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Geometry and Topology of Intrinsic Distances

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Abstract. This exposition provides a detailed accounting of the geometry and topology of the intrinsic length and diameter distances.

Keywords. intrinsic distance, inner distance, length distance, diameter distance, quasiconvex, bounded turning.

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Contents

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1. Introduction

Every metric space $X = (X, |\cdot|)$ supports a natural intrinsic metric, its so-called *length distance*† given by

$$l(x, y) := \inf\{\ell(\gamma) : \gamma \text{ a rectifiable path joining } x, y \text{ in } X\}\,.$$

When X is rectifiably connected, this length distance is a finite metric and (X, l) is a so-called *length space*. A brief list of works discussing length distance includes [Gro99], [BH99], [BBI01], [Pap05].

If X has few rectifiable paths, its length distance is not so useful, and even when X is rectifiably connected, there are some pathological pitfalls that may occur with its length distance. In this setting, and indeed in general, it is easier and more convenient to work with the *diameter distance*‡ which is defined by

$$d(x, y) := \inf\{\operatorname{diam}(\gamma) : \gamma \text{ a path in } X \text{ joining } x, y\}\,.$$

When X is path connected, its diameter distance is always a finite distance, and (X, d) enjoys many of the same properties as (X, l). (We could also consider joining points by continua, or even just by connected sets, and define similar distances in continuumwise-connected spaces, or just connected spaces). The diameter distance is especially useful when working with quasiconformal mappings, since such maps need not preserve rectifiabilty.

A number of recent works have employed diameter distance; cf. [Hei89], [Väi89], [NV91], [GNV94], [FH11], [HM11], [Mey11]. However, the idea of diameter distance is not new; e.g., it appears already in [Why42, p.154] and even in [Maz35], although in these classical papers points are joined via connected sets or by continua rather than by paths.

One striking feature of length distance is that when X is locally compact and complete (and rectifiably connected), the space (X, l) is proper, and therefore geodesic. This is a consequence of the Hopf-Rinow Theorem; see [Gro99, p.9], [BBI01, p.51], [BH99, p.35]. The analogous result for diameter distance does not hold.

Corollary 3.8 reveals the sense in which the length and diameter distances are intrinsic.

The purpose of this article is to provide a written record of the foundational geometric and topologic properties of the metric spaces (X, d) and (X, l). Much of this is folklore, but references are difficult to locate in the literature. To a large extent, (X, d) and (X, l) enjoy similar properties, provided we replace the hypothesis "(locally) path connected" with "(locally) rectifiably connected",

†This is often called the *intrinsic* or *inner* or *internal* length distance.

‡This is also called the *intrinsic* or *inner* or *internal* diameter distance.

but see the paragraph just before Lemma 3.10 for a notable exception to this phenomena. Our primary focus is on diameter distance properties; we state the corresponding results for length distance and exhibit some pertinent examples, but for the most part leave the length distance proofs for the motivated reader.

We also provide proofs for certain assertions made in [FH11] and [Her11].

This document is organized as follows: Section 2 contains preliminary information including basic definitions, notation, terminology and elementary or well-known facts; see especially §2.D. We present properties of the intrinsic distances and exhibit examples in Section 3. We investigate completeness conditions §3.B. In §3.C we provide a geometric realization for the metric boundaries. In §4.A we show that connected sets can be approximated by path connected ones and use this to study their diameter. We examine metric disks in §4.B and verify that their boundaries are metric circles.

2. Preliminaries

Here we set forth our (relatively standard) notation and terminology, provide fundamental definitions, and present basic information.

There are scores of references for metric space geometry. A brief list includes the books: [BH99], [BBI01], [Hei01], [DS97], [Sem99], [Sem01]; see also the references mentioned in these works.

Our notation is relatively standard. We write $C = C(a, \dots)$ to indicate a constant C that depends only on the parameters $a, \dots$; the notation $A \lesssim B$ means there exists a finite constant c with $A \le cB$, and $A \simeq B$ means that both $A \lesssim B$ and $B \lesssim A$ hold. Typically $a, b, c, C, K, \dots$ will be constants that depend on various parameters, and we try to make this as clear as possible often giving explicit values, however, at times C will denote some constant whose value depends only on the data present and which may differ even on the same line of inequalities.

For real numbers a and b,

$$a \wedge b := \min\{a, b\} \quad \text{and} \quad a \vee b := \max\{a, b\}\,.$$

2.A. Metric Space Notation & Terminology. In what follows $(X, |\cdot|)$ always denotes a generic metric space possessing no additional presumed properties; so $|x - y|$ is the distance between the points x and y in X. (We are not assuming the existence of an underlying norm or any sort of group structure. If X is a normed vector space, we write $\|\cdot\|_X$ for the distance obtained from the norm.) It is convenient at times to allow 'infinite distance'—e.g. in definitions—however, when we use the term *metric* we always mean a finite-valued distance function.

Thus, for the record, to say that $|\cdot|$ is a metric means that $|\cdot| : X \times X \to \mathsf{R}$ is non-negative, non-zero off the diagonal of $X \times X$, symmetric, and satisfies the triangle inequality.

The *open ball* and *sphere* of radius r centered at the point x in X are

$$\mathsf{B}(x;r) := \{y : |x-y| < r\} \quad \text{and} \quad \mathsf{S}(x;r) := \{y : |x-y| = r\}$$

and then $\mathsf{B}[x;r] := \mathsf{B}(x;r) \cup \mathsf{S}(x;r)$ is the *closed ball*. When $B = \mathsf{B}(x;r)$ and $\lambda > 0$, $\lambda B := \mathsf{B}(x;\lambda r)$. The *open t-neighborhood* about $A \subset X$ is

$$\mathsf{N}(A;t) := \{x \in X \mid \operatorname{dist}(x,A) < t\} = \bigcup_{a \in A} \mathsf{B}(a;t)\,.$$

The *Hausdorff distance* between two (closed bounded) subsets A, B of X is

$$\mathsf{dist}_{\mathcal{H}}(A,B) := \inf\{t > 0 \mid A \subset \mathsf{N}(B;t) \text{ and } B \subset \mathsf{N}(A;t)\}\,.$$

A metric space is *proper* if it has the Heine-Borel property that every closed bounded subset of X is compact; equivalently, every closed ball is compact; thus, the compact sets are exactly the closed and bounded sets.

A *continuum* is a non-degenerate (so, more than a single point) compact connected space.

Recall that every metric space can be isometrically embedded into a complete metric space. See [Mun00, Theorem 43.7, p.269] or [HY88, Theorem 2-72, p.82]. We let $\bar{X}$ denote the *metric completion* of the metric space X; thus $\bar{X}$ is the closure of the image of X under such an isometric embedding. The *metric boundary* of X is $\partial X := \bar{X} \setminus X$. We write $\bar{\mathsf{B}}(\xi;r)$ to denote the *open* ball in $\bar{X}$ centered at ξ with radius r.

