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# Recent Results on Harmonic and $p$-harmonic Mappings

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**Abstract.** The class of conformal mappings is a natural object in the study of theory of analytic functions. Likewise univalent harmonic mappings (both in the planar and in the higher dimensional case) is natural in the class of harmonic functions. During the last thirty years, the univalent harmonic mappings and many other related investigations become popular in geometric function theory, eg. in the study of minimal surfaces. In this survey, we mainly deal with recent works of the present authors and some related investigations without proof.

**Keywords.** Harmonic mapping, $p$-harmonic mapping, Landau-Bloch's constant, Integral means.

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## Contents



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## 1. Planar Harmonic mappings

A complex-valued function $f$ defined on a domain $D$ is called *harmonic* in $D$ if and only if it is twice continuously differentiable and $\Delta f = 0$, i.e. the real and imaginary parts are real harmonic in $D$, where $\Delta$ represents the usual complex Laplacian operator

$$\Delta = 4\frac{\partial^2}{\partial z \partial \overline{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

A four times continuously differentiable function $f$ is said to be *biharmonic* in $D$ if the Laplacian of $f$, namely $\Delta f$, is harmonic in $D$. The properties of harmonic and biharmonic mappings have been investigated by a number authors (cf. [1, 2, 3, 11, 16, 28, 35]).

An obvious fact is that every harmonic mapping $f$ defined on a simply connected domain $D$ admits a canonical decomposition $f = h + \overline{g}$, where $h$ and $g$ are analytic in $D$. We refer to [28, 34] for the theory harmonic mappings. For harmonic mappings $f$ of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, we use the following standard notations:

$$\Lambda_f(z) = \max_{0 \leq \theta \leq 2\pi} |f_z(z) + e^{-2i\theta} f_{\overline{z}}(z)| = |f_z(z)| + |f_{\overline{z}}(z)|$$

and

$$\lambda_f(z) = \min_{0 \leq \theta \leq 2\pi} |f_z(z) + e^{-2i\theta} f_{\overline{z}}(z)| = \big|\, |f_z(z)| - |f_{\overline{z}}(z)| \,\big|.$$

Then $J_f = \lambda_f \Lambda_f$ if $J_f \geq 0$, where $J_f$ denotes the Jacobian of $f$ given by

$$J_f(z) = |f_z(z)|^2 - |f_{\overline{z}}(z)|^2.$$

**Definition 1.** *We say that $f \in \mathcal{H}_M(\mathbb{D})$ if $f$ is harmonic in $\mathbb{D}$ and $|f(z)| \leq M$ for $z \in \mathbb{D}$. We use the canonical decomposition $f = h + \overline{g}$ with the analytic functions $h$ and $g$ having the power series of the form*

$$h(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n. \tag{1.1}$$

**1.1. Sharp coefficient estimates.** One of the long standing open problems in geometry function theory is to determine the precise value of the schlicht Bloch constant for analytic functions of $\mathbb{D}$. It has attracted much attention, see [50, 57, 58, 59] and references in these articles. By using subordination, the authors obtained the following sharp coefficient estimates which is crucial in obtaining bounds for the Landau-Bloch constants. For example, by applying this theorem, the authors improved many previously known results (see Section 3) on Landau' and Bloch's theorems and also about Landau and Bloch constants for harmonic and biharmonic mappings.

**Theorem 1.** ([18, Theorem 1] *or* [17, Lemma 1]) *Suppose $f \in \mathcal{H}_M(\mathbb{D})$ and $f = h + \overline{g}$. Then $|a_0| \le M$ and for each $n \ge 1$,*

$$|a_n| + |b_n| \le \frac{4M}{\pi}.$$

*The above estimate is sharp for each $n \ge 1$. For each $n \ge 1$, the extremal function is*

$$f_n(z) = \frac{2M\alpha}{\pi} \arg\left(\frac{1+\beta z^n}{1-\beta z^n}\right), \quad |\alpha| = |\beta| = 1$$

*or $f(z) \equiv M$.*

The classical theorem of Landau shows that there exists a $\rho = \rho(M) > 0$ such that every function $f$, analytic in $\mathbb{D}$ with $f(0) = f'(0) - 1 = 0$ and $|f(z)| < M$, is univalent in the disk $\mathbb{D}_\rho = \{z \in \mathbb{C} : |z| < \rho\}$ and in addition, the range $f(\mathbb{D}_\rho)$ contains a disk of radius $M\rho^2$ (cf. [47]). Recently, many authors considered Landau's theorem for planar harmonic mappings (see for example, [8, 11, 16, 31, 39, 48, 49, 74]) and biharmonic mappings (see [1, 11, 16]). As remarked above, Theorem 1 has been proved to be useful. As another application of Theorem 1, one can get the following sharp distortion theorem.

**Corollary 1.** *Suppose $f \in \mathcal{H}_M(\mathbb{D})$. Then*

$$\Lambda_f(z) \le \frac{4}{\pi(1-|z|^2)} \quad \text{for } z \in \mathbb{D}.$$

We remark that Corollary 1 coincides with the result of Colonna [30, Theorem 3] who has discussed harmonic Bloch functions. By applying Corollary 1, we can improve [74, Theorem 2] and [49, Theorem 2.3], respectively, as follows.

**Theorem 2.** ([16, Theorem 1]) *Assume that $f \in \mathcal{H}_M(\mathbb{D})$ with $J_f(0) = 1$. Then $f$ is univalent on a disk $\mathbb{D}_{r_1}$ with*

$$r_1 = \frac{\pi^2 \sqrt{24M^2 + \pi^2 - 4M\sqrt{36M^2 + 2\pi^2}}}{4[\pi^2 + 2M\sqrt{36M^2 + 2\pi^2} + 12M^2]}$$

*and* $f(\mathbb{D}_{r_1})$ *contains a univalent disk* $\mathbb{D}_{R_1}$ *with* $R_1 = \pi r_1/(8M)$.

**Theorem 3.** ([18, Theorem 4]) *Let* $f \in \mathcal{H}_M(\mathbb{D})$ *with* $f(0) = f_{\overline{z}}(0) = f_z(0) - 1 = 0$. *Then* $f$ *is univalent in the disk* $\mathbb{D}_{r_0}$ *with* $r_0 = \phi(M_r)$ *and* $f(\mathbb{D}_{r_0})$ *contains a univalent disk of radius at least*

$$R_0 := \max_{0<r<1} \psi(M_r),$$

*where*

$$\phi(x) = \frac{rx}{(x^2+x-1)}, \ \ \psi(x) = r\left[1 + \left(\frac{x^2-1}{x}\right)\log\left(\frac{x^2-1}{x^2+x-1}\right)\right]$$

*and*

$$M_r = \frac{4M}{\pi(1-r^2)}.$$

**1.2. Differential operator** $L$**.** Let $L$ denote the complex-operator

$$L = z\frac{\partial}{\partial z} - \overline{z}\frac{\partial}{\partial \overline{z}}. \tag{1.2}$$

We see that it is linear and satisfies the usual product rule:

$$L(af+bg) = aL(f) + bL(g) \ \text{ and } \ L(fg) = fL(g) + gL(f),$$

where $a, b$ are complex constants, $f$ and $g$ are $C^1$-(i.e. continuously differentiable) functions. It is easy to see that the operator $L$ possesses a number of interesting properties, e.g. $L$ preserves both *harmonicity* and *biharmonicity*. Many other basic properties are stated for instance in the paper of Mocanu [60] (see also [2, 11]).

It is well-known that a sense-preserving harmonic functions have the open mapping property (i.e. they map every open subset of $\mathbb{D}$ to an open set in $\mathbb{C}$). We call $f$ *univalent* or locally univalent in $\mathbb{D}$ if it is *one-to-one* or locally one-to-one in $\mathbb{D}$, respectively.

Liu [49, Theorem 2.6] proved that for open harmonic mappings $f$ of $\mathbb{D}$ normalized by $f_z(0) = 1$ and $f_{\overline{z}}(0) = 0$, $f(\mathbb{D})$ contains a univalent disk of radius at least $R \approx 0.027735$ which is an improvement of earlier known results [8, Theorem 7] and [39, Theorem 2.5]. It is natural to obtain a similar result but for $L(f)$ defined by (1.2).

**1.3. Landau-Bloch's constant for harmonic mappings.** In our next result, we determine an estimate for the Bloch constant of $L(f)$ when $f$ runs on the class of open harmonic mappings. It is worth pointing out that (see [2, Corollary 1(3)]) the operator $L(f)$ for biharmonic functions behaves much like $zf'$ for analytic functions, for example in the sense that for $f$ univalent and biharmonic, $f$ is starlike in $\mathbb{D}$ if and only if $\mathrm{Re}\,(L(f)(z)/f(z)) > 0$ in $\mathbb{D}$.

