Lipschitz-type Sobolev Spaces in Metric Measure Spaces

Estibalitz Durand-Cartagena

Abstract. We compare different Lipschitz-type spaces defined on a metric space. In the case of a metric measure space, we will also compare these spaces with the Newtonian-Sobolev space $N^{1,\infty}(X)$. If the space is in addition complete, endowed with a doubling measure and supports a ∞-Poincaré inequality, we obtain the equality of all the functional spaces. We also study some aspects of the ∞-Poincaré inequality including a geometric characterization in terms of modulus of curves and an analytic characterization in terms of the equality of all the Lipschitz-type Sobolev spaces.

Keywords. Harmonic mappings, quasiconformal mappings, hyperbolic metric.

2010 MSC. 30C55, 30F45, 31C05, 30C65.

1. Introduction

The study of analysis on metric measure spaces has progressed in recent years to include concepts from first order differential calculus [1],[12],[13],[22]. This theory has applications in several areas of analysis, such as potential theory on Riemannian manifolds, Carnot groups, theory of quasiconformal and quasiregular mappings, degenerate elliptic equations, fractal analysis or analysis on graphs.

Lipschitz functions are the natural class of smooth functions to be considered in a metric space. Actually, in the Euclidean setting, Rademacher's theorem states that Lipschitz continuous functions are differentiable almost everywhere. The Lipschitz condition is a purely geometric condition that makes perfect sense in the metric setting and gives *global information* about the space.

On the other hand, the notion of derivative yields *infinitesimal information*: it measures the infinitesimal oscillations of a function at a given point. However, a metric space is not necessarily endowed with a natural linear or differentiable structure and one does not have a derivative, even in the weak sense of Sobolev spaces. Nevertheless, if f is a real-valued function on a metric space (X, d) and x

is a point in X, one can use similar measurements of sizes of first-order oscillations of f at small scales around x, such as $\operatorname{osc}_r f(x) = \sup\{|f(y) - f(x)|/r : y \in X, d(x, y) \leq r\}$. In fact, if we look at the superior limit of the above expression as r tends to 0 we almost recover in many cases, as in the Euclidean or Riemannian setting, the standard notion of derivative. Furthermore, this limit can play the role of (length of) a gradient.

One interesting problem is to know under which circumstances one can use information that is known infinitesimally, to yield information that holds globally throughout the space. We present here some of the advances in understanding the infinitesimal versus global behavior of Lipschitz functions in the metric setting. The key assumption needed is that the space where the map is defined should be highly connected, meaning that there are many paths joining any part of the space.

Standard assumptions in analysis on metric spaces include that the measure is doubling and that the space supports a p-Poincaré inequality. Both conditions have been instrumental in this development. The first condition is imposed on the measure and allow us for example to talk about Lebesgue points or to define the maximal operator. On the other hand, the Poincaré inequality creates a link between the measure, the metric and the (length of the) gradient. It provides a way to pass from the infinitesimal information which gives the gradient to larger scales. Metric spaces with doubling measure and Poincaré inequality admit first order differential calculus akin to that in Euclidean spaces. Moreover, it implies some kind of connectedness and even something more, the so-called quasiconvexity of the space.

We also present here some recent advances in the study of p-Poincaré inequalities for the case $p = \infty$. This limit case presents nice features such as a geometric characterization in terms of modulus of curves in the space and a purely analytic condition which connects different Lipschitz-type function spaces and Sobolev spaces in the setting of arbitrary metric measure spaces.

2. Notation and Preliminaries.

Let (X, d) be a metric space. For $x \in X$ and $r > 0$ we let $B(x, r) := \{y \in X : d(x, y) < r\}$ be the open ball of radius r centered at x. For $\lambda > 0$ we write $\lambda B(x, r)$ to mean $B(x, \lambda r)$.

By a *curve* γ we will mean a continuous mapping $\gamma : [a, b] \to X$. Recall that the *length* of a continuous curve $\gamma : [a, b] \to X$ in a metric space (X, d) is defined as

$$\ell(\gamma) := \sup \left\{ \sum_{i=0}^{n-1} d(\gamma(t_i), \gamma(t_{i+1})) \right\}$$

where the supremum is taken over all finite partitions $a = t_0 < t_1 < \cdots < t_n = b$ of the interval $[a, b]$. We will say that a curve γ is *rectifiable* if $\ell(\gamma) < \infty$. The integral of a Borel function g over a rectifiable path γ is usually defined via the path length parametrization γ_0 of γ in the following way:

$$\int_\gamma g \, ds = \int_0^{\ell(\gamma)} g \circ \gamma_0(t) dt.$$

Recall here that every rectifiable curve γ admits a parametrization by the arc-length; that is, with $\gamma_0 : [a, b] \to X$, for all $t_1, t_2 \in [a, b]$ with $t_1 \leq t_2$, we have $\ell(\gamma_{0|_{[t_1, t_2]}}) = t_2 - t_1$. Hence from now on we only consider curves that are arc-length parametrized.

A metric space (X, d) is said to be a *length space* if for each pair of points $x, y \in X$ the distance $d(x, y)$ coincides with the infimum of all lengths of curves in X connecting x with y. Another interesting class of metric spaces, which contains length spaces, are the so called *quasiconvex* spaces. Recall that a metric space (X, d) is *quasiconvex* if there exists a constant $C > 0$ such that for each pair of points $x, y \in X$, there exists a curve γ connecting x and y with $\ell(\gamma) \leq C d(x, y)$. As one can expect, a metric space is quasiconvex if, and only if, it is bi-Lipschitz homeomorphic to some length space.