For a subset A of X we write $\mathsf{int}(A), \mathsf{cl}(A), \mathsf{bd}(A)$ to denote the *topological interior, closure, boundary* (respectively) of A. We note that when A is an open subspace of X, $\mathsf{bd}(A) \subset \partial A$ and equality holds if X is complete (but may not hold in general).

A metric space X is *locally complete* provided each point is an interior point of some complete subspace; that is, for each $x \in X$ there is a complete $C \subset X$ with $x \in \mathsf{int}(C)$. Since closed subspaces of complete spaces are complete, it is not hard to check that this is the same as requiring that for all $x \in X$, $\operatorname{dist}(x, \partial X) > 0$. (Equivalently, ∂X is closed in $\bar{X}$, or each point has an open neighborhood whose closure is complete.) For example, this holds when X is locally compact.

When d is some other distance on X, $\bar{X}_d$ and $\partial_d X := \bar{X}_d \setminus X$ denote the metric completion and metric boundary, respectively, of $X_d := (X,d)$. Also, $\mathsf{B}_d(x;r)$ and $\mathsf{S}_d(x;r)$ are the open ball and sphere (of radius r centered at the point x) in

X_d, and $\mathsf{int}_d(A), \mathsf{cl}_d(A), \mathsf{bd}_d(A)$ are the interior, closure, boundary (respectively) of A in X_d.

We require the following result that describes when one distance function is 'comparable' to another. The map ω indicated in item (c) can be chosen to be a homeomorphism; the astute reader recognizes that ω is a *local modulus of continuity*.

2.1. Lemma. *Let* $|\cdot|$ *and* $\|\cdot\|$ *be two distance functions on some set* X*. Fix a point* a *in* X*. The following are equivalent:*

(a) *The identity map* $(X, |\cdot|) \xrightarrow{\mathsf{id}} (X, \|\cdot\|)$ *is continuous at* a*.*
(b) *The map* $x \mapsto \|x-a\|$ *from* $(X, |\cdot|)$ *to* $(\mathsf{R}, \|\cdot\|_{\mathsf{R}})$ *is continuous at* a*.*
(c) *There is an increasing function* $(0,\infty) \xrightarrow{\omega} (0,\infty)$ *with* $\lim_{t\to 0^+} \omega(t) = 0$ *and such that*

$$\forall\, x \in X\,, \quad \|x-a\| \le \omega(|x-a|)\,.$$

Proof. We explain why (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) $\Longrightarrow$ (a). The first implication holds because every distance function is continuous. The last implication is easy to check. We outline an argument to verify the middle implication. Thus we assume that (b) holds. Then for each $t > 0$,

$$I(t) := \{r > 0 \mid |x-a| \le r \implies \|x-a\| \le t\} \ne \emptyset\,.$$

Since $r \in I(t)$ implies $(0, r] \subset I(t)$, we see that $I(t)$ is an interval of the form $(0, s)$ or $(0, s]$ where $s = \sup I(t)$.

Define $[0,\infty) \xrightarrow{\rho} [0,\infty)$ by $\rho(0) := 0$ and for $t > 0$, $\rho(t) := \sup\left(I(t) \cap [0,t]\right)$. Then ρ is increasing (i.e., non-decreasing), right-continuous, and for all $t > 0$, $\rho(t) > 0$.

Next, define $[0,\infty) \xrightarrow{\sigma} [0,\infty)$ by $\sigma(r) := \sup \rho^{-1}([0,r])$. Then σ is increasing (i.e., non-decreasing) and right-continuous with $\sigma(0) = 0$ and for all $r \ge 0$, $\rho(\sigma(r)) \ge r$.

We claim that $\omega(t) := \sigma(2t)$ has the desired properties. First, since σ is right-continuous,

$$\lim_{t\to 0^+} \omega(t) = \lim_{t\to 0^+} \sigma(t) = \sigma(0) = 0\,.$$

Next, given $x \in X$, put $r := |x-a|$ and $s := \omega(r) = \sigma(2r)$. Then

$$r < 2r \le \rho(\sigma(2r)) = \rho(s) \implies r \in I(s) \;\text{ (by the definition of } \rho)\,.$$

Therefore, by the definition of $I(s)$,

$$|x-a| = r \in I(s) \implies \|x-a\| \le s = \omega(|x-a|)\,. \qquad \blacksquare$$

2.B. Paths and Length. A *path* is a continuous map of a compact interval, unless explicitly indicated otherwise, and often we tacitly assume that the parameter interval is $[0,1]$. Thus a *path in* X is a continuous map $\gamma : [0,1] \to X$, unless explicitly indicated otherwise. Such a path γ *joins* $\gamma(0)$ *to* $\gamma(1)$, and when $\gamma(0) = \gamma(1)$ we call γ a *closed path*. Also, $|\gamma| := \gamma([0,1])$ is the *trajectory* (i.e., image) of the path γ. However, for points a, b in a vector space, we write $[a,b]$ both for the line segment joining a and b as well as the affine path $[0,1] \ni t \mapsto a + t(b-a)$; this abuse of notation means that $[a,b] = |[a,b]|$.

When α and β are paths that join a to b and b to c respectively, we write $\alpha \star \beta$ for the concatenation of α and β; so $\alpha \star \beta$ joins a to c. Also, the *reverse of* α is the path $\tilde{\alpha}$ defined by $\tilde{\alpha}(t) := \alpha(1-t)$ and going from $\alpha(1)$ to $\alpha(0)$. Of course, $|\alpha \star \beta| = |\alpha| \cup |\beta|$ and $|\tilde{\alpha}| = |\alpha|$.

We note that every path contains an injective subpath that joins its endpoints; see [Väi94].

The *length* of a path $[0,1] \xrightarrow{\gamma} X$ is defined in the usual way by

$$\ell(\gamma) := \sup \left\{ \sum_{i=1}^{n} |\gamma(t_i) - \gamma(t_{i-1})| \,:\, 0 = t_0 < t_1 < \cdots < t_n = 1 \right\} .$$

For a path $\mathsf{R} \supset I \xrightarrow{\gamma} X$ (with I any interval), $\ell(\gamma)$ is the supremum of all $\ell(\gamma|_K)$ where K is a compact subinterval of I. We call γ *rectifiable* when $\ell(\gamma) < \infty$, and X is *rectifiably connected* provided each pair of points in X can be joined by a rectifiable path. When A is an arc that is the trajectory $|\alpha|$ of some injective path α, we also write $\ell(A) := \ell(\alpha)$.

Väisälä's lecture notes [Väi71] provide an especially nice reference for issues related to paths; while the results are stated for paths in R^n, most still hold in the metric space setting. Other useful references include [BH99], [BBI01], [Pap05].

2.C. Connectivity Properties. Here we recall the notions of local connectivity, local path (or rectifiable) connectivity, quasiconvexity, and bounded turning. For the record, we say that points of X are *separated* by a set A if they lie in different components of $X \setminus A$. And then points are *joined* by A if A is connected and the points belong to A; thus points are joined by a path γ if they are joined by its trajectory $|\gamma|$.