**Theorem 4.** ([18, Theorem 5]) *Let $f$ be an open harmonic mapping of $\mathbb{D}$ normalized by $f_z(0) = 1$ and $f_{\overline{z}}(0) = 0$. Then $L(f)(\mathbb{D})$ contains a univalent disk of radius at least*

$$R = \max_{0<r<1} \varphi(r)$$

*where*

$$\varphi(r) = \frac{r}{\sqrt{2}} \frac{1 - \sqrt{1 - \frac{1}{1+M_r-\frac{1}{M_r}}}}{1 + \sqrt{1 - \frac{1}{1+M_r-\frac{1}{M_r}}}}, \quad M_r = \frac{2(1+r)}{1-r}.$$

*Moreover, $L(f)(\mathbb{D})$ contains a univalent disk of radius at least $R \approx 0.0143328$.*

A harmonic function $f$ is called a *harmonic Bloch mapping* if and only if

$$\sup_{z,w\in\mathbb{D},\ z\neq w} \frac{|f(z) - f(w)|}{\rho(z, w)} < +\infty,$$

where

$$\rho(z, w) = \frac{1}{2} \log \left( \frac{1 + |\frac{z-w}{1-\overline{z}w}|}{1 - |\frac{z-w}{1-\overline{z}w}|} \right) = \operatorname{arctan h} \left| \frac{z - w}{1 - \overline{z}w} \right|$$

denotes the hyperbolic distance between $z$ and $w$ in $\mathbb{D}$. In the following, we denote the hyperbolic disk (resp. circle) with center $a$ and radius $r > 0$ by $\mathbb{D}_h(a, r) = \{z : \rho(a, z) < r\}$ (resp. $\mathbb{S}_h(a, r) = \{z : \rho(a, z) = r\}$). Obviously, for each $a \in \mathbb{D}$, the following are equivalent:

$$(1)\ \rho(a, z) = r; \quad (2)\ \left| \frac{z - a}{1 - \overline{a}z} \right| = \tanh(r); \quad (3)\ \frac{|1 - \overline{a}z|^2}{1 - |z|^2} = \frac{1 - |a|^2}{1 - \tanh^2(r)}.$$

In [30], Colonna proved

$$\sup_{z,w\in\mathbb{D},\ z\neq w} \frac{|f(z) - f(w)|}{\rho(z, w)} = \sup_{z\in\mathbb{D}} (1 - |z|^2)\Lambda_f(z). \tag{1.3}$$

Moreover, the set of all harmonic Bloch mappings, denoted by the symbol $\mathcal{HB}_1$, forms a complex Banach space with the norm $\|\cdot\|$ given by

$$\|f\|_{\mathcal{HB}_1} = |f(0)| + \sup_{z\in\mathbb{D}} (1 - |z|^2)\Lambda_f(z).$$

For $\nu \in (0, \infty)$, a harmonic mapping $f$ is called a *harmonic $\nu$-Bloch mapping* if and only if $\|f\|_{\mathcal{HB}_\nu} < +\infty$, where

$$\|f\|_{\mathcal{HB}_\nu} = |f(0)| + \sup_{z\in\mathbb{D}} (1 - |z|^2)^\nu \Lambda_f(z). \tag{1.4}$$

It can be easily shown that the set of harmonic $\nu$-Bloch mappings with this norm forms a $\nu$-Banach space, which we denote by $\mathcal{HB}_\nu$.

**1.4. Marden's constant for harmonic mappings.** Let $\chi$ denote the chordal metric on the extended complex plane $\mathbb{C}_\infty$. For a geometric purpose we view $\mathbb{C}_\infty$ as the sphere in the 3-dimensional space $\mathbb{R}^3$ with center at $(0, 0, 1/2)$ and radius $1/2$, and have

$$\chi(z,w)=\begin{cases}\dfrac{|z-w|}{(1+|z|^2)^{1/2}(1+|w|^2)^{1/2}} & \text{if } z,w\in\mathbb{C} \text{ with } z\neq w,\\ \dfrac{1}{(1+|z|^2)^{1/2}} & \text{if } w=\infty.\end{cases}$$

**Definition 2.** *A meromorphic harmonic mapping $f$ in $\mathbb{D}$ is called a normal harmonic mapping if $M(f) < \infty$, where*

$$M(f)=\sup_{z,w\in\mathbb{D},\ z\neq w}\frac{\chi(f(z),f(w))}{\rho(z,w)}.$$

By (1.3), we have

$$M(f)=\sup_{z\in\mathbb{D}}\left\{\frac{(1-|z|^2)\Lambda_f(z)}{1+|f(z)|^2}\right\}.$$

Many authors discussed the coefficient estimate, distortion theorem, and the existence of Landau-Bloch and Marden constants for analytic Bloch functions (see [3, 5, 6, 57, 58, 59, 50]). But in literature there are no analogous results for harmonic $\nu$-Bloch mappings. For the following result, the authors have used variational method and subordination to fill this gap.

**Theorem 5.** ([19, Theorem 2.1]) *Let $f = h + \overline{g}$ be a harmonic mapping, where $g$ and $h$ are analytic in $\mathbb{D}$ with the expansions* (1.1). *If $\lambda_f(0) = \alpha$ for some $\alpha \in (0, 1]$ and $\|f\|_{\mathcal{HB}_\nu} \leq M$ for $M > 0$, then*

$$|a_n|+|b_n|\leq A_n(\alpha,\nu,M)=\inf_{0<r<1}\mu(r)\quad \text{for } n\geq 2,$$

*where*

$$\mu(r)=\frac{M^2-\alpha^2(1-r^2)^{2\nu}}{nr^{n-1}(1-r^2)^\nu M}.$$

*In particular, if $\nu = M = \alpha = 1$, then*

$$\mu(r)=\frac{2-r^2}{nr^{n-3}(1-r^2)}$$

*and*

$$|a_n|+|b_n|\leq A_n, \tag{1.5}$$

*where*

$$A_n = A_n(1,1,1) = \begin{cases} 0 & \text{for } n = 2, \\ \frac{1}{3} & \text{for } n = 3, \\ \frac{\sqrt{2}}{2}\left(\frac{3+\sqrt{17}}{(1+\sqrt{17})\sqrt{5-\sqrt{17}}}\right) \approx 1.049889 & \text{for } n = 4, \\ \mu\left(\sqrt{\frac{3n-7-\sqrt{n^2+6n-23}}{2(n-3)}}\right) & \text{for } n \geq 5. \end{cases}$$

*The estimate of* (1.5) *is sharp when* $n \in \{1,2,3\}$. *The extremal functions is*

$$f(z) = \frac{3\sqrt{3}}{4}\left[\left(\frac{z+(\sqrt{3}/3)}{1+(\sqrt{3}/3)z}\right)^2 - \frac{1}{3}\right]$$

*or* $\overline{f(z)}$.

As a consequence of Theorem 5, a computation gives

**Theorem 6.** ([19, Theorem 2.3]) *Let* $f$ *be a harmonic mapping with* $f(0) = \lambda_f(0) - \alpha = 0$ *and* $\|f\|_{\mathcal{HB}_\nu} \leq M$, *where* $M$ *and* $\alpha \in (0,1]$ *are constants. Then* $f$ *is univalent in* $\mathbb{D}_{\rho_0}$, *where*

$$\rho_0 = \varphi(r_0) = \max_{0<r<1} \varphi(r), \quad \varphi(r) = \frac{\alpha r(1-r^2)^\nu M}{\alpha M(1-r^2)^\nu - \alpha^2(1-r^2)^{2\nu} + M^2}.$$

*Moreover,* $f(\mathbb{D}_{\rho_0})$ *contains a univalent disk* $\mathbb{D}_{R_0}$ *with*

$$R_0 = r_0\Big[\alpha + \frac{M^2 - \alpha^2(1-r_0^2)^{2\nu}}{M(1-r_0^2)^\nu} \log \frac{M^2 - \alpha^2(1-r_0^2)^{2\nu}}{\alpha M(1-r_0^2)^\nu - \alpha^2(1-r_0^2)^{2\nu} + M^2}\Big].$$

A special case of Theorem 6 gives the following sharp estimate which is indeed a harmonic analog of [6, Theorem 2] and [6, Corollary 3].

