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Comparison of BV Norms in Weighted Euclidean Spaces

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Abstract. In this paper, we will examine proposed generalizations of the notion of bounded variation by Baldi to weighted Euclidean spaces and Miranda to metric measure spaces. Since weighted Euclidean spaces are metric measure spaces, it is natural to ask whether these two definitions are equivalent or comparable. We will give conditions that ensure equivalency and provide examples of weights for which they are not even comparable.

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1. Introduction

Functions of bounded variation (BV functions) have played an important role in the study of calculus. They have classically been used to study minimal surfaces and more recently to study discontinuity hypersurfaces with applications in image segmentation and fracture mechanics, see for example [10] and [13]. Until recently, the theory of functions of bounded variation has been limited to Euclidean domains. In the past ten years, attempts have been made to generalize this theory to more general spaces such as weighted Euclidean domains and metric measure spaces. Since classical BV theory is built on definitions dependent on Euclidean structure, new definitions must be used to generalize BV theory to abstract spaces. Finding equivalent definitions in Euclidean settings that can be extended to abstract metric measure spaces can also contribute to classical theory by helping to identify the most important features of BV functions.

We first give a brief introduction to BV functions in Euclidean domains (complete discussions can be found in [1] and [4], among other places). For $\Omega \subset \mathbb{R}^n$, $f \in L^1(\Omega)$ belongs to $BV(\Omega)$ if there exists an n-dimensional Radon measure Df with finite total variation $\|Df\|(\Omega)$ such that for all $\varphi \in C_c^\infty(\Omega\colon \mathbb{R}^n)$,

$$\int_\Omega f \,\mathrm{div}\varphi\, dx = -\int_\Omega \varphi \cdot dDf.$$

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The idea here is the integration by parts formula. We can think of BV functions as those having a vector-valued Radon measure acting like its derivative. For differentiable functions f, this measure would be given by $\nabla f\, dx$. BV functions are traditionally defined using the equivalent criteria that $f \in L^1(\Omega)$ is of bounded variation if

$$\sup\left\{\int_\Omega f\ \mathrm{div}\varphi\ dx \;\middle|\; \varphi \in C_c^\infty(\Omega\colon \mathbb{R}^n), |\varphi| \leq 1\right\} < \infty.$$

It has been shown that BV functions can be approximated in a variational sense by smooth functions (see Section 5.2 of [4]). That is, for any $f \in BV(\Omega)$, there exists a sequence of smooth functions $\{f_k\}_{k\in\mathbb{N}}$ satisfying

$$\lim_{k\to\infty} \|f_k - f\|_{L^1(\Omega)} = 0 \quad \text{and} \quad \lim_{k\to\infty} \|Df_k\|(\Omega) = \|Df\|(\Omega).$$

Note that we do not claim $\lim_{k\to\infty} \|D(f_k - f)\|(\Omega) = 0$.

An area of interest is the study of weighted Euclidean spaces, that is an open Euclidean set equipped with a measure obtained by integrating a density function ω against the Lebesgue measure. The Sobolev space $H^{1,p}(\Omega,\omega)$ is the closure of all C^∞ functions φ under the norm

$$\|\varphi\|_{1,p} = \left(\int_\Omega |\varphi|^p\,\omega\ dx\right)^{1/p} + \left(\int_\Omega |\nabla\varphi|^p\,\omega\ dx\right)^{1/p}$$

Given $u \in H^{1,p}(\Omega,\omega)$, there exists a sequence $\{\varphi_k\}_{k=1}^\infty$ from C^∞ and a vector-valued function v, called the weak gradient of u such that

$$\int_\Omega |\varphi_k - u|\,\omega\ dx \to 0 \text{ and } \int_\Omega |\nabla\varphi_k - v|\,\omega\ dx \to 0.$$

By construction, $H^{1,p}(\Omega,\omega)$ is a Banach space. Note that it is not necessary for the gradient defined above to be a distributional derivative for u. However it has been shown by Kilpeläinen that if $\omega \in A_p(\Omega)$, that is $\omega^{1/(1-p)} \in L^1_{\mathrm{loc}}(\Omega)$, then the gradient is a distributional derivative, see [6]. Weighted Sobolev spaces have been studied predominantly for weights ω such that the measure ωdx is doubling and admits a $(1,p)$-Poincaré Inequality. These weights are called *p-admissible weights*. A more thorough discussion of weighted Sobolev spaces can be found in Chapter 1 of [5].

Weighted Euclidean spaces become particularly interesting when two open sets are quasiconformally equivalent. If $f\colon \Omega \to \Omega'$ is a quasiconformal map and u is a nonnegative measurable function on Ω, then Lemma 14.25 of [5] says

$$\int_\Omega u(f(x))J_f(x)dx = \int_{\Omega'} u(x)dx.$$

It follows that $L^p(\Omega') = \{u \mid u \circ f \in L^p(\Omega, J_f)\}$. It is then natural to study weighted spaces when doing analysis on quasiconformally equivalent open sets. It was shown by Gehring that for any quasiconformal map f, the weight $J_f^{1-p/n}$ is a p-admissible weight for every $p > 1$. This then gives us some tools to work with when comparing Sobolev spaces of quasiconformally equivalent open sets. For more information about quasiconformal maps and the weights obtained from their Jacobians, we refer the reader to Chapters 14 and 15 of [5].