A topological space X is *locally connected at* $x \in X$ provided each (open) neighborhood of X contains a connected (open) sub-neighborhood of x. In a metric space this is equivalent to asking that for each $t > 0$ there is an $r > 0$ such that points in $B(x;r)$ can be joined by a connected set in $B(x;t)$, that is, $B(x;r)$ lies in a component of $B(x;t)$. Then X is *locally connected* provided it has this property at each of its points, and a metric space X is *uniformly locally* *connected* if such an r can be chosen—for each $t > 0$—that is independent of location, i.e., that does not depend on the point x.

The definitions of (uniformly) *locally path connected* as well as (uniformly) *locally rectifiably connected* are obtained by replacing "connected" with "path connected" or with "rectifiably connected" in the above definition of (uniformly) locally connected.

A well-known classical result states that every locally connected connected compact space (i.e., every locally connected continuum, a so-called *peano space*) is path connected. The compactness hypothesis can be weakened to local compactness; see [Why64, (4.11), p.27] or [Why42, (5.2), p.38]. The following is probably folklore, at least among point set topologists. A proof can be modeled on the proof of [HY88, Theorem 3-17, p.118], but the reader should be prepared to fill in some details.

2.2. Fact. A connected, locally connected, locally complete, metric space is path connected and locally path connected.

A metric space satisfies the *bounded turning condition* if points can be joined by paths whose diameters are no larger than a fixed constant times the distance between the original points. To be precise, given $C \geq 1$, we say that X has the *C-bounded turning property* if each pair of points $x, y \in X$ can be joined by a path γ satisfying $\operatorname{diam}(\gamma) \leq C\,|x-y|$; we abbreviate this by declaring that X is C-BT. The bounded turning condition has a venerable position in quasiconformal analysis; see the references in [Geh82], [NV91], [Tuk96]. The connection between diameter distance and the bounded turning condition is explained in Lemma 3.10(e).

This property really should be called bounded turning with respect to *paths*, since there are more general related notions where one replaces 'joined by a path' with 'joined by a connected set' or 'joined by a continuum'; cf. [Tuk96]. Below we consider the related condition obtained by replacing 'joined by a path' with 'joined by a rectifiable path' and using arc length in place of diameter. In an ambient length space, for each $\varepsilon > 0$ one can always replace a continuum K that joins two points in some open set by a path γ that joins the same two points in the same open set with $\operatorname{diam}(\gamma) \leq (1+\varepsilon)\operatorname{diam} K$. Tukia established a far more interesting result in [Tuk96].

A metric space is *C-quasiconvex* if each pair of points can be joined by a rectifiable path γ with $\ell(\gamma) \leq C\,|x-y|$. Thus a metric space is a length space if and only if it is C-quasiconvex for each $C > 1$. A 1-quasiconvex metric space is usually called a *geodesic* space. The connection between length distance and quasiconvexity is explained in Lemma 3.11(e).

2.D. Accessible Boundary Points. Let $[0,1) \xrightarrow{\gamma} X$ be a path in X. Suppose there exists a point $\xi \in \partial X$ such that $\lim_{t\to 1^-} |\gamma(t) - \xi| = 0$. Then ξ is called a *path accessible* (metric) boundary point of X. In this situation, we can define $\gamma(1) := \xi$ and thereby obtain a path $\gamma : [0,1] \to X \cup \{\xi\} \subset \bar{X}$. We describe this by saying that γ is *a path in X with terminal endpoint $\xi \in \partial X$*.

We write $\partial^{\mathrm{pa}} X$ for the set of all path accessible boundary points of X and let $\mathcal{P}_{\mathrm{b}}(X)$ denote the collection of all paths in X with terminal endpoints in ∂X (so, in $\partial^{\mathrm{pa}} X$). Thus

$$\mathcal{P}_{\mathrm{b}}(X) := \{[0,1] \xrightarrow{\gamma} \bar{X} \mid \gamma \text{ continuous with } \gamma([0,1)) \subset X\,,\ \gamma(1) \in \partial X\}$$

and then

$$\begin{aligned}\partial^{\mathrm{pa}} X &:= \{\xi \in \partial X \mid \exists \text{ a path } [0,1) \xrightarrow{\gamma} X \text{ with } \lim_{t\to 1^-} |\gamma(t) - \xi| = 0\} \\ &= \{\gamma(1) \mid \gamma \in \mathcal{P}_{\mathrm{b}}(X)\}\,.\end{aligned}$$

In the above discussion we can restrict our attention to rectifiable paths γ which leads us to the *rectifiably accessible* (metric) boundary points of X, denoted by $\partial^{\mathrm{ra}} X$. Here we have

$$\mathcal{R}_{\mathrm{b}}(X) := \{[0,1] \xrightarrow{\gamma} \bar{X} \mid \gamma \text{ rectifiable with } \gamma([0,1)) \subset X\,,\ \gamma(1) \in \partial X\}$$

and then

$$\begin{aligned}\partial^{\mathrm{ra}} X &:= \{\xi \in \partial X \mid \exists \text{ a rect path } [0,1) \xrightarrow{\gamma} X \text{ with } \lim_{t\to 1^-} |\gamma(t) - \xi| = 0\} \\ &= \{\gamma(1) \mid \gamma \in \mathcal{R}_{\mathrm{b}}(X)\}\,.\end{aligned}$$

We call the natural maps $\mathcal{P}_{\mathrm{b}}(X) \xrightarrow{p} \partial^{\mathrm{pa}} X$ and $\mathcal{R}_{\mathrm{b}}(X) \xrightarrow{r} \partial^{\mathrm{ra}} X$, given by $p(\gamma) := \gamma(1)$ and $r(\gamma) := \gamma(1)$, the *terminal endpoint maps*. These maps are always surjective but not necessarily injective; of course $\partial^{\mathrm{ra}} X \subset \partial^{\mathrm{pa}} X \subset \partial X$ and either containment may be strict. We continue this discussion below, in §3.C, as it relates to the diameter distance and length distance boundaries.

We make use of the following information; for topological boundaries this is [HY88, Theorem 3-18, p.119].

2.3. Fact. Suppose X is a locally complete metric space with $\bar{X}$ locally path connected. Then $\partial^{\mathrm{pa}} X$ is dense in ∂X.

To check this, we start with an open neighborhood $U \subset \bar{X}$ of a point $\xi \in \partial X$; an appeal to Fact 2.2 provides a path connected sub-neighborhood V of ξ, so each point x in $X \cap V$ can be joined to ξ via a path γ, and then the first point of γ that lies in ∂X is a point in $\partial^{\mathrm{pa}} X \cap U$.