We can endow our metric spaces with a measure μ in which case, (X, d, μ) will denote a metric measure space, that is, a metric space equipped with a metric d and a Borel regular measure μ defined on the Borel set $\mathcal{B}(X)$, that is, μ is an outer measure on a metric space (X, d) such that all Borel sets are μ-measurable and for each set $E \subset X$ there exists a Borel set F such that $E \subset F$ and $\mu(E) = \mu(F)$.

A measure μ is *doubling* if there is a constant $C_\mu > 0$ such that for all $x \in X$ and $r > 0$,

$$(2.1) \qquad\qquad 0 < \mu(B(x, 2r)) \leq C_\mu \, \mu(B(x, r)) < \infty.$$

We shall denote by C_μ the least constant that satisfies condition (2.1), i.e., we define

$$C_\mu := \sup_B \frac{\mu(2B)}{\mu(B)}.$$

In a complete metric space X, the existence of a doubling measure which is not trivial and is finite on balls implies that X is separable and *proper*. The latter means that closed bounded subsets of X are compact. In particular, X is locally compact. Therefore, the notion of doubling metric spaces is intrinsically finite-dimensional and it is not possible to endow infinite dimensional spaces with doubling measures.

Some of the classical theorems in analysis in the Euclidean setting can be extended to doubling metric measure spaces. The Lebesgue differentiation Theorem is such an example: if f is a locally integrable function on a doubling metric space X, then

$$f(x) = \lim_{r \to 0} \fint_{B(x,r)} f d\mu,$$

for μ-a.e. point in X. In other words, almost every point in X is a *Lebesgue point* for f, see for example [12, Theorem 1.8].

Here for arbitrary $A \subset X$ with $0 < \mu(A) < \infty$ we write

$$f_A = \fint_A f = \frac{1}{\mu(A)} \int_A f d\mu.$$

Furthermore, it is also possible to define the maximal operator and obtain the same continuity properties from $L^p(X, \mu)$ to $L^p(X, \mu)$ as in the Euclidean case (see [12, Theorem 2.2]).

In what follows, $\| \cdot \|_\infty$ will denote the supremum norm whereas $\| \cdot \|_{L^\infty}$ will denote the essential supremum norm, provided we have a measure on X.

3. Pointwise Lipschitz functions on metric measure spaces

Let (X, d) be a metric space. A function $f : X \to \mathbb{R}$ is *C-Lipschitz* if there exists a constant $C > 0$ such that

$$|f(x) - f(y)| \leq Cd(x, y),$$

for each $x, y \in X$. From now on, $\mathrm{LIP}(\cdot)$ will denote the Lipschitz constant:

$$\mathrm{LIP}(f) := \sup_{\substack{x,y \in X \\ x \neq y}} \frac{|f(y) - f(x)|}{d(y, x)}.$$

We denote by $\mathrm{LIP}(X)$(respectively, $\mathrm{LIP}^\infty(X)$) the space of Lipschitz functions (respectively, bounded functions which are in $\mathrm{LIP}(X)$).

Given a function $f : X \to \mathbb{R}$, the *pointwise Lipschitz constant* of f at a non isolated point $x \in X$ is defined as follows:

$$\mathrm{Lip}\, f(x) = \limsup_{\substack{y \to x \\ y \neq x}} \frac{|f(x) - f(y)|}{d(x, y)}.$$

If x is an isolated point we define $\mathrm{Lip}\, f(x) = 0$.

Loosely speaking, the operator $\mathrm{Lip}\, f$ estimates some kind of infinitesimal lipschitzian property around each point. Recently, this functional has played an

important role in several contexts. We just mention here the construction of differentiable structures in the setting of metric measure spaces [4], [16], the theory of upper gradients [14], [24], or Stepanov differentiability theorem [2].

For example, if $f \in C^1(\Omega)$ where Ω is an open subset of Euclidean space, or of a Riemannian manifold, then $\operatorname{Lip} f = |\nabla f|$. On the other hand, if Ω is an open subset of the first Heisenberg group $\mathbb{H}$, $\operatorname{Lip} f = |\nabla_H f|$ where $\nabla_H f$ denotes the horizontal gradient of f.

This concept gives rise to a class of function spaces, *pointwise Lipschitz function spaces*, which contains in some sense infinitesimal information about the functions:
$$D(X) = \{f : X \longrightarrow \mathbb{R} : \| \operatorname{Lip} f \|_\infty < +\infty \}.$$

We also denote by $D^\infty(X)$ the space of bounded functions which are in $D(X)$. It is not difficult to see that for $f \in D(X)$, $\operatorname{Lip} f$ is a Borel function on X and that $\| \operatorname{Lip}(\cdot) \|_\infty$ yields a seminorm in $D(X)$. Moreover, pointwise Lipschitz functions are continuous.

It is clear that if f is a C-Lipschitz function, then $\operatorname{Lip} f(x) \le C$ for every $x \in X$ and so the space $D(X)$ clearly contains the space $\operatorname{LIP}(X)$. The other inclusion is not true in general. For example, if we consider a cusp domain in $\mathbb{R}^2$ endowed with the Euclidean distance, one can define functions $f \in D^\infty(X) \backslash \operatorname{LIP}^\infty(X)$. For a proof of this fact see [5, Example 2.7].

It is well known that $\operatorname{LIP}^\infty(X)$ is a Banach space with the norm
$$\|f\|_{\operatorname{LIP}^\infty(X)} := \|f\|_\infty + \operatorname{LIP}(f).$$

One can also endow $D(X)$ with a norm which arises naturally from the definition of the operator Lip. For each $f \in D^\infty(X)$, let us define
$$\|f\|_{D^\infty} := \max\{\|f\|_\infty, \| \operatorname{Lip} f \|_\infty \}.$$

This norm is not complete in the general case, as it can be seen in Example 3.3. However, there is a wide class of spaces, the *locally radially quasiconvex metric spaces* (see Definition 3.1), for which $D^\infty(X)$ admits a Banach space structure.

