$\operatorname{dist}(\gamma(t_i), a_k + t_{i-1}(b_k - a_k)) = (t_i - t_{i-1})|b_k - a_k| = \operatorname{dist}(a_k + t_{i-1}(b_k - a_k), \gamma(t_{i-1})).$$

Then

$$\begin{aligned}\sum_{i=1}^{m} d_H(\gamma(t_i), \gamma(t_{i-1})) &= \sum_{i=1}^{m} \max\{(t_i - t_{i-1})|b_k - a_k|\colon 1 \le k \le n\} \\ &= \sum_{i=1}^{m} (t_i - t_{i-1}) \max\{|b_k - a_k|\colon 1 \le k \le n\} = D.\end{aligned}$$

Since

$$\sum_{i=1}^{l} d_H(\gamma(s_i), \gamma(s_{i-1})) \le \sum_{i=1}^{m} d_H(\gamma(t_i), \gamma(t_{i-1})) = D,$$

we conclude that $L(\gamma) = D$.

Case 2: There exist an i and a j such that $b_{i+1} - b_i \le 2D$ and $a_{j+1} - a_j \le 2D$. We construct a point a' in $\mathbb{R}^{(n-1)}$ which is obtained from the point a by replacing a_{j+1} with a_j. In the same way we construct a point b' from b by replacing b_{i+1} with b_i. Then for every $k = 1, 2, \cdots, n$ there exists t_k such that $\operatorname{dist}(a_k, b) = |a_k - b_{t_k}|$.

If for a given k we have $t_k \neq i+1$, then $\text{dist}(a_k, b) = \text{dist}(a, b') \leq D$. On the other hand if for some k we have $t_k = i+1$, then

$$|a_k - b_i| \leq |a_k - b_{i+1}| + |b_{i+1} - b_i| \leq \text{dist}(a_k, b) + 2D \leq 3D.$$

So in this case we have $\text{dist}(a_k, b') \leq |a_k - b_i| \leq 3D$. Therefore for every $k = 1, 2, \cdots, n$ we have $\text{dist}(a_k, b') \leq 3D$. Similarly, for every $k = 1, 2, \cdots, n$ we have $\text{dist}(b_k, a') \leq 3D$. Thus, $d_H(a', b') \leq 3D$.

As the points a' and b' lie in $\mathbb{R}^{(n-1)}$ and since $d_H(a', b') \leq 3D$, by the induction hypothesis there exists a curve γ_1 that connects a' to b' and such that $L(\gamma_1) \leq 4^{n-1} d_H(a', b') \leq 3 \cdot 4^{n-1} D$. Note also that $d_H(a, a') = \text{dist}(a_{j+1}, a') \leq |a_{j+1} - a_j| \leq 2D$ and $d_H(b, b') = \text{dist}(b_{i+1}, b') \leq |b_{i+1} - b_i| \leq 2D$. Let

$$\begin{aligned}
\gamma_2(t) &= \{a_1, \ldots, a_j, a_{j+1} + t(a_j - a_{j+1}), a_{j+2}, \ldots, a_n\} \\
\gamma_3(t) &= \{b_1, \ldots, b_i, b_{i+1} + t(b_i - b_{i+1}), b_{i+2}, \ldots, b_i\}.
\end{aligned}$$

be two curves connecting a to a' and b to b', respectively. Then we can easily see that $L(\gamma_2) \leq (a_{j+1} - a_j) \leq 2D$ and $L(\gamma_3) \leq b_{i+1} - b_i \leq 2D$. Therefore the curve $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$ is of length at most $(2 + 2 + 3 \cdot 4^{n-1})D \leqslant 4^n D$.

■

A mapping $\phi \colon X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is said to be *regular* if it is Lipschitz and if there is a constant $C > 0$ such that for each ball B in Y the set $\phi^{-1}(B)$ is covered by at most C balls in X of the same radius as B (see, [10, Defilition 8.1]). Recall that the map ϕ is Lipschitz if there exists $L \geq 1$ such that $d_Y(\phi(x), \phi(y)) \leq L d_X(x, y)$ for all $x, y \in X$.

Theorem 4.2. *The projection $\pi \colon \mathbb{R}^n \to \mathbb{R}^{(n)}$ is regular.*

Proof. Clearly, for all $x, y \in \mathbb{R}^n$ we have $d_H(\pi(x), \pi(y)) \leq |x - y|$. Hence π is Lipschitz. To avoid confusion, we represent elements of $\mathbb{R}^{(n)}$ as subsets of $\mathbb{R}$ of cardinality less than or equal to n and denote them using the set-theoretical braces. Let $a = \{a_1, \ldots, a_n\} \in \mathbb{R}^{(n)}$ be arbitrary point and let $B = \pi^{-1}(B(a, r))$. Put $\mathcal{C} = \{(c_1, \ldots, c_n) \in \mathbb{R}^n \colon c_i \in \{a_1, \ldots, a_n\}\}$. Then the cardinality of $\mathcal{C}$ is at most n^n and hence $\mathcal{C}$ is the collection of p points in $\mathbb{R}^n$, where $p = \text{card}(\mathcal{C})$. Let $b = (b_1, b_2, \ldots, b_n) \in B$ be arbitrary point. Since $d_H(\pi(b), a) < r$, for each i we have $\text{dist}(b_i, a) < r$ and hence there exists k_i such that $|b_i - a_{k_i}| < r$. Let $E(b) = (a_{k_1}, a_{k_2}, \ldots, a_{k_n})$. Then $E(b) \in \mathcal{C}$. Hence b is contained in a cube centered at $E(b)$ of side-length $2r$. Hence B is contained in the union of the cubes centered at points of $\mathcal{C}$ and of side-length $2r$. It is a well-known fact that each one of these cubes (in $\mathbb{R}^n$) can be covered by at most K balls of radius r, where K depends only on n. Thus, the set $\pi^{-1}(B(a, r))$ can be covered by at most Kn^n balls of radius r and hence the map π is C-regular with $C = Kn^n$.

■

We end the paper with the following open problem.

Problem 4.1. Given $n \geq 4$, find the smallest $m > n$ such that there is a biLipschitz embedding of $\mathbb{R}^{(n)}$ into $\mathbb{R}^m$. Also, show that $\mathbb{R}^{(n)}$ is doubling and that its Hausdorff dimension is equal to n.

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