3. Properties of the Intrinsic Spaces

As stated in the Introduction, each metric space X admits natural intrinsic metrics, its *diameter distance*

$$d(x,y) := \inf\{\operatorname{diam}(\gamma) : \gamma \text{ a path in } X \text{ joining } x,y\}$$

and its *length distance*

$$l(x,y) := \inf\{\ell(\gamma) : \gamma \text{ a rectifiable path joining } x,y \text{ in } X\}\,.$$

Evidently, the basic inequalities

$$\forall\, x,y \in X\,, \quad |x-y| \le d(x,y) \le l(x,y) \tag{3.1}$$

always hold. The quantity $d(x,y)$, or $l(x,y)$, is finite precisely when the points x,y can be joined by a path, or by a rectifiable path, in X.

Henceforth, when discussing diameter distance, we tacitly assume our space is path connected, and likewise discussions about length distance will assume rectifiable connectivity. Recall that we write $X_d := (X,d)$, $X_l := (X,l)$ and also, e.g., $\bar{X}_d$ and $\partial_l X$ denote the metric completion and metric boundary of X_d and X_l respectively. The diameter and length distances are *intrinsic* distances in the sense that for any (path connected or rectifiably connected) metric space X we always have

$$(X_d)_d = X_d \quad \text{and} \quad (X_l)_l = (X_l)_d = (X_d)_l = X_l\,;$$

see Corollary 3.8 and also Corollary 4.3.

We could consider metric spaces that fail to be path connected (or rectifiably connected). In this setting, we would still have $X_d = X$, as sets, but now X_d would be the disjoint union of path components of X, and the diameter distance between points in different path components would be infinite. We leave this generalization to the interested reader.

Here we provide a detailed accounting of certain metric, geometric, and topological relations that exist among the spaces X, X_d, X_l. First we exhibit some basic elementary properties. Next we discuss completeness of these spaces. Then we determine the images and pre-images of the metric boundaries under the natural Lipschitz maps h,i,j (described in 3.B). Finally, we discuss a way to realize the metric boundaries $\partial_d X$ and $\partial_l X$ as equivalence classes of certain paths in X.

3.A. Basic Properties. First we record some information mentioned above.

3.2. Facts. Let $X = (X,|\cdot|)$ be a metric space with associated diameter distance and length distance spaces $X_d := (X,d)$ and $X_l := (X,l)$.

(a) When X is path connected, its diameter distance is a metric on X.

(b) When X is rectifiably connected, its length distance is a metric on X.

(c) The topology on X_l is finer than that on X_d which in turn is finer than the topology on X; the l-topology may be strictly finer than the d-topology, which may be strictly finer than the $|\cdot|$-topology.

We always have $l(x,y) = l(y,x) \geq 0$ and $d(x,y) = d(y,x) \geq 0$. According to the basic inequalities in (3.1), $l(x,y) = 0$ or $d(x,y) = 0$ holds if and only if $x = y$. The triangle inequalities follow in the usual way by concatenating paths. Examples 3.3 and 3.6 reveal, among other things, that there can be l-open sets that are not d-open, and d-open sets that are not $|\cdot|$-open.

Before continuing our discussion, we present some elementary examples. In addition to showing that the topologies on X_d or X_l may be strictly finer than the original metric topology on X, these are also used as 'building blocks' for more complicated examples presented later on.

3.3. Example. The *dyadic comb* space is the subspace of R^2 pictured in Figure 1, defined by

$$\mathsf{DC} := ([0,1] \times \{0\}) \cup (\{0\} \times [0,1]) \cup \bigcup_{n=0}^{\infty} (\{1/2^n\} \times [0,1]) \subset \mathsf{R}^2\,,$$

and equipped with Euclidean distance $|\cdot| := \|\cdot\|_{\mathsf{R}^2}$. This space is compact and rectifiably connected but not locally connected at any point $(0,y) \in \{0\} \times (0,1]$.

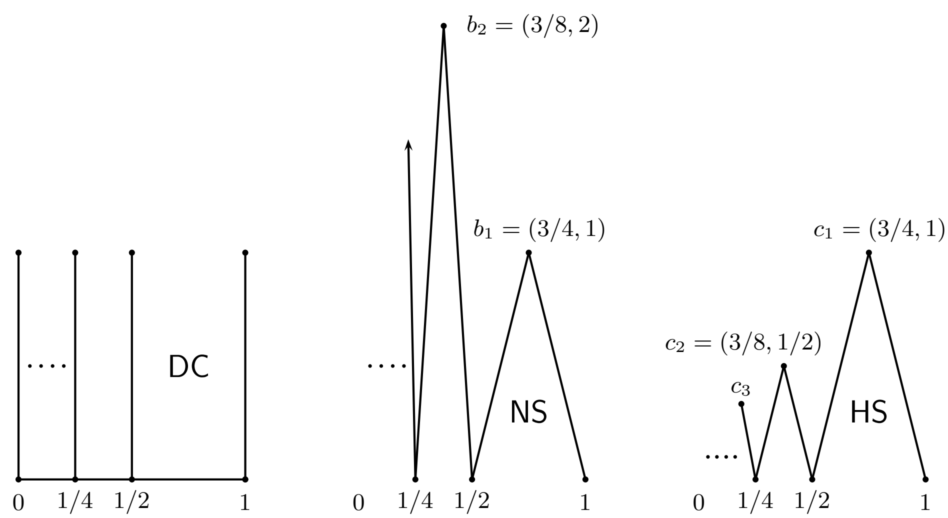

FIGURE 1. Dyadic Comb, Natural Sawtooth, Harmonic Sawtooth

However, DC_d is even locally rectifiably connected. Moreover, each such point $(0,y)$ has a d-open neighborhood that is l-open but is not $|\cdot|$-open.

3.4. Example. The *natural sawtooth* and *harmonic sawtooth* subspaces of R^2 are pictured in Figure 1, have $|\cdot| := \|\cdot\|_{\mathsf{R}^2}$, and are defined by

$$\mathsf{NS} := \bigcup_{n=1}^{\infty}([a_n, b_n] \cup [b_n, a_{n-1}]) \quad \text{and} \quad \mathsf{HS} := \bigcup_{n=1}^{\infty}([a_n, c_n] \cup [c_n, a_{n-1}])$$

where $a_n := (1/2^n, 0)$, $b_n := (3/2^{n+1}, n)$, $c_n := (3/2^{n+1}, 1/n)$. Both of these spaces are non-compact, non-complete, rectifiably connected and locally rectifiably connected.

3.5. Example. The *dyadic sawtooth* space is

$$\mathsf{DS} := |\sigma| \subset \mathsf{R}^2$$

with $|\cdot| := \|\cdot\|_{\mathsf{R}^2}$, where the path $[0,1] \xrightarrow{\sigma} \mathsf{R}^2$ is described in the proof. This space is compact, path connected and locally path connected, but not rectifiably connected; $\mathsf{DS} \setminus \{(0,0)\}$ is rectifiably connected and locally rectifiably connected.