Definition 3.1. Let (X, d) be a metric space. We say that X is *locally radially quasiconvex* if for each $x \in X$, there exist a neighborhood U^x and a constant $K_x > 0$ such that for each $y \in U^x$ there exists a rectifiable curve γ in U^x connecting x and y such that $\ell(\gamma) \le K_x d(x, y)$.

Theorem 3.2. [5, 3.2] *Let (X, d) be a locally radially quasiconvex metric space. Then, $(D^\infty(X), \| \cdot \|_{D^\infty})$ is a Banach space.*

The following example shows that in general $(D^\infty(X), \| \cdot \|_{D^\infty})$ is not a Banach space.

Example 3.3. Consider the connected metric space $X = X_0 \cup \bigcup_{n=1}^{\infty} X_n \cup G \subset \mathbb{R}^2$ with the metric induced by the Euclidean one, where $X_0 = \{0\} \times [0, +\infty)$, $X_n = \{\frac{1}{n}\} \times [0, n]$, $n \in \mathbb{N}$ and $G = \{(x, \frac{1}{x}) : 0 < x \leq 1\}$. For each $n \in \mathbb{N}$ consider the sequence of functions $f_n : X \to [0, 1]$ given by

$$f_n\left(\frac{1}{k}, y\right) = \begin{cases} \frac{k-y}{k\sqrt{k}} & \text{if } 1 \leq k \leq n \\ 0 & \text{if } k > n, \end{cases}$$

and $f_n(x, y) = 0$ if $x \neq \frac{1}{k}$ $\forall k \in \mathbb{N}$. Observe that $f_n(\frac{1}{k}, 0) = \frac{1}{\sqrt{k}}$ and $f_n(\frac{1}{k}, k) = 0$ if $1 \leq k \leq n$. Since $\text{Lip}\, f_n(\frac{1}{k}, y) = \frac{1}{k\sqrt{k}}$ and $\text{Lip}\, f_n(x, y) = 0$ if $x \neq \frac{1}{k}$ $\forall k \in \mathbb{N}$, we have that $f_n \in D^{\infty}(X)$ for each $n \geq 1$. In addition, if $1 < n < m$,

$$\|f_n - f_m\|_{\infty} = \frac{1}{\sqrt{n+1}} \quad \text{and} \quad \|\text{Lip}(f_n - f_m)\|_{\infty} = \frac{1}{(n+1)\sqrt{n+1}}.$$

Thus, we deduce that $\{f_n\}_n$ is a Cauchy sequence in $(D^{\infty}(X), \|\cdot\|_{D^{\infty}})$. However, if $f_n \to f$ in D^{∞} then $f_n \to f$ pointwise. Then $f_m(\frac{1}{n}, 0) = \frac{1}{\sqrt{n}}$ for each $m \geq n$ and so $f(\frac{1}{n}, 0) = \frac{1}{\sqrt{n}}$ and $f(0, 0) = 0$. Thus, we obtain that

$$\text{Lip}(f)(0, 0) \geq \lim_{n \to \infty} \frac{|f((\frac{1}{n}), 0) - f(0, 0)|}{d(\frac{1}{n}, 0)} = \lim_{n \to \infty} \frac{\frac{1}{\sqrt{n}}}{\frac{1}{n}} = +\infty,$$

and so $f \notin D^{\infty}(X)$. This means that $(D^{\infty}(X), \|\cdot\|_{D^{\infty}})$ is not a Banach space. Observe that if one connects X_0 to $\{1\} \times [0, 1]$ by a curve that does not intersect any of the X_n, $n \geq 2$ one obtains a path-connected metric space X such that $(D^{\infty}(X), \|\cdot\|_{D^{\infty}})$ is not complete.

We say that $\text{LIP}^{\infty}(X) = D^{\infty}(X)$ *with comparable energy seminorms* if the two sets are the same and there is a constant $C > 0$ such that for all $f \in \text{LIP}^{\infty}(X)$,

$$\text{LIP}(f) \leq C\|\text{Lip}\, f\|_{\infty}.$$

As shown before, $D^{\infty}(X)$ is not a Banach space in general. However, if $\text{LIP}^{\infty}(X) = D^{\infty}(X)$ *with comparable energy seminorms*, then $D^{\infty}(X)$ is also a Banach space.

The main aim of this section is to see under which conditions a function $f : X \to \mathbb{R}$ is Lipschitz if and only if $\text{Lip}\, f$ is a bounded functional. More precisely, we look for conditions regarding the geometry of the metric space X under which $\text{LIP}^{\infty}(X) = D^{\infty}(X)$ with comparable energy seminorms. As it can be expected, we need some kind of *connectedness*.

Lemma 3.4. [5, 2.3] *Let (X, d) be a metric space and let $f \in D(X)$. Let $x, y \in X$ and suppose that there exists a rectifiable curve $\gamma : [a, b] \to X$ connecting x and y, that is, $\gamma(a) = x$ and $\gamma(b) = y$. Then, $|f(x) - f(y)| \leq \|\text{Lip}\, f\|_{\infty}\, \ell(\gamma)$.*

As a straightforward consequence of the previous result, we deduce

Corollary 3.5. *If (X, d) is a quasiconvex space then $\text{LIP}^\infty(X) = D^\infty(X)$ with comparable energy seminorms.*

Proposition 3.6. *Let (X, d) be a complete locally compact connected metric space. Then X is a quasiconvex space if and only if $\text{LIP}^\infty(X) = D^\infty(X)$ with comparable energy seminorms.*