**Theorem 7.** ([19, Theorem 2.4]) *Let* $f \in \mathcal{HB}_\nu(\alpha)$. *Then for* $z$ *with* $|z| < \frac{a_0+m_0(\alpha)}{1+a_0m_0(\alpha)}$ *and* $a_0 = 1/\sqrt{1+2\nu}$, *we have*

$$\Lambda_f(z) \geq \mathrm{Re}\left(f_z(z) + \overline{f_{\overline{z}}(z)}\right) \geq \frac{\alpha(m_0(\alpha) - |z|)}{m_0(\alpha)(1-m_0(\alpha)|z|)^{2\nu+1}}.$$

*The equalities occur if and only if* $f(z) = e^{i\vartheta}F_\alpha(e^{-i\vartheta}z)$ *for some* $\vartheta \in [0, 2\pi)$, *where*

$$F_\alpha(z) = \frac{\alpha}{m_0(\alpha)} \int_0^z \frac{m_0(\alpha) - \zeta}{(1-m_0(\alpha)\zeta)^{2\nu+1}}\, d\zeta$$

*and $m_0(\alpha)$ satisfies*

$$\sqrt{1+2\nu}\Big(\frac{2\nu+1}{2\nu}\Big)^{\nu} m_0(\alpha)(1-m_0^2(\alpha))^{\nu}=\alpha.$$

*Moreover, $f(\mathbb{D}_{m_0(\alpha)})$ contains a univalent disc of radius $R_0$, where*

$$R_0 \geq \frac{\alpha}{m_0(\alpha)}\int_0^{m_0(\alpha)} \frac{(m_0(\alpha)-t)}{(1-m_0(\alpha)t)^{2\nu+1}}\,dt.$$

*The equality occurs if and only if $f(z)=e^{i\vartheta}F_\alpha(e^{-i\vartheta}z)$ for some $\vartheta\in[0,2\pi)$.*

If $f$ is a normal harmonic mapping and $a\in\mathbb{D}$, then we set

$$s(a,f)=\sup\{r:\ f \text{ is schlicht in the hyperbolic disk } \mathbb{D}_h(a,r)\}$$

and $s(f)=\sup\{s(a,f):\ a\in\mathbb{D}\}$. The Marden constant for the class of normal harmonic mappings $f$ with $M(f)=m>0$ is given by

$$M=\inf\{s(f):\ f \text{ is a normal harmonic function with } M(f)=m\},$$

where $M(f)$ is defined in Definition 2.

**Theorem 8.** ([19, Theorem 4.1]) *Suppose that $f$ is a normal harmonic mapping such that $f(0)=f_{\overline{z}}(0)=f_z(0)-m=0$. Then*

$$M(f)\geq 2\mathrm{arctanh}\left(\frac{1}{\sqrt{3(1+m^2)}}\right).$$

## 2. Modulus continuous and harmonic Hardy space

Let $\Omega$ be a domain in $\mathbb{C}$ and $\rho>0$ a conformal metric in $\Omega$. The Gaussian curvature of the domain is given by

$$K_\rho=-\frac{1}{2}\frac{\Delta\log\rho}{\rho}.$$

We denote $\lambda(z)|dz|^2$ the hyperbolic metric in $\mathbb{D}$, where $\lambda(z)=4/(1-|z|^2)^2$. The following is Ahlfors-Schwarz lemma.

**Lemma 1.** *If $\rho>0$ is a $C^2$ function (metric density) in $\mathbb{D}$ and Gaussian curvature $K_\rho\leq -1$, then $\rho\leq\lambda$.*

A sense-preserving harmonic mapping $f$ in $\mathbb{D}$, is said to be a $K$-quasiregular harmonic mapping if for any $z\in\mathbb{D}$,

$$\frac{\Lambda_f(z)}{\lambda_f(z)}\leq K,$$

where $\Lambda_f=|f_z|+|f_{\overline{z}}|$ and $\lambda_f=|f_z|-|f_{\overline{z}}|$.

In [10], Chen proved the following Schwarz-Pick type theorem for $K$-quasiregular harmonic mappings.

**Theorem 9.** ([10, Theorem 7]) *If $f$ is a sense-preserving and $K$-quasiregular harmonic mapping of $\mathbb{D}$ into itself, then*

$$\Lambda_f(z) \leq \frac{4K}{\pi} \frac{\cos(|f(z)\pi/2|)}{1-|z|^2}$$

*holds for $z \in \mathbb{D}$.*

By using Lemma 1, the authors in [25] improved Theorem 9 as follows.

**Theorem 10.** ([25, Lemma 1]) *Let $f$ be a $K$-quasiregular harmonic mapping in $\mathbb{D}$ with $f(\mathbb{D}) \subset \mathbb{D}$. Then for any $z \in \mathbb{D}$,*

$$\Lambda_f(z) \leq K \frac{1-|f(z)|^2}{1-|z|^2} \leq \frac{4K}{\pi} \frac{\cos(|f(z)\pi/2|)}{1-|z|^2}. \tag{2.1}$$

*Moreover, the first inequality of* (2.1) *is sharp when $K \to 1$.*

**2.1. Local Lipschitz space and $\Lambda_\omega$-extension.** A continuous increasing function $\omega : [0,\infty) \to [0,\infty)$ with $\omega(0) = 0$ is called a *majorant* if $\omega(t)/t$ is non-increasing for $t > 0$. Given a subset $\Omega$ of $\mathbb{C}$ (or $\mathbb{C}^n$), a function $f : \Omega \to \mathbb{C}$ is said to belong to the *Lipschitz space* $\Lambda_\omega(\Omega)$ if there is a positive constant $C$ such that

$$|f(z) - f(w)| \leq C\omega(|z-w|) \ \text{ for all } z,\ w \in \Omega. \tag{2.2}$$

For $\delta_0 > 0$, let

$$\int_0^\delta \frac{\omega(t)}{t}\, dt \leq C \cdot \omega(\delta),\ 0 < \delta < \delta_0 \tag{2.3}$$

and

$$\delta \int_\delta^{+\infty} \frac{\omega(t)}{t^2}\, dt \leq C \cdot \omega(\delta),\ 0 < \delta < \delta_0, \tag{2.4}$$

where $\omega$ is a majorant and $C$ is a positive constant.

A majorant $\omega$ is said to be *regular* if it satisfies the conditions (2.3) and (2.4) (see [35, 63]).

Let $G$ be a proper subdomain of $\mathbb{C}^n$ or $\mathbb{R}^n$. We say that a function $f$ belongs to the *local Lipschitz space* $\mathrm{loc}\Lambda_\omega(G)$ if (2.2) holds, with a fixed positive constant $C$, whenever $z \in G$ and $|z-w| < \frac{1}{2}d(z,\partial G)$, where $d(\cdot,\cdot)$ denotes Euclidean distance (cf. [36, 47]). Moreover, $G$ is said to be a $\Lambda_\omega$-*extension domain* if $\Lambda_\omega(G) = \mathrm{loc}\Lambda_\omega(G)$. The geometric characterization of $\Lambda_\omega$-extension domains was first given by Gehring and Martio [36]. Then Lappalainen [47] extended it to the general case and proved that $G$ is a $\Lambda_\omega$-extension domain if and only if each pair of points $z, w \in G$ can be joined by a rectifiable curve $\gamma \subset G$ satisfying

$$\int_\gamma \frac{\omega(d(z, \partial G))}{d(z, \partial G)}\, ds(z) \leq C\omega(|z-w|)$$

with some fixed positive constant $C = C(G, \omega)$, where $ds$ stands for the arc length measure on $\gamma$. Furthermore, Lappalainen [47, Theorem 4.12] proved that $\Lambda_\omega$-extension domains exist only for majorants $\omega$ satisfying (2.3).

Dyakonov [35] characterized the holomorphic functions of class $\Lambda_\omega$ in terms of their modulus. Later in [63, Theorems A], Pavlović came up with a relatively simple proof of the results of Dyakonov. Recently, many authors considered this topic and generalized Dyakonov's results to pseudo-holomorphic functions and real harmonic functions of several variables for some special majorant $\omega(t) = t^\alpha$, where $\alpha > 0$ (see [46, 54, 56, 52, 53]). By using Theorem 10, the authors in [21] extended [63, Theorems A and B] to planar $K$-quasiregular harmonic mappings as follows.

**Theorem 11.** ([21, Theorem 1]) *Let $\omega$ be a majorant satisfying* (2.3)*, and let $G \subset \mathbb{C}$ be a $\Lambda_\omega$-extension domain. If $f$ is a planar $K$-quasiregular harmonic mapping of $G$ and continuous up to the boundary $\partial G$, then*

$$f \in \Lambda_\omega(G) \Longleftrightarrow |f| \in \Lambda_\omega(G) \Longleftrightarrow |f| \in \Lambda_\omega(G, \partial G),$$

*where $\Lambda_\omega(G, \partial G)$ denotes the class of continuous functions $f$ on $G \cup \partial G$ which satisfy* (1.3) *with some positive constant $C$, whenever $z \in G$ and $w \in \partial G$.*

For any $z_1, z_2 \in G \subset \mathbb{C}$, let

$$d_{\omega,G}(z_1, z_2) := \inf \int_\gamma \frac{\omega(d(z, \partial G))}{d(z, \partial G)}\, ds(z),$$

where the infimum is taken over all rectifiable curves $\gamma \subset G$ joining $z_1$ to $z_2$. We say that $f \in \Lambda_{\omega,\inf}(G)$ whenever for any $z_1, z_2 \in G$,

$$|f(z_1) - f(z_2)| \leq C d_{\omega,G}(z_1, z_2),$$

where $C$ is a positive constant which depends only on $f$ (see [44]).