Proof. For each fixed $n \in \mathsf{N}$, let $[0,1] \xrightarrow{f_n} [0,1]$ be the continuous piecewise linear map whose graph is pictured in Figure 2 and that satisfies $f_n(0) = f_n(1) = 0$ and has slopes

$$f_n' = 2^n \quad \text{on} \quad (0, 1/2^n) \cup (2/2^n, 3/2^n) \cup \cdots \cup ([2^n-2]/2^n, [2^n-1]/2^n)\,,$$
$$f_n' = -2^n \quad \text{on} \quad (1/2^n, 2/2^n) \cup (3/2^n, 4/2^n) \cup \cdots \cup ([2^n-1]/2^n, 1)\,;$$

thus, e.g., for all $i \in \{1, 3, 5, \ldots, 2^n - 1\}$ we have

$$f_n(i/2^n) = 1 \quad \text{and} \quad f_n((i \pm 1)/2^n) = 0\,.$$

The paths $\alpha_n : [0,1] \to \mathsf{R}^2$ defined by $\alpha_n(t) := (t, f_n(t))$ (i.e., the graphs of each f_n) join $(0,0)$ to $(1,0)$ and have $2^n \le \ell(\alpha_n) \le 2^{n+1}$ and $\operatorname{diam}(\alpha_n) \le 2$.

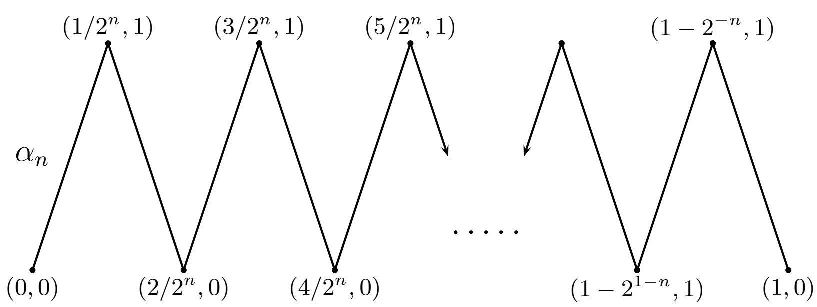

Figure 2. The graph of f_n, which is the path α_n

Define $\beta_n : [0,1] \to \mathsf{R}^2$ by

$$\beta_n(t) := (1/2^n, 0) + 2^{-n}\alpha_n(t) = (2^{-n} + 2^{-n}t, 2^{-n}f_n(t)) \,.$$

Then β_n joins $(1/2^n, 0)$ to $(1/2^{n-1}, 0)$ with $1 \le \ell(\beta_n) \le 2$ and $\operatorname{diam}(\beta_n) \le 2^{1-n}$.

Put $\mathsf{DS} := \{(0,0)\} \cup \bigcup_{n=1}^{\infty} |\beta_n|$; this is compact, path connected and locally path connected. Indeed, the (infinite) concatenation $\sigma := \cdots \star \beta_3 \star \beta_2 \star \beta_1$ is a well-defined continuous map from $(0,1]$ into R^2, since the appropriate endpoints coincide. That is, for all $n \in \mathsf{N}$,

$$\text{for } 1/2^n \le t \le 1/2^{n-1} \text{ we have} \quad \sigma(t) := \beta_n(2^n t - 1) = (t, 2^{-n} f_n(2^n t - 1))$$

and since $\beta_{n+1}(1) = 1/2^n = \beta_n(0)$, $\sigma : (0,1] \to \mathsf{R}^2$ is continuous. As $\operatorname{diam}(\beta_n) \to 0$, we can set $\sigma(0) := (0,0)$ to obtain a path in R^2 (i.e., a continuous map $\sigma : [0,1] \to \mathsf{R}^2$). Evidently, $\mathsf{DS} = |\sigma|$.

Since σ is not rectifiable, DS fails to be rectifiably connected. Evidently, the space $\mathsf{DS} \setminus \{(0,0)\}$ is rectifiably connected and locally rectifiably connected. ∎

It is worthwhile to note that

$$\mathsf{DS} = |\sigma| \subset \bigcup_{n=1}^{\infty} S_n \subset [0,1] \times [0,1/2]$$

where S_n is the square $S_n := [1/2^n, 1/2^{n-1}] \times [0, 1/2^n]$.

Now we combine the above examples to discuss the following.

3.6. Example. The space

$$X := \mathsf{DS} \cup \mathsf{DC}' \subset \mathsf{R}^2 \quad \text{where} \quad \mathsf{DC}' := \mathsf{DC} - (0,1)$$

(with $|\cdot| := \|\cdot\|_{\mathsf{R}^2}$) is compact and rectifiably connected, but not locally connected. Moreover, there is a path $\sigma : [0,1] \to X$ such that the map $\sigma : [0,1] \to X_l$ is not continuous. Hence neither identity map $\mathsf{id} : X_l \to X_d$ nor $\mathsf{id} : X_l \to X$ is a homeomorphism. In particular, the topology on X_l is strictly finer than the topology on X_d which in turn is strictly finer than the original metric topology on X.

Proof. Note that

$$\mathsf{DC}' := \mathsf{DC} - (0,1) := \{z - (0,1) \mid z \in \mathsf{DC}\} \,.$$

We attach this vertically translated dyadic comb space to the bottom of the dyadic sawtooth space DS to ensure that $(0,0)$ is at finite length distance from each point of X, and also to produce d-open sets that are not $|\cdot|$-open.

Let $[0,1] \xrightarrow{\sigma} X$ be the path defined in Example 3.5. Put $\tau_n := 3/2^{n+1}$. Noting that $\sigma(\tau_n) = (\tau_n, 2^{-n} f_n(1/2))$, which is the arclength-midpoint of $|\beta_n|$, we see that

$$l(\sigma(\tau_n), \sigma(0)) \geq \frac{1}{2}\,\ell(\beta_n) \geq \frac{1}{2}\,.$$

Since $\tau_n \to 0$, this reveals that $[0,1] \xrightarrow{\sigma} X_l$ is not continuous.

Finally, $(0,0)$ has an l-open neighborhood that is not d-open, and each point $(0,y)$ in $\{0\} \times (-1,0)$ has a d-open neighborhood that is not $|\cdot|$-open. ■

In view of Example 3.6, it is worthwhile to know when a path in X is also a path in X_d or in X_l.

3.7. Lemma. *Let $I \subset \mathsf{R}$ be an interval and $X = (X, |\cdot|)$ a metric space with associated diameter and length distance spaces $X_d := (X,d)$ and $X_l := (X,l)$. Then*

$$I \xrightarrow{\gamma} X_l \text{ continuous} \implies I \xrightarrow{\gamma} X_d \text{ continuous} \iff I \xrightarrow{\gamma} X \text{ continuous}$$

and when γ is rectifiable

$$I \xrightarrow{\gamma} X \text{ continuous} \implies I \xrightarrow{\gamma} X_l \text{ continuous}$$

but this latter implication need not always hold.