Proof. The fact that if X is a quasiconvex space then $\text{LIP}^\infty(X) = D^\infty(X)$ with comparable energy seminorms is Corollary 3.5. On the other hand, for each $x \in X$ and $\varepsilon > 0$ we define the ε-distance from x to z to be

$$(3.7) \qquad \rho_{x,\varepsilon}(z) := \inf \sum_{i=0}^{N-1} d(z_i, z_{i+1}),$$

where the infimum is taken over all finite ε-chains $(z_i)_{i=0}^N$. For positive integers N we set $\rho_{x,\varepsilon,N} = \min\{N, \rho_{x,\varepsilon}\}$. Since X is connected we see that $\rho_{x,\varepsilon,N}$ is finite-valued everywhere and $|\rho_{x,\varepsilon,N}(z) - \rho_{x,\varepsilon,N}(w)| \le d(z, w)$ when $d(z, w) < \varepsilon$; thus for all $w \in X$ we have $\text{Lip}\,\rho_{x,\varepsilon,N}(w) \le 1$. Hence $\rho_{x,\varepsilon,N}$ belongs to $D^\infty(X)$. Because $\text{LIP}^\infty(X) = D^\infty(X)$ with comparable energy seminorms there is a constant $C > 0$ such that $\text{LIP}(\rho_{x,\varepsilon,N}) \le C$ with C independent of x, ε, N. It follows that for all $y \in X$ and all $\varepsilon > 0$,

$$(3.8) \qquad |\rho_{x,\varepsilon,N}(y)| = |\rho_{x,\varepsilon,N}(y) - \rho_{x,\varepsilon,N}(x)| \le \text{LIP}(\rho_{x,\varepsilon,N})d(x,y) \le Cd(x,y).$$

Using a standard limiting argument, which involves Arzela-Ascoli's theorem and inequality (3.8), we can construct a 1-Lipschitz rectifiable curve connecting x and y with length at most $Cd(x, y)$. Since x and y were arbitrary this completes the proof. For further details about the construction of the curve we refer the reader to [18, Theorem 3.1]. ∎

4. Sobolev spaces on metric measure spaces

Our aim in this section is to compare the function spaces $D^\infty(X)$ and $\text{LIP}^\infty(X)$ with certain Sobolev spaces on metric-measure spaces. There are several possible extensions of the classical theory of Sobolev spaces to the setting of metric spaces equipped with a Borel measure. Hajłasz defined in [11] the spaces $M^{1,p}$ for $1 \le p \le \infty$ in connection with maximal operators. Shanmugalingam in [24] introduced, using the notion of upper gradient (and more generally weak upper gradient) the Newtonian spaces $N^{1,p}(X)$ for $1 \le p < \infty$. There are another interesting notions of Sobolev spaces to the context of metric measure spaces (see for example [9],[8],[15],[19]). However under suitable conditions, all the approaches turn to be equivalent ([25, 26]). The overview article [10] by Hajłasz

presents further generalizations of Sobolev spaces in metric measure spaces. It should be pointed out, that if the space supports a p-Poincaré inequality, $1 < p < \infty$ (see definition 5.1), all the approaches to Sobolev spaces described in [10] are also equivalent (see [17, Theorem 1.0.6]).

Following [10] we record the definition of the Hajłasz-Sobolev space $M^{1,p}(X)$.

For $0 < p \leq \infty$ the space $\widetilde{M}^{1,p}(X, d, \mu)$ is defined as the set of all functions $f \in L^p(X)$ for which there exists a function $0 \leq g \in L^p(X)$ such that

$$(4.1) \qquad |f(x) - f(y)| \leq d(x, y)(g(x) + g(y)) \qquad \mu\text{-a.e.}$$

As usual, we get the space $M^{1,p}(X, d, \mu)$ after identifying any two functions $u, v \in \widetilde{M}^{1,p}(X, d, \mu)$ such that $u = v$ almost everywhere with respect to μ. The space $M^{1,p}(X, d, \mu)$ is equipped with the norm

$$\|f\|_{M^{1,p}} = \|f\|_{L^p} + \inf_g \|g\|_{L^p},$$

where the infimum is taken over all functions $0 \leq g \in L^p(X)$ that satisfy the requirement (4.1).

In particular, if $p = \infty$ it can be shown that $M^{1,\infty}(X, d, \mu)$ coincides with $\mathrm{LIP}^\infty(X)$ provided that $\mu(B) > 0$ for every open ball $B \subset X$ (see remark 5.1.4 in [1]). In addition, we also have that $1/2\| \cdot \|_{\mathrm{LIP}^\infty} \leq \| \cdot \|_{M^{1,\infty}} \leq \| \cdot \|_{\mathrm{LIP}^\infty}$. In this case we obtain that $M^{1,\infty}(X) = \mathrm{LIP}^\infty(X) \subseteq D^\infty(X)$.

Another interesting generalization of Sobolev spaces to general metric spaces are the so-called Newtonian Spaces $N^{1,p}$, introduced by Shanmungalingam [24, 25]. Its definition is based on the notion of *upper gradient.* This concept was introduced by Heinonen and Koskela [14] and serves the role of derivatives in a metric space.

A non-negative Borel function g on X is said to be an *upper gradient* for an extended real-valued function f on X, if

$$(4.2) \qquad |f(\gamma(a)) - f(\gamma(b))| \leq \int_\gamma g$$

for every rectifiable curve $\gamma : [a, b] \to X$, when both $f(\gamma(a))$ and $f(\gamma(b))$ are finite, and $\int_\gamma g = \infty$ otherwise.

Observe that $g \equiv \infty$ is an upper gradient of every function on X and if there are no rectifiable curves in X then $g \equiv 0$ is an upper gradient of every function on X. If f is Lipschitz, then $g = \mathrm{LIP}(f)$ is an upper gradient for f. Moreover, the pointwise Lipschitz constant $\mathrm{Lip}\, f$ provides us with a smaller upper gradient than the global Lipschitz constant. On the other hand, each function $f \in W^{1,p}(\mathbb{R}^n)$ has a representative that has a p-integrable upper gradient (see [24]).