The structure of this paper is as follows: In Section 2, we give two candidate definitions for weighted BV functions and show that the BV norms are lower semicontinuous. In Section 3, we give a discussion on lower semicontinous envelopes and their importance related to the norm studied in [2]. This section culminates with Theorem 3.4, giving a formula to compute this weighted BV norm using the classical BV norm. In Section 4, we discuss what happens on the set where the weight equals zero and prove the coarea formula for the norm in [2]. Section 5 features conditions on the weight which ensure that the candidate weighted BV norms are equivalent, see Theorem 5.6.

2. Definitions and Preliminary Results

Let $\Omega \subset \mathbb{R}^n$ be an open set and $\omega\colon \Omega \to [0, \infty)$ be a locally integrable weight function. Spaces of Lipschitz functions, weighted L^p, and Sobolev spaces will be defined as follows:

$$L^p(\Omega, \omega) = \{f\colon \Omega \to \mathbb{R} \text{ measurable } \mid |f|^p\, \omega \in L^1(\Omega)\},$$
$$\mathcal{W}^{1,p}(\Omega, \omega) = \{f \in L^p(\Omega, \omega) \mid \nabla f \text{ exists weakly and } |\nabla f| \in L^p(\Omega, \omega)\},$$
$$\mathrm{Lip}_{\mathrm{c}}(\Omega\colon \mathbb{R}^n) = \{f\colon \Omega \to \mathbb{R}^n \mid f \text{ is Lipschitz and } \mathit{supt}(f) \subset\subset \Omega\},$$
$$\mathrm{Lip}_{\mathrm{loc}}(\Omega) = \{f\colon \Omega \to \mathbb{R} \mid \Omega \text{ is covered by open sets on which } f \text{ is Lipschitz}\}.$$

When we say that ∇f exists weakly, we refer to the usual weak derivative in the unweighted Euclidean space: for all $\varphi \in C_{\mathrm{c}}(\Omega\colon \mathbb{R}^n)$, $\int_\Omega \varphi \cdot \nabla f \; dx = -\int_\Omega f \,\mathrm{div}\varphi \; dx$. In the special case where $\omega \geq c > 0$, for all measurable $f\colon \Omega \to \mathbb{R}$ and all $1 \leq p \leq \infty$, $L^p(\Omega, \omega) \subseteq L^p(\Omega)$ and $\mathcal{W}^{1,p}(\Omega, \omega) \subseteq \mathcal{W}^{1,p}(\Omega)$. Similiarly, if $\omega \leq C < \infty$, then for all $1 \leq p \leq \infty$, $L^p(\Omega) \subseteq L^p(\Omega, \omega)$ and $\mathcal{W}^{1,p}(\Omega) \subseteq \mathcal{W}^{1,p}(\Omega, \omega)$.

We will study the two norms listed below and use them to define functions of bounded variation. The norm $\|D_\omega f\|_B$ has been studied in [2] and has the advantage of providing a vector-valued measure which can be used to study the structure of sets of finite perimeter. The norm $\|D_\omega f\|_M$ has been studied in

[7] for the more general metric space setting and has the advantage of being applicable in the more general metric space setting where the norm of [2] is unavailable. It also provides a norm-approximation of BV functions by locally Lipschitz functions, again a property not in general available with respect to the norm of [2] (see Example 5.4). In this case, the metric space is a Euclidean set with the measure ωdx. We are interested in comparing these norms to see when they are equal and can thus be used interchangeably. For $f \in L^1_{\text{loc}}(\Omega, \omega)$, define

$$\|D_\omega f\|_B(\Omega) = \sup\left\{ \int_\Omega f \,\operatorname{div}\varphi \,dx \;\middle|\; \varphi \in \operatorname{Lip_c}(\Omega\colon \mathbb{R}^n), |\varphi| \le \omega \right\},$$

$$\|D_\omega f\|_M(\Omega) = \inf\left\{ \liminf_{k\to\infty} \int_\Omega |\nabla f_k|\,\omega \,dx \;\middle|\; (f_k - f) \to 0 \text{ in } L^1(\Omega,\omega), f_k \in \operatorname{Lip_{loc}}(\Omega) \right\}.$$

Since $\omega \in L^1_{\text{loc}}(\Omega)$, we see that $\operatorname{Lip_{loc}}(\Omega) \subset L^1_{\text{loc}}(\Omega, \omega)$. It should be noted that in the language of calculus of variations, $\|D_\omega f\|_M(\Omega)$ is the relaxation of the functional $I(f) = \int_\Omega |\nabla f|\,\omega\,dx$.

Under certain conditions, these norms are lower semicontinuous. Such a property will be useful in proving the results outlined in the abstract. We will at times assume that ω is Lipschitz and strictly positive. Sometimes, in preliminary stages, we will need to assume that there exists a constant c such that $\omega \ge c > 0$. When necessary, we may also assume that Ω is bounded. In such instances, we will explicitly state these assumptions.

Proposition 2.1. *If ω is locally bounded away from zero, then $\|D_\omega\cdot\|_B(\Omega)$ is lower semicontinuous with respect to convergence in $L^1_{loc}(\Omega,\omega)$. By this we mean that if $f_k \to f$ in $L^1_{loc}(\Omega,\omega)$, then $\|D_\omega f\|_B(\Omega) \le \liminf_{k\to\infty} \|D_\omega f\|_B(\Omega)$.*

See Proposition 1.3.1 of [3] for a proof.