**Theorem 12.** ([21, Theorem 2]) *Let $\omega$ be a majorant satisfying* (2.3)*, and let $G$ be a domain in $\mathbb{C}$. If $f$ is a planar $K$-quasiregular harmonic mapping in $G$, then*

$$f \in \Lambda_{\omega,\inf}(G) \Longleftrightarrow |f| \in \Lambda_{\omega,\inf}(G).$$

**2.2. Harmonic Hardy space.** Many authors discussed the relationships between Hardy classes of holomorphic functions and integral means (see [43, 62]). In order to derive an analogous result of [43, Theorem 1] for the setting of harmonic mappings, we need to introduce some notation.

For a harmonic function $f$ in $\mathbb{D}$, $p > 0$ and $0 \leq r < 1$, we define

$$I_p(r, f) = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{1/p}$$

and say that $f$ belongs to the *harmonic Hardy class* $\mathcal{H}_h^p$ if

$$\|f\|_p = \sup_{0<r<1} I_p(r, f) < \infty.$$

For a harmonic mapping $f$ in $\mathbb{D}$, the *generalized harmonic area function* $A_h(r, f)$ is defined by

$$A_h(r, f) = \int_{\mathbb{D}_r} |\nabla f(z)|^2 \, dA(z),$$

where $dA$ denotes the normalized Lebesgue measure on $\mathbb{D}$ and

$$|\nabla f| = (|f_z|^2 + |f_{\overline{z}}|^2)^{1/2}, \;\; \nabla f = (f_z, f_{\overline{z}}) \;\; \text{and} \;\; I_\infty(r, f) = \max_{|z|=r} |f(z)|.$$

The following theorem is an analogous result of [43, Theorem 1].

**Theorem 13.** ([21, Theorem 3]) *Let $f$ be harmonic in $\mathbb{D}$ and $\delta > 0$. Then for $1 < p \leq 2$,*

$$f \in \mathcal{H}_h^p(\mathbb{D}) \Rightarrow \int_0^1 A_h^{\frac{p}{2}}(r, f)(1 - r)^{\frac{\delta(2-p)}{2}} \, dr < +\infty,$$

*while for $p > 2$,*

$$\int_0^1 A_h^{\frac{p}{2}}(r, f)(1 - r)^{\frac{\delta(2-p)}{2}} \, dr < +\infty \Rightarrow f \in \mathcal{H}_h^p(\mathbb{D}).$$

**Theorem 14.** ([21, Theorem 4]) *Let $f \in \mathcal{H}_h^p(\mathbb{D})$ and $\delta > 0$. If $1 < p \leq 2$, then*

$$\lim_{r \to 1-} (1 - r)^{\frac{\delta(2-p)+2}{p}} A_h(r, f) = 0.$$

In [21], the following version of Landau's theorem for a class of harmonic Hardy mappings is also established.

**Theorem 15.** ([21, Theorem 5]) *Let $f$ be a harmonic in $\mathbb{D}$ with $\|f\|_p \leq M$ and $f(0) = \lambda_f(0) - 1 = 0$, where $M$ is a positive constant, $\lambda_f(z) = \big||f_z(z)| - |f_{\overline{z}}(z)|\big|$ and $p \geq 1$. Then $f$ is univalent in $\mathbb{D}_{\rho_0}$, where*

$$\rho_0 = \varphi(r_0) = \max_{0<r<1} \varphi(r), \quad \varphi(r) = r \left( 1 - \sqrt{\frac{t}{1+t}} \right),$$

*with*

$$t=\frac{4}{\pi}\cdot\frac{2^{\frac{1}{p}}M}{r(1-r)^{\frac{1}{p}}}.$$

*Moreover,* $f(\mathbb{D}_{\rho_0})$ *contains a univalent disk* $\mathbb{D}_{R_0}$ *with*

$$R_0=\frac{r_0\varphi(r_0)}{2r_0-\varphi(r_0)}.$$

In Theorem 15, we remark that $\max_{0<r<1}\varphi(r)$ does exist, since

$$\lim_{r\to 0+}\varphi(r)=\lim_{r\to 1-}\varphi(r)=0.$$

**2.3. Integral mean problem of Girela and Peláez.** A classical result of Hardy and Littlewood asserts that if $p\in(0,\infty]$, $\alpha\in(1,\infty)$ and $f$ is an analytic function in $\mathbb{D}$, then

$$I_p(r,f')=O\left(\frac{1}{1-r}\right)^{\alpha}\quad\text{as } r\to 1$$

if and only if

$$I_p(r,f)=O\left(\log\frac{1}{1-r}\right)^{\alpha-1}\quad\text{as } r\to 1,$$

where

$$I_p(r,f)=\left(\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p\,d\theta\right)^{1/p},\ r\in(0,1).$$

We refer to [29, 32, 33, 38, 40, 41, 43, 62] for many other related discussions concerning the case of analytic functions. Indeed the above result of Hardy and Littlewood provides a close relationship between the integral means of analytic functions and those of its derivative [32, 40, 41]). In [38, Theorem 1(a) ], Girela and Peláez refined the above result for the case $\alpha=1$ as follows.

**Theorem 16.** *If* $p\in(0,\infty)$ *and* $f$ *is an analytic function in* $\mathbb{D}$ *such that*

$$I_p(r,f')=O\left(\frac{1}{1-r}\right)\quad\text{as } r\to 1$$

*then*

$$I_p(r,f)=O\left(\log\frac{1}{1-r}\right)^{\beta}\quad\text{as } r\to 1,\ \text{for all } \beta>\tfrac{1}{2}. \tag{2.5}$$

In [38, p.464, Equation (26)], Girela and Peláez asked whether or not $\beta$ in (2.5) can be substituted by $1/2$. In [23], the present authors settle this problem affirmatively for a more general class of harmonic mappings in the unit disk. In addition, we also obtain several analogous results for harmonic mappings (see also [14]).

Later in [23], the authors generalized Theorem 16. In the case of analytic function $f$, $\nabla f$ equals $f'(z)$ and therefore, the following theorem also contains a solution to the open problem of Girela and Peláez [38, p.464, Equation (26)].

**Theorem 17.** ([23, Theorem 1]) *If $p \in (2, \infty)$ and $f$ is harmonic function in $\mathbb{D}$ such that*

$$I_p(r, \nabla f) = O\left(\frac{1}{1-r}\right) \quad \text{as } r \to 1,$$

*then*

$$I_p(r, f) = O\left(\log \frac{1}{1-r}\right)^{1/2} \quad \text{as } r \to 1.$$

**Theorem 18.** ([23, Theorem 2]) *If $p \in (2, \infty)$ and $f$ is harmonic function in $\mathbb{D}$ such that*

$$I_p(r, \nabla f) = O\left(\frac{1}{1-r}\right) \quad \text{as } r \to 1, \tag{2.6}$$

*then*

$$I_\infty(r, f) = O\left(\log \frac{1}{1-r}\right)^{1/p} \quad \text{as } r \to 1.$$

The following result is a generalization of [38, Theorem 2 ] for the harmonic case.

**Theorem 19.** ([23, Theorem 3]) *If $p \in (0, 2]$ and $f$ is a harmonic function in $\mathbb{D}$ satisfying the condition* (2.6), *then*

$$I_p(r, f) = O\left(\log \frac{1}{1-r}\right)^{1/p} \quad \text{as } r \to 1.$$

*Moreover, this result is sharp and one of the extremal functions is $f(z) = 1/(1-z)^{1/p}$, where $p \in (0, 2]$.*

Next two theorems provide coefficient estimates and a distortion theorem for harmonic Hardy mappings.

**Theorem 20.** ([23, Theorem 4]) *Let $f$ be a harmonic in $\mathbb{D}$ such that*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \overline{z}^n,$$

*and* $\|f\|_p < M$ *for some* $p \in [1, \infty)$ *and* $M > 0$. *Then* $|a_0| \leq \|f\|_p$ *and for any* $n \geq 1$,

$$|a_n| + |b_n| \leq \frac{2^{(1/p)+2} M(1 + np)^{n+(1/p)}}{\pi(pn)^n}.$$

*In particular,*

$$|a_n| + |b_n| \leq \frac{4M}{\pi} \tag{2.7}$$

*as* $p \to \infty$. *For* $p = \infty$, *the estimates of* (2.7) *is sharp and the extremal functions are*

$$f_n(z) = \frac{2M\alpha}{\pi} \arg\left(\frac{1 + \beta z^n}{1 - \beta z^n}\right),$$

*where* $|\alpha| = |\beta| = 1$.

**Theorem 21.** ([23, Theorem 5]) *Let* $f$ *be a harmonic in* $\mathbb{D}$ *with* $\|f\|_p < M$ *for some* $p \in [1, \infty)$ *and* $M > 0$. *Then, for any* $n \geq 1$,

$$|f_z(z)| + |f_{\overline{z}}(z)| \leq \frac{2^{(1/p)+2} M(1 + p)^{1+(1/p)}}{\pi p(1 - |z|^2)}.$$

*In particular,*

$$|f_z(z)| + |f_{\overline{z}}(z)| \leq \frac{4M}{\pi(1 - |z|^2)} \tag{2.8}$$

*as* $p \to \infty$. *For* $p = \infty$, *the estimates of* (2.8) *is sharp and the extremal functions are*

$$f(z) = \frac{2M\alpha}{\pi} \arg\left(\frac{1 + \phi(z)}{1 - \phi(z)}\right),$$

*where* $|\alpha| = 1$ *and* $\phi$ *is a conformal automorphism of* $\mathbb{D}$.