The upper gradient plays the role of a derivative in the formula (4.2) which is similar to the one related to the Fundamental Theorem of Calculus. The point is that using upper gradients we may have many of the properties of ordinary Sobolev spaces even though we do not have derivatives of our functions.

If g is an upper gradient of u and $\widetilde{g} = g$ almost everywhere, then it may happen that $\widetilde{g}$ is no longer an upper gradient for f. We do not want our upper gradients to be sensitive to changes on small sets. To avoid this unpleasant situation the notion of *weak upper gradient* is introduced as follows. First we need a way to measure how large a family of curves is. The most important point is if a family of curves is small enough to be ignored. This kind of problem was first approached in [7]. In what follows let $\Upsilon \equiv \Upsilon(X)$ denote the family of all non-constant rectifiable curves in X. It may happen that $\Upsilon = \emptyset$, but we will be mainly concerned with metric spaces for which the space Υ is large enough. If E is a subset of X then Γ_E^+ is the family of curves γ such that $\mathscr{L}^1(\gamma^{-1}(\gamma \cap E)) > 0$.

From now on, will focus on the case $p = \infty$.

Definition 4.3. For $\Gamma \subset \Upsilon$, let $F(\Gamma)$ be the family of all Borel measurable functions $\rho : X \to [0, \infty]$ such that

$$\int_\gamma \rho \geq 1 \ \text{ for all } \ \gamma \in \Gamma.$$

We define the ∞-*modulus of* Γ by

$$\mathrm{Mod}_\infty \Gamma = \inf_{\rho \in F(\Gamma)} \{\|\rho\|_{L^\infty}\}.$$

If some property holds for all curves $\gamma \notin \Gamma$ for some $\Gamma \subset \Upsilon$ that satisfies $\mathrm{Mod}_\infty \Gamma = 0$, then we say that the property holds for ∞-*a.e. curve*.

It can be easily checked that Mod_∞ is an outer measure as it is for $1 \leq p < \infty$, see for example [11, Theorem 5.2].

The following lemma provides a characterization of path families whose ∞-modulus is zero.

Lemma 4.4. [5, 5.7] *Let $\Gamma \subset \Upsilon$. The following conditions are equivalent:*

(a) $\mathrm{Mod}_\infty \Gamma = 0$.
(b) *There exists a Borel function $0 \leq \rho \in L^\infty(X)$ such that $\int_\gamma \rho = +\infty$, for each $\gamma \in \Gamma$.*
(c) *There exists a Borel function $0 \leq \rho \in L^\infty(X)$ such that $\int_\gamma \rho = +\infty$, for each $\gamma \in \Gamma$ and $\|\rho\|_{L^\infty} = 0$.*

Remark 4.5. An important feature of the ∞-modulus of curves is that if we have two measures μ and λ defined on X with the same zero measure sets, then

the ∞- modulus of a family of curves is the same, independent of the measure we use to compute it.

Lemma 4.6. *Let $E \subset X$. If $\mu(E) = 0$, then $\mathrm{Mod}_\infty(\Gamma_E^+) = 0$.*

Definition 4.7. A non-negative Borel function g on X is an ∞-*weak upper gradient* of an extended real-valued function f on X if for ∞-a.e. curve $\gamma \in \Upsilon$,

$$|f(\gamma(a)) - f(\gamma(b))| \leq \int_\gamma g$$

when both $f(\gamma(a))$ and $f(\gamma(b))$ are finite, and $\int_\gamma g = \infty$ otherwise.

Let $\widetilde{N}^{1,\infty}(X, d, \mu) = \widetilde{N}^{1,\infty}(X)$ be the class of all Borel functions $u \in L^\infty(X)$ for which there exists an ∞-weak upper gradient g in $L^\infty(X)$. For $f \in \widetilde{N}^{1,\infty}(X, d, \mu)$ we set

$$\|f\|_{\widetilde{N}^{1,\infty}} = \|f\|_{L^\infty} + \inf_g \|g\|_{L^\infty},$$

where the infimum is taken over all ∞-weak upper gradients g of f.

Definition 4.8. We define an equivalence relation in $\widetilde{N}^{1,\infty}(X)$ by $f_1 \sim f_2$ if and only if $\|f_1 - f_2\|_{\widetilde{N}^{1,\infty}} = 0$. The space $N^{1,\infty}(X, d, \mu) = N^{1,\infty}(X)$ denotes the quotient $\widetilde{N}^{1,\infty}(X, d, \mu)/\sim$ and it is equipped with the norm

$$\|f\|_{N^{1,\infty}} = \|f\|_{\widetilde{N}^{1,\infty}}.$$

Theorem 4.9. [5, 5.18] $N^{1,\infty}(X)$ *is a Banach space.*

Lemma 4.10. *If $f \in D^\infty(X)$ then $\mathrm{Lip}(f)$ is an upper gradient of f.*