Remark 2.2. Positive lower semicontinuous weights are locally bounded away from zero. Later sections of this paper deal with lower semicontinuous weights, so Proposition 2.1 will apply. It will also come up later that for $\omega \ge 0$, $\|D_\omega f\|_B(\Omega) = \|D_\omega f\|_B(\Omega_0)$ where $\Omega_0 = \{x \in \Omega \mid \omega(x) > 0\}$. Thus Proposition 2.1 will then apply to any nonnegative lower semicontinuous weight.

Lower semicontinuity of norm $\|D_\omega\cdot\|_M(\Omega)$ can be proven in greater generality. In the following proposition (found in Proposition 3.6 of [7]), we only assume ω to be nonnegative and measurable.

Proposition 2.3. *The norm $\|D_\omega\cdot\|_M(\Omega)$ is lower semicontinuous with respect to convergence in $L^1(\Omega,\omega)$.*

Note that for a given f, both norms define a Borel regular outer measure on Ω. It is not difficult to prove that $\|D_\omega f\|_B$ satisfies the axioms of an outer measure on the collection of open subsets of Ω. The proof that $\|D_\omega f\|_M$ is an outer measure is not trivial and is provided in the more general metric measure space setting in [7].

Remark 2.4. We extend both BV norms to the collection of all subsets of Ω using the Carathéodory construction described in Section 12.2, Theorem 8 of [11]. For general sets $E \subset \Omega$,

$$\|D_\omega f\|_B(E) = \inf\{\|D_\omega f\|_B(V) \mid E \subset V,\ V \text{ open}\},$$
$$\|D_\omega f\|_M(E) = \inf\{\|D_\omega f\|_M(V) \mid E \subset V,\ V \text{ open}\}.$$

This construction ensures a Borel regular outer measure as long as countable subadditivity is satisfied by open sets, which is the case here.

3. The Lower Semicontinuous Envelope

The focus of this section is on lower semicontinuous weights. We will show that in studying the norm $\|D_\omega f\|_B$, we can always assume that the weight ω is lower semicontinuous. We will also show how to approximate lower semicontinuous functions with Lipschitz functions. Thus in many cases we will be able to work with Lipschitz weights. Finally, this section concludes with Theorem 3.4 which provides a formula for $\|D_\omega f\|_B(\Omega)$ using the weight ω and the classical total variation measure $\|Df\|$ of f.

For any function $g\colon \Omega \to (-\infty, \infty]$, define $g^*\colon \Omega \to [-\infty, \infty]$ by

$$g^*(x) = \sup\{\varphi(x) \mid \varphi \in \mathrm{Lip}(\Omega), \varphi \le g\}.$$

The function g^* is lower semicontinuous, $g^* \le g$, and will be referred to as the *lower semicontinuous envelope* of g. Note that g^* will be identically negative infinity if there are no Lipschitz functions $\varphi \le g$. ($g(x) = -x^2$ is an easy example since its derivative is not bounded below). This will not be an issue for us since we will be looking at lower semicontinuous envelopes of weight functions, which are nonnegative.

Proposition 3.1. *For any measurable $f\colon \Omega \to \mathbb{R}$ and any weight $\omega \ge 0$,*

$$\|D_\omega f\|_B(\Omega) = \|D_{\omega^*} f\|_B(\Omega).$$

Proof. Since each test function $\varphi \in \mathrm{Lip_c}(\Omega\colon \mathbb{R}^n)$ is Lipschitz, then $|\varphi| \le \omega$ if and only if $|\varphi| \le \omega^*$. So $\|D_\omega f\|_B(\Omega)$ and $\|D_{\omega^*} f\|_B(\Omega)$ are the supremums of the same set of numbers and hence are equal. ■

Thus when working with the norm $\|D_\omega f\|_B$, we can always replace ω with its lower semicontinuous envelope ω^*. Theorem 3.2 shows that a lower semicontinuous function is its own lower semicontinuous envelope. Therefore when dealing with these norms, it is natural to assume that the weight is lower semicontinuous. Note that a function and its lower semicontinuous envelope can be quite different. Consider $g = \chi_{\mathbb{R}\setminus\mathbb{Q}}$. Notice that $g = 1$ almost everywhere, but since $g \equiv 0$ on the dense subset $\mathbb{Q}$, $g^* \equiv 0$. This will be very important to remember when comparing this norm to $\|D_\omega f\|_M$.

It is useful to approximate lower semicontinuous functions from below with Lipschitz functions. Theorem 3.2 will show us a way to do this. The method employed to approximate Lipschitz functions is inspired by the Lipschitz extensions developed in Theorem 1 of [9]. The following theorem is from Example 9.11 in [12]. Below, when we say $f_k \nearrow f$, we mean that $\{f_k\}_{k=1}^\infty$ is a pointwise monotone increasing sequence of functions converging to f.

Theorem 3.2. *Let (X, d) be a metric space and $f\colon X \to (-\infty, +\infty]$ be any function. For each $k > 0$, define*

$$f_k(x) = \inf\{f(w) + kd(x, w) \mid w \in X\}.$$

(a) Either $f_k \equiv -\infty$ or f_k is k-Lipschitz.
(b) If f is lower semicontinuous and there exists $k_0 > 0$ such that $f_{k_0} \not\equiv -\infty$, then $f_k(x) \nearrow f(x)$ for all $x \in X$.