We remark that Theorem 21 is a generalization of [30, Theorems 3 and 4].

## 3. Real harmonic functions

Let $\mathbb{R}^n = \{x = (x_1, \ldots, x_n) : x_1, \ldots, x_n \in \mathbb{R}\}$ denote the real vector space of dimension $n$. It is often convenient to identify each point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ with an $n \times 1$ column matrix so that

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

For $a = (a_1, \ldots, a_n),\ x \in \mathbb{R}^n$, we define the Euclidean inner product $\langle \cdot, \cdot \rangle$ by

$$\langle x, a \rangle = x_1 a_1 + \cdots + x_n a_n$$

so that the Euclidean length in $\mathbb{R}^n$ is defined by

$$|x| = \langle x, x\rangle^{1/2} = (|x_1|^2 + \cdots + |x_n|^2)^{1/2}.$$

Denote a ball in $\mathbb{R}^n$ with center $x'$ and radius $r$ by

$$\mathbb{B}^n(x', r) = \{x \in \mathbb{R}^n : |x - x'| < r\}.$$

In particular, $\mathbb{B}^n$ denotes the unit ball $\mathbb{B}^n(0, 1)$. Set $\mathbb{B}^2 = \mathbb{D}$, the open unit disk in the complex plane $\mathbb{C}$.

As in the plane case, a function $f$ of an open subset $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}$ is called *harmonic* if $\Delta f = 0$, where $\Delta$ denotes also for the $n$-dimensional Laplacian operator

$$\Delta = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}.$$

**3.1. Schwarz lemma for real harmonic functions.** Recently, Knezević and Mateljević [55] proved that if $f$ is a $K$-quasiconformal mapping of $\mathbb{D}$ into itself, then

$$|f_z(z)| + |f_{\overline{z}}(z)| \leq K \frac{1 - |f(z)|^2}{1 - |z|^2}.$$

Kalaj and Vuorinen [45] proved that if $u$ is a real harmonic function of $\mathbb{D}$ into $(-1, 1)$, then

$$|\nabla u(z)| \leq \frac{4}{\pi} \frac{1 - u^2(z)}{1 - |z|^2}, \ z \in \mathbb{D},$$

where $\nabla u = (u_x, u_y)$ denotes the gradient of $u$. There are also many other authors discussed Schwarz's lemma for harmonic functions (resp. harmonic mappings), see for examples [9, 16, 17, 18, 27, 45, 55, 64].

The classical Schwarz-Pick lemma tell us that if $f$ is an analytic function from $\mathbb{D}$ into itself, then

$$\rho(f(z), f(w)) \leq \rho(z, w) \quad \text{for } z, w \in \mathbb{D},$$

where $\rho(z, w)$ denotes the hyperbolic distance between $z$ and $w$ in $\mathbb{D}$ given by

$$\rho(z, w) = \frac{1}{2} \log \left( \frac{1 + |\frac{z-w}{1-\overline{z}w}|}{1 - |\frac{z-w}{1-\overline{z}w}|} \right) = \operatorname{arctanh} \left| \frac{z - w}{1 - \overline{z}w} \right|.$$

For an analytic function $f$ on $\mathbb{D}$ into itself, Beardon and Minda [4] defined

$$f^*(z, w) = \frac{[f(z), f(w)]}{[z, w]}$$

as the hyperbolic difference quotient, where

$$[f(z), f(w)] = \frac{f(z) - f(w)}{1 - f(z)\overline{f(w)}} \text{ and } [z, w] = \frac{z - w}{1 - z\overline{w}},$$

$z, w \in \mathbb{D}$. They also investigated some properties of hyperbolic derivative of $f$. In [14], the authors discussed the hyperbolic difference quotient for real harmonic functions. It may now be appropriate to call next result as a Schwarz-Pick lemma for real harmonic functions.

**Theorem 22.** ([14, Theorem 1]) *Let $u$ be a real harmonic function of $\mathbb{D}$ into $(-1, 1)$. Then*

$$\mathcal{L}_h(u) = \sup_{z \neq w} \left[\frac{\rho(u(z), u(w))}{\rho(z, w)}\right] = \sup_{z \in \mathbb{D}} \left[\frac{(1 - |z|^2)|\nabla u(z)|}{1 - u^2(z)}\right] \leq \frac{4}{\pi}.$$

*Moreover, $\mathcal{L}_h(u) = 4/\pi$ if and only if*

$$u(z) = \frac{2}{\pi} \operatorname{Im}\left(\frac{1 + \phi(z)}{1 - \phi(z)}\right),$$

*where $\phi$ is an automorphism of the unit disk $\mathbb{D}$.*

The following result is a generalization of [45, Theorem 1.12].

**Theorem 23.** ([14, Theorem 2]) *Suppose that $u : D \to (-1, 1)$ is real harmonic on planar hyperbolic domain $D$. Then*

$$\rho_{\mathbb{D}}(u(z), u(w) \leq \frac{4}{\pi} \rho_D(z, w), \ \ z, w \in D.$$

Let $\sigma(z, w) = |[z, w]|$ denote the pseudo-hyperbolic distance between $z$ and $w$ in the plane hyperbolic domain $D$.

**Theorem 24.** ([14, Theorem 3]) *Let $u$ be a real harmonic function of a planar hyperbolic domain $D$ into $(-1, 1)$. Then, for $z, w \in D$,*

$$\sigma_{\mathbb{D}}\big(u(z), u(w)\big) \leq \frac{4}{\pi} \arctan(\sigma_D(z, w)) \leq \frac{4}{\pi} \sigma_D(z, w).$$

The following result is an analogy of [37, Theorem 1].

**Theorem 25.** ([14, Theorem 4]) *Let $u$ be a real harmonic function of $\mathbb{D}$ into $\mathbb{R}$. If*

$$B_u = \sup_{z \neq w} \left[\frac{|u(z) - u(w)|}{\rho(z, w)}\right] < \infty,$$

*then, for $z, w \in \mathbb{D}$,*

$$\big|(1 - |z|^2)|\nabla u(z)| - (1 - |w|^2)|\nabla u(w)|\big| \leq \frac{3\sqrt{3}}{2} B_u \sigma(z, w).$$

We remark that the constant $3\sqrt{3}/2$ in Theorem 25 is better than the constant 3.31 in [37, Theorem 1].

Next, we generalize [42, Lemma] into the following form.

**Theorem 26.** ([14, Theorem 5]) *Let $f:\mathbb{D}\to\mathbb{B}^n\subset\mathbb{C}^n$ be a harmonic mapping with $f(0)=0$. Then*

$$|f(z)|\le\frac{4}{\pi}\arctan|z|$$

*and this inequality is sharp for each point $z\in\mathbb{D}$.*

The following two results are the generalization of [63, Theorems A and B].

**Theorem 27.** ([14, Theorem 6]) *Let $\omega$ be a majorant satisfying* (2.3)*, and let $G\subset\mathbb{R}^n$ be a $\Lambda_\omega$-extension in $\mathbb{R}^n$. If $f$ is a real harmonic function of $G$ into $\mathbb{R}$ and continuous up to the boundary $\partial G$, then*

$$f\in\Lambda_\omega(G)\Longleftrightarrow|f|\in\Lambda_\omega(G)\Longleftrightarrow|f|\in\Lambda_\omega(G,\partial G),$$

*where $\Lambda_\omega(G,\partial G)$ denotes the class of continuous functions $f$ on $G\cup\partial G$ which satisfy* (2.2) *with some positive constant $C$, whenever $z\in G$ and $w\in\partial G$.*

For any $x_1,x_2\in G\subset\mathbb{R}^n$, let

$$d_{\omega,G}(x_1,x_2):=\inf\int_\gamma\frac{\omega(d(x,\partial G))}{d(x,\partial G)}\,ds(z),$$

where the infimum is taken over all rectifiable curves $\gamma\subset G$ joining $x_1$ to $x_2$. We say that $f\in\Lambda_{\omega,\inf}(G)$ whenever for any $x_1,x_2\in G$,

$$|f(x_1)-f(x_2)|\le Cd_{\omega,G}(x_1,x_2),$$

where $C$ is a positive constant which depends only on $f$ (see [44]).