Proof. Let $\gamma : [a, b] \to X$ be a rectifiable curve, parametrized by arc-length, which connects x and y. It can be checked that γ is 1-Lipschitz. The function $f \circ \gamma$ is a pointwise Lipschitz function and by Stepanov's differentiability theorem (see [27]), it is differentiable a.e. Note that $|(f \circ \gamma)'(t)| \leq \mathrm{Lip}\, f(\gamma(t))$ at every point of $[a, b]$ where $(f \circ \gamma)$ is differentiable. Now, we deduce that

$$|f(x) - f(y)| \leq \left| \int_a^b (f \circ \gamma)'(t)dt \right| \leq \int_a^b \mathrm{Lip}(f(\gamma(t)))\, dt$$

as wanted.   ∎

Now suppose that $\mu(B) > 0$ for every open ball $B \subset X$. It is clear by Lemma 4.10 that $D^\infty(X) \subset \widetilde{N}^{1,\infty}(X)$ and that the map

$$\phi \;:\; \begin{array}{ccc} D^\infty(X) & \longrightarrow & N^{1,\infty}(X) \\ f & \longrightarrow & [f]. \end{array}$$

is an inclusion. Indeed, if $f_1, f_2 \in D^\infty(X)$ with $0 = [f_1 - f_2] \in N^{1,\infty}(X)$, we have $f_1 - f_2 = 0$ μ-a.e. Thus $f_1 = f_2$ in a dense subset and since f_1, f_2 are continuous we obtain that $f_1 = f_2$. Therefore we have the following chain of inclusions:

$$(4.11) \qquad \mathrm{LIP}^\infty(X) = M^{1,\infty}(X) \subset D^\infty(X) \subset N^{1,\infty}(X),$$

and $\| \cdot \|_{N^{1,\infty}} \leq \| \cdot \|_{D^\infty} \leq \| \cdot \|_{\mathrm{LIP}^\infty} \leq 2 \, \| \cdot \|_{M^{1,\infty}}$.

Observe that in general, $D^\infty(X) \neq N^{1,\infty}(X)$. Indeed, the path-connected metric space mentioned in Example 3.3 gives an example in which $D^\infty(X)$ is not a Banach space whereas $N^{1,\infty}(X)$ is a Banach space and so $D^\infty(X) \neq N^{1,\infty}(X)$.

In the next section, we will look for conditions under which the spaces $\mathrm{LIP}^\infty(X)$ and $N^{1,\infty}(X)$ coincide. In particular, this will gives us the equality of all the spaces in the chain (4.11) above. The main idea to get the equality is to construct curves not "too long" which avoid zero-measure sets. More precisely, given two points $x, y \in X$ and a zero measure set E, one should be able to construct a quasiconvex curve γ connecting the two points and such that $\mathscr{L}^1(\gamma^{-1}(\gamma \cap E)) = 0$. For a proof along this line see [5, Theorem 5.2].

5. The ∞-Poincaré inequality in metric measure spaces

It is known that if X is doubling and supports a p-Poincaré inequality then we have equality of the Sobolev spaces $M^{1,p}(X)$ and $N^{1,p}(X)$ [25, Theorem 4.9],[17, Theorem 1.0.6]. Furthermore, under the same assumptions, Lipschitz functions are dense in $N^{1,p}(X)$ [25, Theorem 4.1]. Since we are looking for the equality of the spaces $M^{1,\infty}$ and $N^{1,\infty}$ it would be reasonable to look for an ∞-version of the Poincaré inequality.

The classical Poincaré inequality allows one to obtain integral bounds on the oscillation of a function using integral bounds on its derivatives. It is worth mentioning that in this type of inequalities the derivative itself is not needed, but only the size of the gradient of the function is really used; a nice discussion of this can be found in [23]. This is the idea behind many generalizations of Poincaré inequalities, in spaces where we may not have a linear structure. The idea of Poincaré inequalities makes sense in the more general setting of metric measure spaces.

The following Poincaré inequality is now standard in literature on analysis in metric measure spaces.

Definition 5.1. Let $1 \leq p < \infty$. We say that (X, d, μ) supports a *weak p-Poincaré inequality* if there exist constants $C_p > 0$ and $\lambda \geq 1$ such that for every

Borel measurable function $f : X \to \mathbb{R} \cup \{-\infty, \infty\}$ and every upper gradient $g : X \to [0, \infty]$ of f, the pair (f, g) satisfies the inequality

$$(5.2) \qquad \fint_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \leq C_p \, r \left(\fint_{B(x,\lambda r)} g^p d\mu \right)^{1/p}$$

for each ball $B(x, r) \subset X$. The word *weak* refers to the possibility that λ may be strictly greater than 1.

To require that inequality (5.2) holds in X is to require that the space has plenty of rectifiable curves, uniformly at all scales. For that reason for a space to support a Poincaré inequality one needs some kind of connectedness. Indeed, let us consider the space $X = A \cup B$ with $A, B \subset \mathbb{R}^n$ with A, B bounded open sets in $\mathbb{R}^n$ which are a positive distance apart and $\mathscr{L}^n(A), \mathscr{L}^n(B) > 0$. The function $f = \chi_A$ if Lipschitz on X, $|\nabla f| = 0$ but

$$\int_X |f - f_X| d\mathscr{L}^n = \frac{2 \mathscr{L}^n(A) \mathscr{L}^n(B)}{\mathscr{L}^n(A) + \mathscr{L}^n(B)} > 0.$$

A further geometric implication of the p-Poincaré inequality is the fact that if a complete doubling metric measure space supports a p-Poincaré inequality then there exists a constant such that each pair of points can be connected with a curve whose length is at most the constant times the distance between the points (see [22] or [14]), that is, the space is *quasiconvex*.

There is a long list of metric spaces supporting a Poincaré inequality, including some standard examples such as $\mathbb{R}^n$, Riemannian manifolds with non-negative Ricci curvature, Carnot groups (in particular the Heisenberg group), but also other non-Riemannian metric measure spaces of fractional Hausdorff dimension, see for example [20], [13] and references therein. Metric spaces with doubling measure and p-Poincaré inequality admit a first order differential calculus theory akin to that in Euclidean spaces [4],[16]. One surprising fact is that some geometric consequences of this condition seem to be independent of the parameter p and the picture is not yet clear.