Approximating lower semicontinuous weights with Lipschitz weights in this way can be used to give us a formula for the norm $\|D_\omega f\|_B$. The following lemma is useful a useful step along the way.

Lemma 3.3. *If $\omega \ge 0$ is lower semicontinuous on Ω and $f \in L^1_{loc}(\Omega, \omega)$, then there exist Lipschitz weights $\{\omega_k\}_{k=1}^\infty$ such that $\omega_k \nearrow \omega$ pointwise in Ω and*

$$\|D_\omega f\|_B(\Omega) = \lim_{k\to\infty} \|D_{\omega_k} f\|_B(\Omega).$$

Proof. From Theorem 3.2, there exist Lipschitz weights $\{\tilde{\omega}_k\}_{k=1}^\infty$ such that $\tilde{\omega}_k \nearrow \omega$ pointwise everywhere in Ω. There also exist test functions $\varphi_k \in \mathrm{Lip_c}(\Omega\colon \mathbb{R}^n)$ such that $|\varphi_k| \le \omega$ and

$$\|D_\omega f\|_B(\Omega) = \lim_{k\to\infty} \int_\Omega f\, \mathrm{div}\varphi_k dx.$$

For each $k \in \mathbb{N}$, let $\omega_k = \max\{\tilde{\omega}_k, |\varphi_1|, \dots, |\varphi_k|\}$. Each ω_k is Lipschitz and $\omega_k \nearrow \omega$ pointwise everywhere in Ω. Since $\{\omega_k\}_{k\in\mathbb{N}}$ is an increasing sequence, it is clear that $\{\|D_{\omega_k} f\|_B(\Omega)\}_{k\in\mathbb{N}}$ is also increasing and hence $\lim_{k\to\infty} \|D_{\omega_k} f\|_B(\Omega)$ exists. We see that

$$\|D_\omega f\|_B(\Omega) = \lim_{k\to\infty} \int_\Omega f \operatorname{div}\varphi_k dx \le \lim_{k\to\infty} \|D_{\omega_k} f\|_B(\Omega) \le \|D_\omega f\|_B(\Omega).$$

Therefore $\|D_\omega f\|_B(\Omega) = \lim_{k\to\infty} \|D_{\omega_k} f\|_B(\Omega)$. ■

The main result of this section, Theorem 3.4, states that $\|D_\omega f\|_B(\Omega)$ is finite only if $f \in BV_{\mathrm{loc}}(\Omega)$, that is f is locally of bounded variation in Ω in the classical sense. Since $f \in BV_{\mathrm{loc}}(\Omega)$, f has a distributional derivative Df which is a vector-valued Radon measure. The total variation of this measure, also a Radon measure itself, is denoted by $\|Df\|$. The BV measures studied in this paper have been denoted with the weight as a subscript so as to not confuse them with the classical BV measure.

Theorem 3.4. *Assume $\omega > 0$ is lower semicontinuous. Then $\|D_\omega f\|_B(\Omega) < \infty$ if and only if $f \in BV_{loc}(\Omega)$ and $\omega \in L^1(\Omega, \|Df\|)$. When these conditions are true,*

$$\|D_\omega f\|_B(\Omega) = \int_\Omega \omega \, d\|Df\|.$$

In fact for all Borel sets $E \subset \Omega$,

$$\|D_\omega f\|_B(E) = \int_E \omega \, d\|Df\|.$$

Proof. If $f \in BV_{\mathrm{loc}}(\Omega)$, the corresponding vector-valued variational measure Df exists. Hence for any $\varphi \in \mathrm{Lip_c}(\Omega\colon \mathbb{R}^n)$ with $|\varphi| \le \omega$,

$$\int_\Omega f \operatorname{div}\varphi \, dx = -\int_\Omega \varphi \cdot dDf \le \int_\Omega \omega \, d\|Df\|.$$

Thus $\|D_\omega f\|_B(\Omega) \le \int_\Omega \omega \, d\|Df\|$. If we assume in addition that $\omega \in L^1(\Omega, \|Df\|)$, then $\|D_\omega f\|_B(\Omega) < \infty$.

For the reverse inequality, the idea for the proof is to approximate the weight ω from below with a sequence of Lipschitz weights $\{\omega_k\}_{k=1}^\infty$, and then for every $\psi \in \mathrm{Lip_c}(\Omega\colon \mathbb{R}^n)$ with $|\psi| \le 1$,

$$\int_\Omega \psi \cdot \omega \, dDf = \lim_{k\to\infty} \int_\Omega \psi \cdot \omega_k dDf = \lim_{k\to\infty} -\int_\Omega f \operatorname{div}(\psi\omega_k) dx \le \|D_\omega f\|_B(\Omega).$$

Thus $\int_\Omega \omega \, d \left\| Df \right\| \leq \left\| D_\omega f \right\|_B (\Omega)$ and hence

$$\int_\Omega \omega \, d \left\| Df \right\| = \left\| D_\omega f \right\|_B (\Omega). \tag{3.5}$$

The rigorous details for this can be found in Theorem2.1.5 of [3].

Since $\left\| D_\omega f \right\|_B$ and $E \mapsto \int_E \omega \, d \left\| Df \right\|$ are Borel measures that agree on all open sets, then for all Borel sets $E \subset \Omega$, $\left\| D_\omega f \right\|_B (E) = \int_E \omega \, d \left\| Df \right\|$. ■

It is crucial to draw the reader's attention to one consequence of Theorem 3.4.