**Theorem 28.** ([14, Theorem 7]) *Let $\omega$ be a majorant satisfying* (2.3)*. If $f$ is a real harmonic function of $G\subset\mathbb{B}^n$ into $\mathbb{R}$, then*

$$f\in\Lambda_{\omega,\inf}(G)\Longleftrightarrow|f|\in\Lambda_{\omega,\inf}(G).$$

**3.2. Generalization of a result of Girela and Peláez.** Let $f=(f_1,\dots,f_n)$ be a *vector-valued and real harmonic function* from $\mathbb{B}^n$ into $\mathbb{R}^n$, that's, for each $i\in\{1,2,\dots,n\}$, $f_i$ is a harmonic function from $\mathbb{B}^n$ into $\mathbb{R}$. We denote the Jacobian of $f$ by $J_f$, where

$$J_f=\left(\frac{\partial f_i}{\partial x_j}\right)_{n\times n}.$$

Let $\mathcal{H}(\mathbb{B}^n,\mathbb{R}^n)$ be the set of all real harmonic functions of $f$ from $\mathbb{B}^n$ into $\mathbb{R}^n$. Also, let $\mathcal{H}^p(\mathbb{B}^n,\mathbb{R}^n)$ denote the harmonic Hardy class consisting of all functions $f\in\mathcal{H}(\mathbb{B}^n,\mathbb{R}^n)$ such that

$$\|f\|_p=\sup_{0<r<1}M_p(f,r)<\infty,$$

where $p\in(0,\infty)$, $M_p^p(f,r)=\int_{\partial\mathbb{B}^n}|f(r\zeta)|^p d\sigma(\zeta)$ and $d\sigma$ is the normalized surface measure on $\partial\mathbb{B}^n$.

The following result is a generalization of Theorem 17.

**Theorem 29.** *If $p\in(2,\infty)$ and $f\in\mathcal{H}(\mathbb{B}^n,\mathbb{R})$ $(n\geq 3)$ such that*

$$M_p(r,\nabla f)=O\left(\frac{1}{1-r}\right)\quad as\ r\to 1,$$

*then*

$$M_p(r,f)=O\left(\left(\log\frac{1}{1-r}\right)^{1/2}\right)\quad as\ r\to 1.$$

**3.3. Landau-Bloch's theorems for holomorphic mappings.** For general holomorphic mappings of more than one complex variable, no Landau-Bloch constant exists (cf. [73]). In order to obtain some analogs of Landau-Bloch's theorem for mappings with several complex variables, it is necessary to restrict the class of mappings considered, see [20, 25, 26, 51, 72, 73].

By using the Schwarz's lemma and distortion theorems, we can obtain a schlicht Bloch constant for vector-valued harmonic functions in the Hardy space.

**Theorem 30.** ([14, Theorem 5]) *Suppose that $f\in\mathcal{H}^p(\mathbb{B}^n,\mathbb{R}^n)$ satisfies $\|f\|_p\leq K_0$ for some constant $K_0>0$ and $J_f(0)-1=|f(0)|=0$, where $p\geq 1$. Then $f(\mathbb{B}^n)$ contains a univalent ball $\mathbb{B}^n(0,R)$, where*

$$R\ \geq\ \max_{0<r<1}\varphi(r)\geq\max_{0<r<1}\left\{\frac{r}{2(nK(r))^{n-1}\sqrt{2[1+m^2(nK(r))^{2(n-1)}]}}\right\},$$

$$\varphi(r)=r\rho(r)\left[\frac{1}{(nK(r))^{n-1}}-m(\sqrt{2}/2-\sqrt{1/2-\rho^2(r)})\right]$$

*with*

$$\rho(r)=1/\sqrt{2[1+m^2(nK(r))^{2(n-1)}]},\ K(r)=2^{1/p}K_0/[r(1-r)^{(n-1)/p}]$$

*and*

$$m=\frac{4M}{\pi}[(3+\sqrt{2})n+2\sqrt{2}]n.$$

We remark that, as

$$\lim_{r\to 0+}\varphi(r)=\lim_{r\to 1-}\varphi(r)=0,$$

the maximum of $\varphi(r)$ in Theorem 30 does exist.

## 4. Some properties of planar $p$-harmonic mappings

A $2p$ times continuously differentiable complex-valued function $f=u+iv$ in a domain $D\subseteq\mathbb{C}$ is *$p$-harmonic* if $f$ satisfies the $p$-harmonic equation $\underbrace{\Delta\cdots\Delta}_{p} f=0$, where $p$ is a positive integer and $\Delta$ represents the complex Laplacian operator defined in the introduction.

It is known that a mapping $f$ is $p$-harmonic in a simple connected domain $D$ if and only if $f$ has the following representation:

$$f(z)=\sum_{k=1}^{p}|z|^{2(k-1)}G_{p-k+1}(z), \tag{4.1}$$

where $\Delta G_{p-k+1}(z)=0$ in $D$ for each $k\in\{1,\ \ldots,\ p\}$ (cf. [15, Proposition 2.1]).

Obviously, when $p=1$ (resp. 2), $f$ is harmonic (resp. biharmonic). Throughout the section we consider $p$-harmonic mappings in the unit disk $\mathbb{D}$.

**Definition 3.** *We say that a univalent $p$-harmonic mapping $f$ with $f(0)=0$ is starlike with respect to the origin if the curve $f(re^{it})$ is starlike with respect to the origin for each $r\in(0,1)$.*

**Proposition 1.** *If $f$ is univalent, $f(0)=0$ and $\frac{d}{dt}(\arg f(re^{it}))>0$ for $z\neq 0$, then $f$ is starlike with respect to the origin (cf. [65]).*

**Definition 4.** *We say that a univalent $p$-harmonic mapping $f$ with $f(0)=0$ and $\frac{\partial f(re^{it})}{\partial t}\neq 0$ whenever $0<r<1$, is said to be convex if the curve $f(re^{it})$ is convex for each $r\in(0,1)$.*

**Proposition 2.** *If $f$ is univalent, $f(0)=0$ and $\frac{d}{dt}(\arg\frac{d}{dt}f(re^{it}))>0$ for $z\neq 0$, then $f$ is convex (cf. [65]).*

Our results in this direction are Theorems 31 and 32.

**Theorem 31.** ([24, Theorem 3.1]) *Let $f$ be a univalent $p$-harmonic mapping of $\mathbb{D}$ with the form*

$$f(z)=\sum_{k=1}^{p}\lambda_k|z|^{2(k-1)}G(z),$$

*where $G$ is a locally univalent harmonic mapping and $\lambda_k$ $(k = 1, 2, \dots, p)$ are complex constants. Then $f$ is convex (resp. starlike) if and only if $G$ is convex (resp. starlike).*

**Theorem 32.** ([24, Theorem 3.2]) *Let $f$ be a $p$-harmonic mapping of $\mathbb{D}$ satisfying $f(z) = |z|^{2(p-1)}G(z)$, where $G$ is harmonic, orientation preserving and starlike. Then $f$ is starlike univalent.*

### 4.1. Regions of variabilities of certain class of $p$-harmonic mappings.

Recently, Yanagihara [75, 76], and Ponnusamy and Vasudevarao [66, 67] have discussed the regions of variability for certain classes of univalent analytic functions in $\mathbb{D}$. Then Chen and Huang [13] generalized the corresponding results of [66, 76] to general cases.

**Definition 5.** *Let $\mathcal{H}_p$ denote the set of all $p$-harmonic mappings of $\mathbb{D}$ with the normalization $f_{z^{p-1}}(0) = (p-1)!$ and $|f| \leq 1$. Here we prescribe that $\mathcal{H}_0 = \emptyset$. For a fixed point $z_0 \in \mathbb{D}$, we also let*

$$V_p(z_0) = \{f(z_0) : f \in \mathcal{H}_p \setminus \mathcal{H}_{p-1}\}.$$

Then we have

**Theorem 33.** ([24, Theorem 4.1])

(1) *If $p = 1$, then $V_1(z_0) = \{1\}$;*
(2) *If $p \geq 2$, $V_p(z_0) = \overline{\mathbb{D}}$.*

By the same proof of (1) in Theorem 33, we obtain the following generalization of Cartan's uniqueness Theorem (see [7] or [69, page 23]) for harmonic mappings.

**Theorem 34.** ([24, Theorem 4.2]) *Let $f$ be a harmonic mapping in $\mathbb{D}$ with $f(\mathbb{D}) \subseteq \mathbb{D}$ and $f_z(0) = 1$. Then $f(z) = z$ in $\mathbb{D}$.*

**Lemma 2.** ([49, Lemma 2.1]) *Suppose that $f(z) = h(z) + \overline{g(z)}$ is a harmonic mapping of $\mathbb{D}$ with $h(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ for $z \in \mathbb{D}$. If $J_f(0) = 1$ and $|f(z)| < M$, then*

$$|a_n|,\ |b_n| \leq \sqrt{M^2 - 1},\ n = 2, 3, \dots,$$

$$|a_n| + |b_n| \leq \sqrt{2M^2 - 2},\ n = 2, 3, \dots$$

*and $\lambda_f(0) \geq \lambda_0(M)$, where*

$$\lambda_0(M) = \begin{cases} \dfrac{\sqrt{2}}{\sqrt{M^2-1} + \sqrt{M^2+1}} & \text{if } 1 \leq M \leq \dfrac{\pi}{2\sqrt[4]{2\pi^2 - 16}} \\ \dfrac{\pi}{4M} & \text{if } \quad M > \dfrac{\pi}{2\sqrt[4]{2\pi^2 - 16}} \end{cases}.$$

**4.2. Landau's theorems for $p$-harmonic mappings.** By using Theorem 1 and Lemma 2, the authors obtained the following results.