It follows from Hölder's inequality that if a space admits a p-Poincaré inequality, then it admits a q-Poincaré inequality for each $q \geq p$. Recently Keith and Zhong [17] proved a self-improving property for Poincaré inequalities, that is, if X is a complete metric space equipped with a doubling measure satisfying a p-Poincaré inequality for some $1 < p < \infty$, then there exists $\varepsilon > 0$ such that X supports a q-Poincaré inequality for all $q > p - \varepsilon$. The strongest of all these inequalities would be the 1-Poincaré inequality, which is closely related to relative isoperimetric inequalities. For example, it is well known that the 1-Poincaré inequality is equivalent to the relative isoperimetric property [21], [3]. A natural question that arises at this point is the following: what would be the weakest

version of p-Poincaré inequality that would still give reasonable information on the geometry of the metric space?

Definition 5.3. We say that (X, d, μ) supports a *weak ∞-Poincaré inequality* if there exist constants $C > 0$ and $\lambda \geq 1$ such that for every Borel measurable function $f : X \to \mathbb{R} \cup \{\infty\}$ and every upper gradient $g : X \to [0, \infty]$ of f, the pair (f, g) satisfies the inequality

$$\fint_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \leq C \, r \|g\|_{L^\infty(B(x,\lambda r))}$$

for each ball $B(x, r) \subset X$.

First let us notice that there exist spaces with a weak ∞-Poincaré inequality which do not admit a weak p-Poincaré inequality for any finite p.

Example 5.4. Let T be a non-degenerate triangular region in $\mathbb{R}^2$ and let T' be an identical copy of T. Let X be the metric space obtained by identifying a vertex V of T with a vertex V' of T' ($V = V' = \{0\}$) and the metric defined by

$$d(x, y) = \begin{cases} |x - y| & \text{if } x, y \in T \text{ or } x, y \in T', \\ |x - V| + |V' - y| & \text{if } x \in T \text{ and } y \in T'. \end{cases}$$

The space is a bow-tie shaped subset of $\mathbb{R}^2$. We equip the space with the weighted measure μ given by $d\mu(x) = \omega(x) d\mathcal{L}^2(x)$, where $\omega(x) = e^{-\frac{1}{|x|^2}}$. It is already known that this space equipped with the Lebesgue measure $\mathcal{L}^2$ admits a p-Poincaré inequality for $p > 2$ (see for example [24]). However (X, d, μ) does not admit a weak p-Poincaré inequality for any finite p but admits a weak ∞-Poincaré inequality. See [6] for further details.

If X is only known to support an ∞-Poincaré inequality then the space is still quasiconvex, as demonstrated by Proposition 3.4 in [6]. But in fact a ∞-Poincaré inequality gives us a stronger geometric implication: every pair of sets of positive measure, which are a positive distance apart, can be connected by a "thick" family of quasiconvex curves in the sense that the modulus of this family of curves is positive. The following definition makes this idea more precise.

Definition 5.5. A metric measure space (X, d, μ) is said to be a *∞-thick quasiconvex* space if there is a constant $C \geq 1$ such that for all $x, y \in X$, all $0 < \varepsilon < \frac{1}{4} d(x, y)$, and all measurable sets $E \subset B(x, \varepsilon)$ and $F \subset B(y, \varepsilon)$ satisfying $\mu(E)\mu(F) > 0$, we have that

$$\text{Mod}_\infty(\Gamma(E, F, C)) > 0.$$

Here $\Gamma(E, F, C)$ denotes the set of all curves $\gamma_{p,q}$ connecting $p \in E$ and $q \in F$ with $\ell(\gamma_{p,q}) \leq C d(p, q)$.

Remark 5.6. Note that every complete ∞-thick quasiconvex space supporting a doubling measure is quasiconvex. The converse is not true in general. The Sierpinski carpet is a quasiconvex space which is not ∞-thick quasiconvex ([6, Corollary 4.15]).

In what follows we say that $\mathrm{LIP}^\infty(X) = N^{1,\infty}(X)$ *with comparable energy seminorms* if there is a constant $C > 0$ such that for all $f \in N^{1,\infty}(X)$ there exists $f_0 \in \mathrm{LIP}^\infty(X)$ with $f = f_0$ μ-a.e. and

$$\mathrm{LIP}(f_0) \leq C \inf_g \|g\|_{L^\infty},$$

where the infimum is taken over all ∞-weak upper gradients g of f.

The following theorem gives the equality of all the spaces in (4.11). Moreover, it is a geometric and analytic characterization of ∞- Poincaré inequality. A complete proof of this theorem can be found in [6].

Theorem 5.7. *Suppose that X is a connected complete metric space supporting a doubling Borel measure μ which is non-trivial and finite on balls. Then the following conditions are equivalent:*

(a) *X supports a weak ∞-Poincaré inequality.*
(b) *X is thick quasiconvex.*
(c) *$\mathrm{LIP}^\infty(X) = N^{1,\infty}(X)$ with comparable energy seminorms.*
(d) *X supports a weak ∞-Poincaré inequality for functions in $N^{1,\infty}(X)$.*

The equivalence of Condition (c) with the other three conditions needs the additional assumption of connectedness of X since the example of the union of two disjoint planar discs satisfies (c) but fails the other three conditions. The other three conditions directly imply that X is connected.

Theorem 5.8. *Suppose that X is a connected complete metric space supporting a doubling Borel measure μ which is non-trivial and finite on balls. Then the following conditions are equivalent:*

(a) *X supports an ∞-Poincaré inequality for locally Lipschitz continuous functions with continuous upper gradients.*
(b) *X is quasiconvex.*
(c) *$\mathrm{LIP}^\infty(X) = D^\infty(X)$ with comparable energy seminorms.*

Observe that the implication $(b) \iff (c)$ follows from Proposition 3.6. Just observe that the doubling measure implies the locally compactness of the space and the result follows.