Remark 3.6. The value of $\left\| D_\omega f \right\|_B (\Omega)$ is sensitive to changes in the weight ω on sets of positive $\left\| Df \right\|$ measure. If $\left\| Df \right\|$ has a nonzero singular component with respect to Lebesgue measure, then changes in the weight on sets of Lebesgue measure zero can change the value of $\left\| D_\omega f \right\|_B (\Omega)$ even though the measure induced by the weight remains unchanged. It is therefore important for the weight to be clearly defined at all points in Ω because the BV norm depends on the weight and not the measure induced by the weight.

4. Removal of the Zero Set of the Weight and the Coarea Formula

In some of the preceding results (such as Theorem 3.4), it has been helpful to assume that the weight is positive. In this section we will give conditions under which we can shrink the space to the set where the weight is positive. In other words, when do the BV norms as outer measures live on the set Ω_0 defined below? Along the way, we will also give the coarea formula for the norm $\left\| D_\omega f \right\|_B (\Omega)$ (The coarea formula for $\left\| D_\omega f \right\|_M (\Omega)$ is proven in Proposition 4.2 of [7]). Given a weight $\omega \colon \Omega \to [0, \infty)$ in $L^1_{\mathrm{loc}}(\Omega)$, set

$$\Omega_0 = \{x \in \Omega \mid \omega(x) > 0\}.$$

Recall from Remark 2.4 that our BV norms have been defined on all sets using the Carathéodory Construction. We will frequently work with the set Ω_0. If ω is not lower semi-continuous, then Ω_0 may not be open. We will mostly focus on lower semi-continuous weights, so this is not a major issue. If the proof can easily be adapted to any weight, then we will do so. So in this section, the only assumptions on ω are that it is a Borel measurable non-negative function in $L^1_{\mathrm{loc}}(\Omega)$.

Proposition 4.1. *If $f\colon \Omega \to \mathbb{R}$ is measurable, then*

$$\|D_\omega f\|_M(\Omega) = \|D_\omega f\|_M\left(\Omega \cap \overline{\Omega_0}\right).$$

Proof. Since $\|D_\omega f\|_M$ is a Borel measure,

$$\|D_\omega f\|_M(\Omega) = \|D_\omega f\|_M\left(\Omega \cap \overline{\Omega_0}\right) + \|D_\omega f\|_M\left(\Omega \setminus \overline{\Omega_0}\right).$$

Since $\Omega \setminus \overline{\Omega_0}$ is open and $\omega \equiv 0$ on $\Omega \setminus \overline{\Omega_0}$, it is clear that $\|D_\omega f\|_M\left(\Omega \setminus \overline{\Omega_0}\right) = 0$.
■

In some cases, we can do a little better and show that the relation $\|D_\omega f\|_M(\Omega) = \|D_\omega f\|_M(\Omega_0)$ holds. See Example 5.4 for a case when this is not true. Section 5 will give some conditions under which $\|D_\omega f\|_M(\Omega) = \|D_\omega f\|_B(\Omega)$. Under such conditions, Theorem 4.6 and Proposition 5.1 will ensure that $\|D_\omega f\|_M(\Omega) = \|D_\omega f\|_M(\Omega_0)$. We will state here two other conditions that will ensure the zero set of the weight can be removed.

Theorem 4.2. *Let $\omega \geq 0$ be a lower semicontinuous weight on Ω. For each $\varepsilon > 0$, let $\Omega_\varepsilon = \{x \in \Omega \mid dist(x, \mathbb{R}^n \setminus \Omega_0) < \varepsilon\}$. If there exists a sequence $\varepsilon_k \searrow 0$ such that*

$$\lim_{k\to\infty} \frac{1}{\varepsilon_k} \int_{\Omega_{\varepsilon_k}} \omega dx = 0,$$

then $\|D_\omega f\|_M(\Omega) = \|D_\omega f\|_M(\Omega_0)$.

Remark 4.3. The condition on the weight can interpreted to say that the weighted codimension 1 lower Minkowski content of $\partial\Omega_0 \cap \Omega$ is zero (see Section 5.5 of [8]).

Theorem 4.4. *If $\omega \geq 0$ is upper semicontinuous and $f \in BV_{loc}(\Omega)$, then*

$$\|D_\omega f\|_M(\Omega) = \|D_\omega f\|_M(\Omega_0).$$

Proofs of the above two theorems can be found in Theorems 3.1.3 and 3.1.5 of [3].

For any measurable weight ω, we can remove the entire zero set of the weight for norm $\|D_\omega f\|_B$. The proof is not trivial, and requires a little classical BV theory along with the coarea formula. The classical coarea formula is stated below (see Section 5.5, Theorem 1 of [4] for a proof) and is used along with Theorem 3.4 to prove the coarea formula for $\|D_\omega \cdot\|_B$.

Theorem 4.5 (Classical Coarea Formula)**.** *If $f \in L^1_{loc}(\Omega)$, and for each $t \in \mathbb{R}$ we set $E_t = \{x \in \Omega \mid f(x) > t\}$ and $\|\partial E_t\| = \|D\chi_{E_t}\|$, then*

$$\|Df\|(\Omega) = \int_{-\infty}^{\infty} \|\partial E_t\|(\Omega) dt.$$

This result can be generalized to weighted Euclidean spaces using the norm $\|D_\omega f\|_B$.