**Theorem 35.** ([24, Theorem 5.1]) *Let $f(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z)$ be $p$-harmonic mapping of $\mathbb{D}$ satisfying $\Delta G_{p-k+1}(z) = f(0) = G_p(0) = J_f(0) - 1 = 0$ and for any $z \in \mathbb{D}$, $|G_{p-k+1}(z)| \leq M$, where $M \geq 1$. Then there is a constant $\rho$ $(0 < \rho < 1)$ such that $L(f)$ is univalent in $\mathbb{D}_\rho$, where $\rho$ satisfies the following equation:*

$$\lambda_0(M) - \frac{T(M)}{(1-\rho)^2} \sum_{k=2}^{p} (2k-1)\rho^{2(k-1)} - \sum_{k=1}^{p} \frac{2T(M)\rho^{2k-1}}{(1-\rho)^3} - \frac{16M}{\pi^2} s_0 \arctan \rho = 0$$

*with $s_0 \approx 4.2$, $\lambda_0(M)$ as above, and*

$$T(M) = \begin{cases} \sqrt{2M^2 - 2} & \text{if } 1 \leq M \leq \dfrac{\pi}{\sqrt{\pi^2 - 8}} \\ \dfrac{4M}{\pi} & \text{if } \quad M > \dfrac{\pi}{\sqrt{\pi^2 - 8}} \end{cases}.$$

*Moreover, the range $L(f)(\mathbb{D}_\rho)$ contains a univalent disk $\mathbb{D}_{R_0}$, where*

$$R_0 = \rho \left[ \lambda_0(M) - \sum_{k=2}^{p} \frac{T(M)\rho^{2(k-1)}}{(1-\rho)^2} - \frac{16M}{\pi^2} s_0 \arctan \rho \right].$$

**Theorem 36.** ([24, Theorem 5.7]) *Let $f(z) = |z|^{2(p-1)} G(z)$ be a $p$-harmonic mapping of $\mathbb{D}$ satisfying $G(0) = J_G(0) - 1 = 0$ and $|G| \leq M$, where $M \geq 1$ and $G$ is harmonic. Then there is a constant $\rho$ $(0 < \rho < 1)$ such that $L(f)$ is univalent in $\mathbb{D}_\rho$, where $\rho$ satisfies the following equation:*

$$\lambda_0(M) - \frac{48M}{\pi^2} s_0 \arctan \rho - \frac{2T(M)\rho}{(1-\rho)^3} = 0.$$

*Moreover, the range $L(f)(\mathbb{D}_\rho)$ contains a univalent disk $\mathbb{D}_{R_0}$, where*

$$R_0 = \rho^{2p-1} \left[ \lambda_0(M) - \frac{16M}{\pi^2} s_0 \arctan \rho \right],$$

*$s_0$, $\lambda_0(M)$ and $T(M)$ are the same as in Theorem 35. Especially, if $M = 1$, then $f(z) = |z|^{2(p-1)} z$ is univalent.*

## 5. Landau's theorem for $p$-harmonic mappings in $\mathbb{C}^n$

As usual, $C(X)$ denotes the set of all continuous functions $f : X \longrightarrow \mathbb{C}$, where $X$ is a topological space.

**Definition 6.** *Suppose $\Omega$ is an open domain of $\mathbb{C}^n$. A function $f = (f_1, \ldots, f_m) : \Omega \to \mathbb{C}^m$ is said to be $p$-harmonic in $\Omega$ provided*

(a) *for each* $i \in \{1, \ldots, m\}$, $f_i \in C(\Omega)$ *and*
(b) *each component* $f_i$ *of* $f$ *is* $p$*-harmonic with respect to each variable separately.*

In particular, when $p = 1$ (resp. 2), $f$ is called *harmonic* (resp. *biharmonic*).

By Definition 6 and (4.1), we have

**Proposition 3.** $f : \mathbb{B}^n \to \mathbb{C}^m$ *is* $p$*-harmonic if and only if* $f$ *has the following representation:*

$$f(z) = f(z_1, \ldots, z_n) = \sum_{k_1, \ldots, k_n = 1}^{p} |z_1|^{2(k_1-1)} \cdots |z_n|^{2(k_n-1)} G_{p-k_1+1, \ldots, p-k_n+1}(z),$$

*where all* $G_{p-k_1+1, \ldots, p-k_n+1} : \mathbb{B}^n \to \mathbb{C}^m$ *are harmonic for* $k_1, \ldots, k_n \in \{1, \ldots, p\}$*.*

Let $\mathbb{C}^n = \{z = (z_1, \ldots, z_n) : z_1, \ldots, z_n \in \mathbb{C}\}$ the complex space of dimension $n$ and $\overline{z}$ the conjugate of $z$, that is, $\overline{z} = (\overline{z}_1, \ldots, \overline{z}_n)$. We define its length by

$$|z| = \langle z, z \rangle^{\frac{1}{2}} = (|z_1|^2 + \cdots + |z_n|^2)^{\frac{1}{2}}.$$

Denote a ball in $\mathbb{C}^n$ with center $z'$ and radius $r$ by

$$\mathbb{B}^n(z', r) = \{z \in \mathbb{C}^n : |z - z'| < r\}.$$

In particular, $\mathbb{B}^n$ denotes the unit ball $\mathbb{B}^n(0, 1)$. Obviously, $\mathbb{B}^1 = \mathbb{D}$, the open unit disk.

For a mapping $f = (f_1, \ldots, f_m)$ of a domain in $\mathbb{C}^n$ into $\mathbb{C}^m$, we denote by $\partial f/\partial z_k$ the column vector formed by $\partial f_1/\partial z_k, \ldots, \partial f_m/\partial z_k$ and by

$$f_z = \left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right)$$

the matrix formed by these column vectors. Let

$$f_{\overline{z}} = \left(\frac{\partial f}{\partial \overline{z}_1}, \ldots, \frac{\partial f}{\partial \overline{z}_n}\right).$$

For an $n \times n$ matrix $A$, we introduce the operator norm

$$|A| = \sup_{x \neq 0} \frac{|Ax|}{|x|} = \max\{|A\theta| : \theta \in \partial\mathbb{B}^n\}.$$

The authors established the following two lemmas to prove the next main result.

**Lemma 3.** ([22, Lemma 2.1]) *Let* $A = \big(a_{i,j}(z)\big)_{n\times n}$ *be a matrix-valued harmonic mapping of* $\mathbb{B}^n(0,r)$ *into the space of all* $n\times n$ *complex matrices, that's, each* $a_{i,j}(z)$ *is a harmonic mapping of* $\mathbb{B}^n(0,r)$ *into* $\mathbb{C}$. *If* $A(0)=0$ *and* $|A(z)|\le M$ *for* $z\in\mathbb{B}^n(0,r)$, *then*

$$|A(z)| \le \frac{4M}{\pi}\frac{|z|}{r}\left(1+\frac{2(n-1)r}{\sqrt{r^2-|z|^2}}\right).$$

**Lemma 4.** ([22, Lemma 2.2]) *Let* $f$ *be a harmonic mapping of* $\mathbb{B}^n$ *into* $\mathbb{C}^n$ *with* $|f(z)|\le M$ *for* $z\in\mathbb{B}^n$, *where* $M$ *is a positive constant. Then*

$$\max\big\{|f_z(z)|,\ |f_{\overline{z}}(z)|\big\} \le \frac{M[|z|+n(1+|z|)]}{1-|z|^2}.$$

By applying Lemmas 3 and 4, the authors obtained the following result.

**Theorem 37.** ([22, Theorem 2.1]) *Let*

$$f(z)=\sum_{k_1,\dots,k_n=1}^{p}|z_1|^{2(k_1-1)}\cdots|z_n|^{2(k_n-1)}G_{p-k_1+1,\dots,p-k_n+1}(z),$$

*where all* $G_{p-k_1+1,\dots,p-k_n+1}$ *are harmonic for* $k_1,\dots,k_n\in\{1,\dots,p\}$. *Suppose* $f(0)=0$, $|\det f_z(0)|-\alpha=|f_{\overline{z}}(0)|=0$ *and for any* $z\in\mathbb{B}^n$, $k_1,\dots,k_n\in\{1,\dots,p\}$,

$$|G_{p-k_1+1,\dots,p-k_n+1}(z)|<M,$$

*where* $\alpha$ *and* $M$ *are positive constants. Then there is a constant* $\rho_0\in(0,1)$ *such that* $f$ *is univalent in* $|z|<\rho_0$. *In specific* $\rho_0$ *satisfies*

$$\begin{aligned}0 \;=\;& \frac{\alpha}{(nM)^{n-1}}-\frac{4M[5n+2\sqrt{2}(n+1)](2n-1)\rho}{\pi\sqrt{\frac{1}{2}-\rho^2}}\\ &-2\sum_{(k_1,\dots,k_n)\neq(1,\dots,1)}^{p}\Big[\Big(\sum_{i=1}^{n}(k_i-1)^2\Big)^{\frac{1}{2}}\rho^{2(k_1+\cdots+k_n)-2n-1}M\\ &+\frac{[n+(n+1)\rho]M\rho^{2(k_1+\cdots+k_n)-2n}}{(1-\rho^2)}\Big].\end{aligned}$$