Proof. The first two $\implies$ implications follow from continuity of the maps $X_l \xrightarrow{\mathsf{id}} X_d \xrightarrow{\mathsf{id}} X$. Suppose $\mathsf{R} \supset I \xrightarrow{\gamma} X$ is continuous. Fix $t_0 \in I$ and let $\varepsilon > 0$ be given. Choose $\delta > 0$ so that

$$t \in I \cap (t_0 - \delta, t_0 + \delta) \implies |\gamma(t) - \gamma(t_0)| < \varepsilon/2\,.$$

Let $t \in I \cap (t_0 - \delta, t_0 + \delta)$. Then for all $r, s \in [t, t_0]$,

$$|\gamma(r) - \gamma(s)| \leq |\gamma(r) - \gamma(t_0)| + |\gamma(s) - \gamma(t_0)| < \varepsilon/2 + \varepsilon/2 = \varepsilon\,.$$

Thus $d(\gamma(t), \gamma(t_0)) \leq \operatorname{diam}(\gamma|_{[t,t_0]}) < \varepsilon$. So, $I \xrightarrow{\gamma} X_d$ is continuous.

A similar argument gives continuity of $I \xrightarrow{\gamma} X_l$ when γ is rectifiable. Example 3.6 reveals that this need not hold if γ is not rectifiable. ■

The following explains why the diameter and length distances are said to be *intrinsic*. See also Corollary 4.3 and the comments that immediately follow.

3.8. Corollary. *For any (path/rectifiably connected) metric space X,*

$$(X_d)_d = X_d \quad \textit{and} \quad (X_l)_l = (X_l)_d = (X_d)_l = X_l\,.$$

Proof. For a given metric ρ, we write d_ρ and l_ρ for the ρ-diameter and ρ-length distances. Thus the basic inequalities in (3.1) say that $\rho \le d_\rho \le l_\rho$. This provides immediate information about the (possibly different) diameters and lengths of a path, computed with respect to each of the (probably different) metrics ρ, d_ρ, l_ρ.

Since each path in X is a path in X_d, X_d is path connected whenever X is, so as sets, $X = X_d = (X_d)_d$. We claim that the identity map $(X_d)_d \to X_d$ is an isometry. Let $\gamma : [0,1] \to X_d$ be a path. Then γ is also a path in X. For any $s, t \in [0,1]$,

$$d(\gamma(s), \gamma(t)) \le \operatorname{diam}(\gamma|_{[s,t]}) \le \operatorname{diam}(\gamma)\,, \quad \text{so} \quad \operatorname{diam}_d(\gamma) \le \operatorname{diam}(\gamma) \le \operatorname{diam}_d(\gamma)\,.$$

Taking an infimum over all paths joining given points $x, y \in X_d$ we obtain $d_d(x,y) = d(x,y)$.

Since each rectifiable path in X is rectifiable in X_d and in X_l, whenever X is rectifiably connected, so are X_d and X_l. Let $\gamma : [0,1] \to X$ be a rectifiable path. For any partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of $[0,1]$,

$$\begin{aligned}\sum_1^n |\gamma(t_i) - \gamma(t_{i-1})| &\le \sum_1^n d(\gamma(t_i), \gamma(t_{i-1})) \le \sum_1^n l(\gamma(t_i), \gamma(t_{i-1})) \\ &\le \sum_1^n \ell(\gamma|_{[t_i, t_{i-1}]}) \le \ell(\gamma)\,.\end{aligned}$$

Taking a supremum over all such partitions gives $\ell(\gamma) = \ell_d(\gamma) = \ell_l(\gamma)$. Taking an infimum over all rectifiable paths joining given points $x, y \in X_d$ yields $l(x,y) = l_d(x,y) = l_l(x,y)$.

Finally, by taking $\rho = l$ in the basic inequality $\rho \le d_\rho \le l_\rho$ we deduce that $l \le d_l \le l_l = l$. ∎

3.9. Remark. Notice that in the above proof of Corollary 3.8 we demonstrated that for any (rectifiable) path γ in X,

$$\operatorname{diam}(\gamma) = \operatorname{diam}_d(\gamma) \quad \text{and} \quad \ell(\gamma) = \ell_d(\gamma) = \ell_l(\gamma)\,.$$

In light of Lemma 3.7 above, it is worthwhile to know when one of the three identity maps

$$X_l \xrightarrow{\mathsf{id}} X_d \quad \text{and} \quad X_d \xrightarrow{\mathsf{id}} X \quad \text{and} \quad X_l \xrightarrow{\mathsf{id}} X$$

is a homeomorphism. Needless to say, in each case here the question is whether or not the appropriate map id^{-1} is continuous. Lemma 2.1 provides various ways to check if this holds.

Interestingly, while local path connectivity ensures that $\mathsf{id} : X_d \to X$ is a homeomorphism, the obvious analog (i.e., local rectifiable connectivity) does not guarantee that $\mathsf{id} : X_l \to X$ is a homeomorphism; see Example 3.12. It is true that local rectifiable connectivity is a necessary condition for $\mathsf{id} : X_l \to X$ to be a homeomorphism; this follows easily from Lemma 3.11 stated below. Moreover, if $\mathsf{id} : X_l \to X_d$ is a homeomorphism, then X_d is locally rectifiably connected, but it could be that X is not locally connected; for example, the dyadic comb space DC enjoys these properties.

3.10. Lemma. *Let $X = (X, |\cdot|)$ be a path connected metric space with associated diameter distance space $X_d := (X, d)$.*

(a) *For each $C > 1$, X_d is C-bounded turning.*

(b) *Each open ball $\mathsf{B}_d(a; r)$ in X_d is path connected. In particular, X_d is "1"-uniformly locally path connected.*

(c) *The identity map $X_d \xrightarrow{\mathsf{id}} X$ is a homeomorphism (i.e., id^{-1} is continuous) if and only if X is locally path connected.*

(d) *The inverse of the identity map $X_d \xrightarrow{\mathsf{id}} X$ (i.e., id^{-1}) is uniformly continuous if and only if X is uniformly locally path connected.*

(e) *The identity map $X_d \xrightarrow{\mathsf{id}} X$ is bilipschitz (i.e., id^{-1} is Lipschitz) if and only if X is bounded turning. If X is C-bounded turning, then this map is C-bilipschitz, and if the map is K-bilipschitz, then for each $C > K$, X_d is C-bounded turning.*

Proof. If $d(x, y) < r$, there is a path γ in X that joins x, y with $\operatorname{diam}(\gamma) < r$; this gives (**a**), (**b**), and the latter half of (**e**).

To verify (**c**), first suppose X is locally path connected. Fix a point $a \in X$ and let $t > 0$. Since $\mathsf{B}(a; t/2)$ is open, there is a path connected open set U with $a \in U \subset \mathsf{B}(a; t/2)$. Choose $r > 0$ so that $\mathsf{B}(a; r) \subset U$. Let $x \in \mathsf{B}(a; r)$. Then there is a path γ joining a, x in U. Clearly $d(x, a) \le \operatorname{diam}(|\gamma|) < t$. Thus id^{-1} is continuous.