Theorem 4.6. *For any weight* $\omega \geq 0$ *and* $f \in L^1_{loc}(\Omega, \omega)$, *set* $E_t = \{x \in \Omega \mid f(x) > t\}$ *and* $\|\partial_\omega E_t\|_B = \|D_\omega \chi_{E_t}\|_B$. *Then*

$$\|D_\omega f\|_B(\Omega_0) = \|D_\omega f\|_B(\Omega) = \int_{-\infty}^{+\infty} \|\partial_\omega E_t\|_B(\Omega)dt$$

In particular, if $\|D_\omega f\|_B(\Omega)$ *is finite, then for almost every* $t \in \mathbb{R}$ *the set* E_t *has finite weighted perimeter. Furthermore, for every Borel set* $A \subset \Omega$,

$$\|D_\omega f\|_B(A) = \int_{-\infty}^{+\infty} \|\partial_\omega E_t\|_B(A)dt$$

Proof. Let $\varphi \in \mathrm{Lip_c}(\Omega\colon \mathbb{R}^n)$ with $|\varphi| \leq \omega$. Suppose $f \geq 0$. Then

$$\begin{aligned}\int_\Omega f\, \mathrm{div}\varphi\, dx &= \int_\Omega \int_0^\infty \chi_{E_t}(x)dt\, \mathrm{div}\varphi\, dx \\ &= \int_0^\infty \int_{E_t} \mathrm{div}\varphi\, dxdt \\ &\leq \int_0^\infty \|\partial_\omega E_t\|_B(\Omega)dt.\end{aligned}$$

A similar result holds for $f \leq 0$, and then for general f,

$$\|D_\omega f\|_B(\Omega) \leq \int_{-\infty}^\infty \|\partial_\omega E_t\|_B(\Omega)dt. \tag{4.7}$$

Therefore the theorem is proven for the case when $\|D_\omega f\|_B(\Omega) = \infty$.

Now assume $\|D_\omega f\|_B(\Omega) < \infty$. Furthermore, assume for now that $\omega > 0$ in Ω and ω is lower semicontinuous. For each $s \geq 0$, define $\Omega_s = \{x \in \Omega \mid \omega(x) > s\}$. Since ω is lower semicontinuous, each Ω_s is open. Using Theorem 3.4, the Cavalieri Principle, Tonelli's Theorem, and the classical coarea formula (Theorem 4.5), we show that

$$\begin{aligned}\|D_\omega f\|_B(\Omega) &= \int_\Omega \omega\, d\|Df\| = \int_0^\infty \|Df\|(\Omega_s)ds = \int_0^\infty \int_{-\infty}^\infty \|\partial E_t\|(\Omega_s)dtds \\ &= \int_{-\infty}^\infty \int_0^\infty \|\partial E_t\|(\Omega_s)dsdt = \int_{-\infty}^\infty \int_\Omega \omega\, d\|\partial E_t\|\, dt = \int_{-\infty}^\infty \|\partial_\omega E_t\|_B(\Omega)dt.\end{aligned}$$

Using Proposition 3.1 and Lemma 3.1.8 of [3], we can remove the assumptions that ω is lower semicontinuous and positive. This gives us the desired result of

$$\|D_\omega f\|_B(\Omega_0) = \|D_\omega f\|_B(\Omega) = \int_{-\infty}^\infty \|\partial_\omega E_t\|_B(\Omega)dt.$$

This proof works for all open subsets of Ω as well. To show the coarea formula holds for all Borel subsets of Ω, use Carathéodory's criterion (Theorem 1.7 of [8]) to show that

$$\mu(A) = \int_{-\infty}^{\infty} \|\partial_\omega E_t\|_B (A) dt$$

is a Borel measure that agrees with $\|D_\omega f\|_B$ on all open subsets of Ω.

Complete details can be found in Theorem 3.1.13 of [3]. ■

Since the zero set of the weight can be removed without changing $\|D_\omega f\|_B(\Omega)$, some of the previous results in which the weight was assumed to be positive can be extended to nonnegative weights. An example is the lower semicontinuity result in Proposition 2.1. A useful corollary of the coarea formula is that any $f \in L^1_{\mathrm{loc}}(\Omega, \omega)$ can be approximated in a variational sense by truncations.

Corollary 4.8. *For any $f \in L^1_{loc}(\Omega, \omega)$, let f_k be the truncation of f at k and $-k$.*

$$f_k = \max\{-k, \min\{f, k\}\}.$$

Then $\|D_\omega f_k\|_B(\Omega) \to \|D_\omega f\|_B(\Omega)$ and $\|D_\omega f_k\|_M(\Omega) \to \|D_\omega f\|_M(\Omega)$.

5. Conditions that Ensure Equality of the BV Norms

In this section we will explore conditions under which $\|D_\omega f\|_B$ and $\|D_\omega f\|_M$ are equal. We will also give examples of weights that do not satisfy this condition and for which the two norms in question are not even comparable.

Recall from our discussion of lower semicontinuous weights in Section 3 that $\|D_\omega f\|_B = \|D_{\omega^*} f\|_B$, while in general we do not have $\|D_\omega f\|_M = \|D_{\omega^*} f\|_M$. Therefore when comparing $\|D_\omega f\|_B$ and $\|D_\omega f\|_M$, we can only hope for equality when the weight is lower semicontinuous. Even this may not ensure equality, and we will give examples (Examples 5.4 and 5.5) of when the two norms are not even equivalent in this case. We will see that equality is indeed obtained when the weight is continuous and positive. This first inequality holds for all weights.