*and* $f(\mathbb{B}^n)$ *contains a univalent ball of radius at least* $R_0$, *where*

$$R_0 \;=\; \rho_0\left\{\frac{\alpha}{[(2n+1)M]^{n-1}}-\frac{2M[5n+2\sqrt{2}(n+1)](2n-1)\rho}{\pi\sqrt{\frac{1}{2}-\rho^2}}\right.$$

$$-\sum_{(k_1,\ldots,k_n)\neq(1,\cdots,1)}^{p}\Big[\frac{\big(\sum_{i=1}^{n}(k_i-1)^2\big)^{\frac{1}{2}}\rho_0^{2(k_1+\ldots+k_n)-2n-1}M}{k_1+\cdots+k_n-n}$$
$$+\frac{2[n+(n+1)\rho_0]M\rho_0^{2(k_1+\cdots+k_n)-2n}}{(1-\rho_0^2)[2(k_1+\cdots+k_n)-2n+1]}\Big]\Bigg\}.$$

**5.1. Pluriharmonic mappings.** A continuous complex-valued function $f$ defined on a domain $\Omega\subset\mathbb{C}^n$ is said to be *pluriharmonic* if for each fixed $z\in\Omega$ and $\theta\in\partial\mathbb{B}^n$, the function $f(z+\theta\zeta)$ is harmonic on the complex variable $\zeta$, for $|\zeta|$ smaller than the distance of $z$ from $\partial\Omega$ (cf. [70, 77]). Moreover, a mapping $f$ of $\mathbb{B}^n$ into $\mathbb{C}^n$ is pluriharmonic if and only if $f$ has a representation $f=h+\overline{g}$, where $g$ and $h$ are holomorphic mappings. Let $\mathcal{H}_n^n$ denote the set of all pluriharmonic mappings of $\mathbb{B}^n$ into $\mathbb{C}^n$. It is not difficult to know that $f\in\mathcal{H}_n^n$ is harmonic. Especially, functions in $\mathcal{H}_1^1$ are planar harmonic mappings.

Let $f:\Omega\to\mathbb{R}^n$ be a continuous mapping of domain $\Omega\subset\mathbb{R}^n$. Then $f$ is called quasiregular if $f\in W^1_{n,loc}(\Omega)$ and

$$|f'(x)|^n\leq KJ_f(x)\ \text{ for almost all }\ x\in\Omega,$$

where $K$ ($K\geq 1$) is a constant. Here $f\in W^1_{n,loc}(\Omega)$ means that the distributional derivatives $\partial f_j/\partial x_k$ of the coordinates $f_j$ of $f=(f_1,\ldots,f_n)$ are locally in $L^n$ and $J_f(x)$ denotes its determinant (cf. [68, 71]).

**Definition 7.** *A pluriharmonic mapping $f$ of $\mathbb{B}^n$ into $\mathbb{C}^n$ is said to be a $(K,K_1)$-quasiregular pluriharmonic mapping if for any $z\in\mathbb{B}^n$ and $\theta\in\partial\mathbb{B}^n$,*

$$|f_z(z)|^n\leq K|\det f_z(z)|\ \textit{and}\ K_1|f_{\overline{z}}(z)\overline{\theta}|\leq|f_z(z)\theta|,$$

*where $K\geq 1$ and $K_1>1$ are constant.*

Obviously, a $(K,K_1)$-quasiregular pluriharmonic mapping $f$ of $\mathbb{B}^n$ into $\mathbb{C}^n$ is $K$-quasiregular mapping if $f_{\overline{z}}\equiv 0$ (see [73]).

The following is a Landau's Theorem for a class of $(K,K_1)$-quasiregular pluriharmonic mappings.

**Lemma 5.** ([22, Lemma 3.2]) *Let $f\in\mathcal{H}_n^n$ with $|f(z)|<M$ for $z\in\mathbb{B}^n$. Then*

$$\max\big\{|f_z(z)|,\ |f_{\overline{z}}(z)|\big\}\leq\frac{4M}{\pi(1-|z|^2)}$$

*and*

$$\max\big\{|\det f_z(z)|,\ |\det f_{\overline{z}}(z)|\big\}\leq\frac{(4M)^n}{\pi^n(1-|z|^2)^{\frac{n+1}{2}}}.$$

**Lemma 6.** ([22, Lemma 3.3]) *Let* $A = \big(a_{i,j}(z)\big)_{n\times n}$ *be a matrix-valued holomorphic mapping of* $\mathbb{B}^n(0,r)$ *into the space of all* $n\times n$ *complex matrices, that's, each* $a_{i,j}(z)$ *is a holomorphic mapping of* $\mathbb{B}^n(0,r)$ *into* $\mathbb{C}$. *If* $A(0)=0$ *and* $|A(z)|\leq M$ *for* $z\in\mathbb{B}^n(0,r)$, *then*

$$|A(z)|\leq \frac{M}{r}|z|.$$

By using Lemmas 5 and 6, we have

**Theorem 38.** ([25, Theorem 3]) *Suppose* $f$ *is a* $(K,K_1)$*-quasiregular pluriharmonic mapping of* $\mathbb{B}^n$ *into* $\mathbb{C}^n$ *with* $|\det f_z(0)| = 1$. *Then* $f(\mathbb{B}^n)$ *contains a schlicht ball with radius at least*

$$R_0 = \frac{(1-\frac{1}{K_1})^2\pi^{2n-1}}{2^{4n-1}m(M^*)^{2n-1}}\max_{0<t<1}\varphi(t) \geq \frac{(1-\frac{1}{K_1})^2\pi^{2n-1}}{2^{4n-1}m(M^*)^{2n-1}}\varphi\Big(\frac{1}{2}\Big),$$

*where*

$$\varphi(t)=\frac{t^{2n}}{[-\log(1-t)]^{2n-1}},\ M^*=\frac{K^{\frac{1}{n}}(1+K_1)}{K_1}\quad \textit{and}\quad m\approx 4.199556.$$

Here $\max_{0<t<1}\varphi(t)$ does exist, since

$$\lim_{t\to 0+}\varphi(t)=\lim_{t\to 1-}\varphi(t)=0.$$

**Theorem 39.** ([22, Theorem 3.1]) *Let*

$$f(z)=\sum_{k_1,\ldots,k_n=1}^{p}|z_1|^{2(k_1-1)}\cdots|z_n|^{2(k_n-1)}G_{p-k_1+1,\ldots,p-k_n+1}(z),$$

*where all* $G_{p-k_1+1,\ldots,p-k_n+1}\in\mathcal{H}_n^n$ *for* $k_1,\ldots,k_n\in\{1,\ldots,p\}$. *Suppose* $f(0)=0$, $|\det f_z(0)|-\alpha=|f_{\overline{z}}(0)|=0$ *and for any* $z\in\mathbb{B}^n$, $k_1,\ldots,k_n\in\{1,\ldots,p\}$,

$$|G_{p-k_1+1,\ldots,p-k_n+1}(z)|<M,$$

*where* $\alpha$ *and* $M$ *are positive constants. Then there is a constant* $\rho_0\in(0,1)$ *such that* $f$ *is univalent in* $|z|<\rho_0$. *In specific* $\rho_0$ *satisfies*

$$\frac{\alpha\pi^{n-1}}{(4M)^{n-1}}-\frac{4(m_3+m_4)M\rho}{\pi}$$
$$-2\sum_{(k_1,\ldots,k_n)\neq(1,\ldots,1)}^{p}\Big[\Big(\sum_{i=1}^{n}(k_i-1)^2\Big)^{\frac{1}{2}}\rho^{2(k_1+\cdots+k_n)-2n-1}M+\frac{4M\rho^{2(k_1+\cdots+k_n)-2n}}{\pi(1-\rho^2)}\Big]=0.$$

*and* $f(\mathbb{B}^n)$ *contains a univalent ball of radius at least* $R_0$, *where*

$$R_0 = \rho_0\left\{\frac{\alpha\pi^{n-1}}{(4M)^{n-1}}-\frac{2(m_1+m_2)M\rho_0}{\pi}\right.$$

$$- \sum_{(k_1,\ldots,k_n)\neq(1,\ldots,1)}^{p} \Big[\frac{\big(\sum_{i=1}^{n}(k_i-1)^2\big)^{\frac{1}{2}}\rho_0^{2(k_1+\cdots+k_n)-2n-1}M}{k_1+\cdots+k_n-n} + \frac{8M\rho_0^{2(k_1+\cdots+k_n)-2n}}{\pi(1-\rho_0^2)[2(k_1+\cdots+k_n)-2n+1]}\Big]\Bigg\}.$$

*$m_3 \approx 4.200$ and $m_4 \approx 2.598$ are constants.*

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