The implication $(a) \Rightarrow (b)$ is given by the proof of Proposition 3.4 in [6]. We only need to apply the Poincaré inequality to the locally Lipschitz continuous function $\rho_{x,\varepsilon}$ (see definition 3.7) and its continuous upper gradient 1. The

implication $(b) \Rightarrow (a)$ follows from the argument that if g is a continuous upper gradient of a locally Lipschitz continuous function f, then for $x, y \in X$, by choosing a quasiconvex path γ connecting x to y, we get

$$|f(x) - f(y)| \leq \int_{\gamma} g \, ds \leq C \, d(x,y) \sup_{z \in B(x, Cd(x,y))} g(z).$$

So if B is a ball in X and x, y are points in B, then

$$\fint_B \fint_B |f(x) - f(y)| \, d\mu(x) \, d\mu(y) \leq C \mathrm{rad}(B) \sup_{z \in CB} g(z) = C \mathrm{rad}(B) \|g\|_{L^{\infty}(CB)}.$$

References

[1] L. Ambrosio, P. Tilli, Topics on Analysis in Metric Spaces. Oxford Lecture Series in Mathematics and its Applications **25**. *Oxford University Press, Oxford*, (2004).

[2] Z. M. Balogh, K. Rogovin, T. Zürcher, The Stepanov Differentiability Theorem in Metric Measure Spaces. *J. Geom. Anal.* **14**. no. 3, (2004), 405–422.

[3] S. G. Bobkov, C. Houdré, Some connections between isoperimetric and Sobolev-type inequalities. (English summary) *Mem. Amer. Math. Soc.* **129**, no. 616 (1997).

[4] J. Cheeger, Differentiability of Lipschitz Functions on metric measure spaces. *Geom. Funct. Anal.* **9** (1999), 428–517.

[5] E. Durand-Cartagena, J. A. Jaramillo, Pointwise Lipschitz functions on metric spaces. *J. Math. Anal. Appl.* **363**(2010), 525–548.

[6] E. Durand-Cartagena, J. A. Jaramillo, and N. Shanmugalingam, The ∞-Poincaré inequality in metric measure spaces. To appear in *Michigan Math. J.* (2011).

[7] B. Fuglede, Extremal length and functional completion. *Acta. Math.* **98**(1957), 171–219.

[8] M. Fukushima, Y. Ōshima, Yōichi and M. Takeda, *Dirichlet forms and symmetric Markov processes,* de Gruyter Studies in Mathematics **19**. Walter de Gruyter & Co., Berlin (1994).

[9] V. Goldshtein and M. Troyanov, Axiomatic theory of Sobolev spaces, *Expo. Math.* **19**(4)(2001), 289–336.

[10] P. Hajłasz, Sobolev spaces on metric-measure spaces. *Contemp. Math.* **338**(2003), 173–218.

[11] P. Hajłasz, Sobolev spaces on an arbitrary metric space. *Potential Anal.* **5**(1996), 403–415.

[12] J. Heinonen, *Lectures on Analysis on Metric Spaces*, Springer (2001).

[13] J. Heinonen, Nonsmooth calculus, *Bull. Amer. Math. Soc.* **44**(2007), 163–232.

[14] J. Heinonen and P. Koskela, Quasiconformal maps in metric spaces with controlled geometry, *Acta Math.* **181** (1998), 1–61.

[15] J. Jost, Generalized Dirichlet forms and harmonic maps, *Calc. Var.* **5**(1997), 1–19.

[16] S. Keith, A differentiable structure for metric measure spaces. *Adv. Math.* **183**(2004), 271–315.

[17] S. Keith and X. Zhong, The Poincaré inequality is an open ended condition, *Ann. Math.* **167**(2)(2008), 575–599.

[18] R. Korte, *Geometric implications of the Poincaré inequality,* Licentiate's thesis, Helsinki University of Technology, 2006.

[19] J.N. Korevaar and R.M. Schoen, Sobolev spaces and harmonic maps for metric space targets, *Comm. Anal. Geom.* **1**(1993), 561–659.

[20] T.J. Laakso, Ahlfors Q-regular spaces with arbitrary $Q > 1$ admitting weak Poincaré inequality, *Geom. Funct. Anal.* **10**, no. 1, (2000), 111–123.

[21] M. Miranda, Functions of bounded variation on "good" metric spaces, *J. Math. Pures Appl.* **9**(83)(2003), 975–1004.

[22] S. Semmes, Some Novel Types of Fractal Geometry, Oxford Science Publications, The Clarendon Press Oxford University Press (2001).

[23] S. Semmes, Finding Curves on General Spaces through Quantitative Topology, with Applications to Sobolev and Poincaré Inequalities, *Selecta Math., New Series* Vol. **2**, no. 2, (1996), 155–295.

[24] N. Shanmugalingam, *Newtonian Spaces: An extension of Sobolev spaces to Metric Measure Spaces*, Ph. D. Thesis, University of Michigan (1999), http: math.uc.edu/ nages/papers.html.

[25] N. Shanmugalingam, Newtonian Spaces: An extension of Sobolev spaces to Metric Measure Spaces, *Rev. Mat. Iberoamericana*, **16**(2000), 243–279.

[26] N. Shanmugalingam, A universality property of Sobolev spaces in metric measure spaces, Sobolev spaces in mathematics, I *Int. Math. Ser.* (N. Y.) Vol. 8. Springer. New York (2009),345–359.

[27] W. Stepanoff, Über totale Differenzierbarkeit, *Math. Ann.* **90**(1923), 318–320.

Estibalitz Durand-Cartagena   E-mail: `estibalitzdurand@mat.ucm.es`
Address:
Departamento de Análisis Matemático,
Universidad Complutense de Madrid,
Plaza de Ciencias 3, Madrid, 28040,
Spain