Conversely, suppose id^{-1} is continuous. Again, fix a point $a \in X$ and let $t > 0$. Choose $r > 0$ so that $x \in \mathsf{B}(a; r)$ implies $d(x, a) < t/2$. Then given $x \in \mathsf{B}(a; r)$, there is a path γ joining a, x with $\operatorname{diam}(|\gamma|) < t/2$, so $|\gamma| \subset \mathsf{B}(a; t)$. Thus $\mathsf{B}(a; r)$ lies in a path component of $\mathsf{B}(a; t)$. This means that for each open neighborhood U of a there is a path connected set P with $a \in \mathsf{int}(P)$ and $P \subset U$. It now follows that path-components of open sets are open. This is equivalent to local path connectedness.

The assertion in (**d**) follows by similar reasoning. It remains to establish (**e**). If X_d is C-bounded turning, then for all $x, y \in X$ there is a path γ in X that joins x, y with

$$d(x, y) \le \operatorname{diam}(\gamma) \le C\,|x - y| \le C\,d(x, y)$$

so the identity map is C-bilipschitz. Conversely, if this map is K-bilipschitz and $C > K$, then for all $x, y \in X$

$$d(x,y) \le K\,|x-y| < C\,|x-y|$$

so there is a path γ in X that joins x, y with $\text{diam}(\gamma) \le C\,|x-y|$ and therefore X_d is C-bounded turning. ∎

Since a uniformly continuous map has a uniformly continuous extension to the completions of its domain and target spaces, in (d) and (e) above we see that id would have a homeomorphic extension mapping $\bar{X}_d$ onto $\bar{X}$. In particular, for a bounded turning space X we find that $\bar{X}$ is also bounded turning. We also find that all boundary points are path accessible. See Proposition 3.18(d,e). Similar comments apply to the result below; e.g., the metric completion of a quasiconvex space is quasiconvex, and all of its boundary points are rectifiably accessible.

Here is the length distance version of the above. As already noted, item (c) below is significantly different from that above; this condition is readily verified by recalling the definition of the length distance and appealing to Lemma 2.1.

3.11. Lemma. *Let $X = (X, |\cdot|)$ be a rectifiably connected metric space with associated length distance space $X_l := (X, l)$.*

(a) *X_l is a length space.*

(b) *Each open ball $\mathsf{B}_l(a;r)$ in X_l is rectifiably connected. In particular, X_l is "1"-uniformly locally rectifiably connected.*

(c) *The identity map $X_l \xrightarrow{\mathsf{id}} X$ is a homeomorphism (i.e., id^{-1} is continuous) if and only if for each point $a \in X$ there is an increasing function $\omega : (0,\infty) \to (0,\infty)$ with $\lim_{t\to 0^+} \omega(t) = 0$ and such that for each point $x \in X$ there exists a rectifiable path γ in X that joins a, x with $\ell(\gamma) \le \omega(|x-a|)$.*

(d) *The inverse of the identity map $X_l \xrightarrow{\mathsf{id}} X$ (i.e., id^{-1}) is uniformly continuous if and only if the necessary condition in* (c) *holds uniformly, meaning that there is one $\omega : (0,\infty) \to (0,\infty)$ that 'works' for each point a.*

(e) *The identity map $X_l \xrightarrow{\mathsf{id}} X$ is bilipschitz (i.e., id^{-1} is Lipschitz) if and only if X is quasiconvex. If X is C-quasiconvex, then this map is C-bilipschitz, and if the map is K-bilipschitz, then for each $C > K$, X_d is C-quasiconvex.*

3.12. Example. There is a space $X \subset \mathsf{R}^2$ (with $|\cdot| := \|\cdot\|_{\mathsf{R}^2}$) that is compact, rectifiably connected, and also *locally rectifiably connected*, but neither identity map $\mathsf{id} : X_l \to X_d$ nor $\mathsf{id} : X_l \to X$ is a homeomorphism.

Proof. Let $[0,1] \xrightarrow{\alpha_n} \mathsf{R}^2$ be defined by $\alpha_n(t) := (t, f_n(t))$ (i.e., the graphs of each f_n) as in Example 3.5 and pictured in Figure 2; so α_n joins $(0,0)$ to $(1,0)$ in $[0,1]\times[0,1]$ with $2^n \le \ell(\alpha_n) \le 2^{n+1}$ and $\operatorname{diam}(\alpha_n) \le 2$. Define $\beta_n : [0,1] \to \mathsf{R}^2$ by

$$\beta_n(t) := (0, 1/2^{n-1}) + 2^{-n}\alpha_n(t) = \left(2^{-n}t, 2^{1-n} + 2^{-n} f_n(t)\right);$$

so β_n joins $(0, 1/2^{n-1})$ to $(1/2^n, 1/2^{n-1})$ inside $[0, 1/2^n] \times [1/2^{n-1}, 3/2^n]$ with $1 \le \ell(\beta_n) \le 2$ and $\operatorname{diam}(\beta_n) \le 2^{1-n}$. Thus, for distinct $m, n \in \mathsf{N}$, $|\beta_m| \cap |\beta_n| = \emptyset$.

Now put

$$X := (\{0\}\times[0,1]) \cup \bigcup_{n=1}^{\infty} \left(|\beta_n| \cup \left(\{1/2^n\} \times [0, 1/2^{n-1}]\right)\right) .$$

This is compact, rectifiably connected, and also locally rectifiably connected. Since the points $a_n := (1/2^n, 0)$, $a_0 = (0,0)$ satisfy

$$|a_n - a_0| \to 0 \ \text{ as } n \to \infty\,, \quad \text{whereas for all } n \in \mathsf{N}\,,\ l(a_n, a_0) \ge \ell(\beta_n) \ge 1\,,$$

we deduce that $X \xrightarrow{\mathsf{id}^{-1}} X_l$ is not continuous at a_0. ∎

3.B. Completeness and Images of Boundaries. The identity maps

$$X_l \xrightarrow{\mathsf{id}} X_d \xrightarrow{\mathsf{id}} X$$

are 1-Lipschitz (according to the basic inequalities in (3.1)) and therefore have 1-Lipschitz extensions $h, i, j = i \circ h$ to the metric completions as pictured below. In general, h, i, j need not be surjective nor injective. However, we always have

$$i(\partial_d X) = \partial^{\mathrm{pa}} X \quad \text{and} \quad j(\partial_l X) = \partial^{\mathrm{ra}} X\,.$$

$$\begin{array}{ccc} \bar{X}_d & \xrightarrow{\ i\ } & \bar{X} \\ {\scriptstyle h}\nwarrow & & \nearrow{\scriptstyle j} \\ & \bar{X}_l & \end{array}$$

(Recall from §2.D that $\partial^{\mathrm{pa}} X$ and $\partial^{\mathrm{ra}} X$ are the sets of *path accessible* and *rectifiably accessible* points of ∂X.) These identities provide a means for determining when X_d or X_l is complete. These facts, and others, are corroborated below. Before pursuing this, we illustrate how the maps h, i, j may be non-injective or non-surjective.

The spaces given in Examples 3.13 and 3.14 are pictured in Figure 3.