Proposition 5.1. *Assume that $\omega \geq 0$ is measurable. For $f \in L^1_{loc}(\Omega, \omega)$,*

$$\|D_\omega f\|_B(\Omega) \leq \|D_\omega f\|_M(\Omega).$$

Proof. We will first assume that ω is lower semicontinuous and positive. Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathrm{Lip}_{\mathrm{loc}}(\Omega)$ with $(f_k - f) \to 0$ in $L^1(\Omega, \omega)$. Choose any $\varphi \in \mathrm{Lip}_c(\Omega\colon \mathbb{R}^n)$ with $|\varphi| \leq \omega$. Since φ has compact support, $\mathrm{div}\varphi$ also has compact support. Since $\omega > 0$ is lower semicontinuous, there exists $c > 0$ such that $\omega \geq c$ on $\mathrm{supt}(\mathrm{div}\varphi)$. It follows that $(f_k - f) \to 0$ in $L^1(\mathrm{supt}(\mathrm{div}\varphi))$. Since φ is Lipschitz, $\mathrm{div}\varphi$ is bounded. Hence we see that

$$\int_\Omega f \,\mathrm{div}\varphi\, dx = \lim_{k\to\infty} \int_\Omega f_k \,\mathrm{div}\varphi\, dx = -\lim_{k\to\infty} \int_\Omega (\nabla f_k \cdot \varphi) dx$$
$$\leq \liminf_{k\to\infty} \int_\Omega |\nabla f_k|\, |\varphi|\, dx \leq \liminf_{k\to\infty} \int_\Omega |\nabla f_k|\, \omega\, dx.$$

Then by taking a supremum over all such φ, we get that

$$\|D_\omega f\|_B (\Omega) \leq \liminf_{k\to\infty} \int_\Omega |\nabla f_k|\, \omega\, dx.$$

Since this holds for all such sequences $\{f_k\}_{k\in\mathbb{N}}$, it follows that

$$\|D_\omega f\|_B (\Omega) \leq \|D_\omega f\|_M (\Omega).$$

For general ω, let $\Omega^* = \{x \in \Omega \mid \omega^*(x) > 0\}$ where ω^* is the lower semicontinuous envelope of ω. Note that as $\omega^* \leq \omega$, $\|D_{\omega^*} f\|_M (\Omega) \leq \|D_\omega f\|_M (\Omega)$. Theorem 4.6 and the above result give us that

$$\|D_\omega f\|_B (\Omega) = \|D_{\omega^*} f\|_B (\Omega^*) \leq \|D_{\omega^*} f\|_M (\Omega^*) \leq \|D_{\omega^*} f\|_M (\Omega) \leq \|D_\omega f\|_M (\Omega).$$

■

We are nearly ready to prove one of our main results, that the two norms are equal when the weight is continuous and positive. To do so, we will need the following lemma.

Lemma 5.2. *Let $\omega > 0$ be continuous and bounded in Ω. If $f \in BV(\Omega)$, then*

$$\|D_\omega f\|_B (\Omega) = \|D_\omega f\|_M (\Omega) = \int_\Omega \omega\, d\, \|Df\|.$$

Proof. Since $f \in BV(\Omega)$, the measure $\|Df\|$ exists and is finite. There also exist smooth functions $\{f_k\}_{k=1}^\infty$ such that $f_k \to f$ in $L^1(\Omega)$, $\|Df_k\|(\Omega) \to \|Df\|(\Omega)$ and $\|Df_k\| \rightharpoonup \|Df\|$ (i.e. for compactly supported continuous functions φ on Ω, $\lim_{k\to\infty} \int_\Omega \varphi\, d\, \|Df_k\| = \int_\Omega \varphi\, d\, \|Df\|$).

Let V be any open set with $V \subset\subset \Omega$, and for all $t \geq 0$, $A_t = \{x \in V \mid \omega(x) > t\}$. A proof can then be written to show that

$$\begin{aligned}\|D_\omega f\|_M(V) &\leq \lim_{k\to\infty} \int_V |\nabla f_k|\, \omega\, dx = \lim_{k\to\infty} \int_V \omega\, d\, \|Df_k\| = \lim_{k\to\infty} \int_0^\infty \|Df_k\|(A_t)dt \\ &= \int_0^\infty \lim_{k\to\infty} \|Df_k\|(A_t)dt = \int_0^\infty \|Df\|(A_t)dt = \int_V \omega\, d\, \|Df\| \\ &= \|D_\omega f\|_B(V) \leq \|D_\omega f\|_M(V).\end{aligned}$$

Details of this proof can be found in Lemma 3.2.2 of [3]. Since Ω can be approximated from within by open sets with compact closure, we conclude that $\|D_\omega f\|_B(\Omega) = \|D_\omega f\|_M(\Omega)$. ∎

Theorem 5.3. *If $\omega > 0$ is continuous and $f \in L^1_{loc}(\Omega, \omega)$, then*

$$\|D_\omega f\|_B(\Omega) = \|D_\omega f\|_M(\Omega) = \int_\Omega \omega d\, \|Df\|.$$

Proof. If $\|D_\omega f\|_B(\Omega) = \infty$, the result is trivial by Proposition 5.1 and Theorem 3.4. If $\|D_\omega f\|_B(\Omega) < \infty$, Theorem 3.4 tells us that $f \in BV_{\mathrm{loc}}(\Omega)$. This along with ω being continuous give us the existence of a sequence of open sets

$$U_1 \subset U_2 \subset \cdots \subset U_k \subset \cdots$$

such that $\Omega = \bigcup_{k=1}^\infty U_k$ and for each $k \in \mathbb{N}$, ω is bounded on U_k and $f \in BV(U_k)$. Lemma 5.2 then gives us

$$\|D_\omega f\|_B(\Omega) = \lim_{k\to\infty} \|D_\omega f\|_B(U_k) = \lim_{k\to\infty} \|D_\omega f\|_M(U_k) = \|D_\omega f\|_M(\Omega).$$

∎

We next give examples of lower semicontinuous weights and weighted BV functions such that $\|D_\omega f\|_B(\Omega) < \|D_\omega f\|_M(\Omega)$. Consequently, these functions cannot be approximated in the $\|D_\omega \cdot\|_B(\Omega)$ norm in a variational sense by locally Lipschitz functions.

Example 5.4. Let $U \subset \mathbb{R}^n$ be any bounded open set with smooth boundary and *nonzero* finite unweighted perimeter, that is $\chi_U \in BV(\mathbb{R}^n)$ with $\|D\chi_U\|(\mathbb{R}^n) > 0$. Let $f = \chi_U$ and $\omega = 2 - \chi_{\partial U}$. Notice that ω is positive and lower semicontinuous, $\mathcal{L}^n(\partial U) = 0$, $\omega = 1$ on ∂U, and $\|Df\|$ is supported on ∂U. Theorem 3.4 tells us that

$$\|D_\omega f\|_B(\mathbb{R}^n) = \int_{\mathbb{R}^n} \omega d\, \|Df\| = \int_{\partial U} \omega d\, \|Df\| = \|Df\|(\mathbb{R}^n) > 0.$$

Since $\omega = 2$ almost everywhere,

$$\|D_\omega f\|_M(\mathbb{R}^n) = 2\|Df\|(\mathbb{R}^n).$$

Example 5.5. We now construct a positive lower semicontinuous weight for which $\|D_\omega f\|_B(\Omega)$ and $\|D_\omega f\|_M(\Omega)$ are not even comparable. Let $\Omega = \mathbb{R}^n$ and define the weight by

$$\omega(x) = \begin{cases} 1/k & \text{if } |x-2k| = 1/k,\ k \in \mathbb{N}, \\ 1 & \text{otherwise.} \end{cases}$$

This weight is indeed lower semicontinuous. For each $k \in \mathbb{N}$, let $B_k = B(2k, 1/k)$ and $f_k = \chi_{B_k}$. Then

$$\|D_\omega f_k\|_B(\mathbb{R}^n) = \int_{\mathbb{R}^n} \omega d\|Df_k\| = (1/k)\mathcal{H}^{n-1}(\partial B_k).$$

Since $\omega = 1$ almost everywhere,

$$\|D_\omega f_k\|_M(\mathbb{R}^n) = \|Df_k\|(\mathbb{R}^n) = \mathcal{H}^{n-1}(\partial B_k) = k\|D_\omega f_k\|_B(\mathbb{R}^n).$$

Therefore, these two norms are not comparable.

The proof of Theorem 5.3 showing equality of the norms cannot be used in general with lower semicontinuous weights. Theorem 5.6 gives two conditions that allow us to loosen the requirement that ω be continuous on all of Ω by limiting how big the set of discontinuities can be. One involves the codimension one Hausdorff measure $\mathcal{H}^h$. This is a Hausdorff-type measure using the gauge function $\frac{1}{r}\int_{B(x,r)} \omega\, dx$. More precisely,

$$H^h(E) = \lim_{\delta \to 0^+} \inf \left\{ \sum_{j=1}^{\infty} \frac{1}{r_j} \int_{B(x_j,r_j)} \omega\, dx \,\middle|\, E \subset \bigcup_{j=1}^{\infty} B(x_j, r_j), r_j \leq \delta \right\}.$$

Note that in an unweighted space, $\mathcal{H}^h$ is a constant multiple of $\mathcal{H}^{n-1}$.

Theorem 5.6. *Let there be a relatively closed set $E \subset \Omega \subset \mathbb{R}^n$ such that $\omega > 0$ on $\Omega \setminus E$, and ω is continuous in $\Omega \setminus E$. If either of the following conditions are satisfied, then for all $f \in L^1_{loc}(\Omega, \omega)$, $\|D_\omega f\|_B(\Omega) = \|D_\omega f\|_M(\Omega)$.*

(i) *$\mathcal{H}^{n-1}(E) = 0$ and ω is locally bounded in a neighborhood of E,*

(ii) *$\mathcal{H}^h(E) = 0$ and the measure induced by ω is doubling.*

The proof of this theorem can be found in Theorem 3.2.6 of [3].

The conditions above ensuring equality of the two norms all require that the set where the weight is zero, $\Omega \setminus \Omega_0$, be very small. A large zero set will not be a problem though if $\|D_\omega f\|_M(\Omega) = \|D_\omega f\|_M(\Omega_0)$, for example when the hypothesis of Theorem 4.2 is satisfied. In this case as long as the weight satisfies one of the criteria outlined above on Ω_0, we obtain equality of the two norms on Ω.

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