3.13. Example. Let $X := \mathsf{D} \setminus ([0,1]\times\{0\})$ where D is the open Euclidean unit disk in R^2. Then each point $(r,0) \in (0,1] \times \{0\}$ corresponds to two distinct points in $\partial_d X$ or in $\partial_l X$. Thus the maps $i : \bar{X}_d \to \bar{X}$ and $j : \bar{X}_l \to \bar{X}$ fail to be injective on $\partial_d X$ and on $\partial_l X$, respectively.

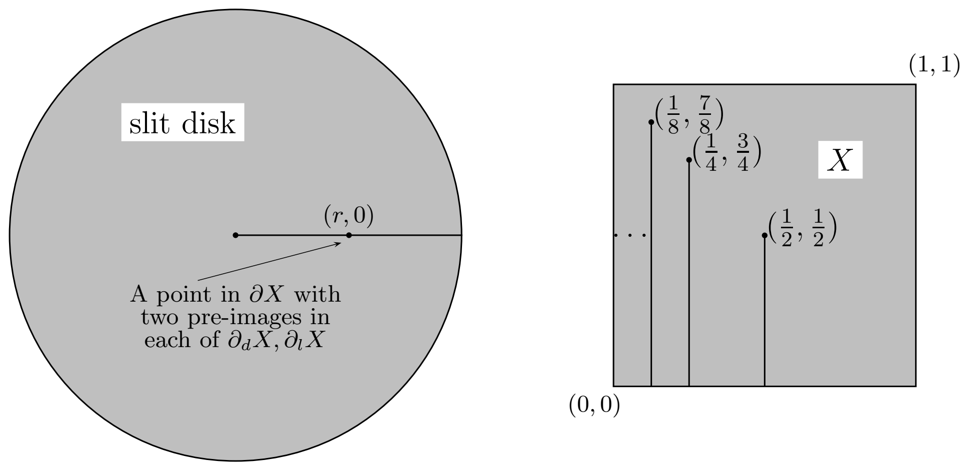

FIGURE 3. Spaces with i and j non-injective and/or non-surjective

3.14. Example. Let $X := (0,1)^2 \setminus \bigcup_n I_n$ where $(0,1)^2$ is the open Euclidean unit square in R^2 and for each $n \in \mathsf{N}$, $I_n := \{2^{-n}\} \times [0, 1-2^{-n}]$. As above, each point of I_n, except for its tip, corresponds to two points in $\partial_d X$ or in $\partial_l X$. Moreover, there are no points of $\partial_d X$, nor of $\partial_l X$, that correspond to the points $(0,y) \in \{0\} \times [0,1) \subset \partial X$. Thus here $i : \bar{X}_d \to \bar{X}$ and $j : \bar{X}_l \to \bar{X}$ are both neither injective nor surjective.

3.15. Example. Let $X := \mathsf{DS} \setminus \{(0,0),(1,0)\}$. Then $\partial X = \{(0,0),(1,0)\} = \partial_d X$ while $\partial_l X = \{(1,0)\}$. Thus here $h : \bar{X}_l \to \bar{X}_d$ and $j : \bar{X}_l \to \bar{X}$ are not surjective.

3.16. Example. Let

$$X := \Big(([-1,0) \cup (0,1]) \times \{0\} \Big) \cup (\{-1,1\} \times [0,1]) \cup ([-1,1] \times \{1\})$$
$$\bigcup_{n=1}^{\infty} \big(\{-1/2^n, 1/2^n\} \times [0, 1/2^n] \big) \cup \bigcup_{n=1}^{\infty} \mathsf{DDS}_n \,.$$

Thus we start by removing the origin from the rectangle $\partial([-1,1] \times [0,1])$, then we attach dyadic 'sticks', and then we attach the spaces DDS_n. Here DDS_n is a similarity copy of the the *doubled dyadic saw* space

$$\mathsf{DDS} := \mathsf{DS} \cup \mathsf{DS}^* \quad \text{where } \mathsf{DS}^* := \{(-x,y) \mid (x,y) \in \mathsf{DS}\}\,.$$

To get DDS_n, we first scale DDS by $1/2^n$ and then translate it so that DDS_n joins the point $(-1/2^n, 1/2^n)$ to $(1/2^n, 1/2^n)$.

Then $\partial X = \{(0,0)\} = \partial_d X$ whereas $\partial_l X$ consist of exactly two points. Thus here neither $h : \bar{X}_l \to \bar{X}_d$ nor $j : \bar{X}_l \to \bar{X}$ is injective.

Below we examine completeness properties of the spaces X, X_d, X_l. For example, completeness of X implies that X_d is complete which in turn implies completeness of X_l (see Remarks 3.19(b) and 3.23(b)). On the other hand, it is easy to see that the natural sawtooth space NS (see Example 3.4) is non-complete, whereas both NS_d and NS_l are complete. A bounded example of such a space is provided by removing the "limit tooth" of the dyadic comb; $\mathsf{DC}^* := \mathsf{DC} \setminus (\{0\} \times (0,1])$ is non-complete but both DC^*_d and DC^*_l are complete. It is also easy to check that the harmonic sawtooth space HS (see Example 3.4) and its associated diameter distance space HS_d are both non-complete, whereas HS_l is complete. A similar example is obtained by removing the origin from the dyadic sawtooth space (see Example 3.5): the space $\mathsf{DS} \setminus \{(0,0)\}$ has a complete length distance, but both its Euclidean and diameter distances are non-complete.

We make repeated appeals to the following elementary fact. Recall that $\mathcal{P}_{\rm b}(X)$ is the collection of all paths in X with terminal endpoints in ∂X. See §2.D.

3.17. Lemma. *Let $X = (X, |\cdot|)$ be a path connected metric space with associated diameter distance space $X_d = (X, d)$. Suppose $[0,1) \xrightarrow{\gamma} X$ is a path in X. The following are equivalent.*

(a) $\displaystyle\lim_{s,t\to 1^-} |\gamma(s) - \gamma(t)| = 0\,.$

(b) $\displaystyle\lim_{s,t\to 1^-} d(\gamma(s), \gamma(t)) = 0\,.$

(c) $\displaystyle\lim_{s,t\to 1^-} \operatorname{diam}(\gamma|_{[s,t]}) = 0\,.$

(d) *There exists a point $\xi \in \bar{X}$ such that* $\displaystyle\lim_{t\to 1^-} |\gamma(t) - \xi| = 0\,.$

(e) *There exists a point $\zeta \in \bar{X}_d$ such that* $\displaystyle\lim_{t\to 1^-} d(\gamma(t), \zeta) = 0\,.$

When these hold, we have paths

$$[0,1] \xrightarrow{\gamma} X \cup \{\xi\} \subset \bar{X} \quad \text{and} \quad [0,1] \xrightarrow{\gamma_d} X \cup \{\zeta\} \subset \bar{X}_d \quad \text{with} \quad \gamma = i \circ \gamma_d$$

that are obtained by defining

$$\gamma(1) := \xi \quad \text{and} \quad \gamma_d(t) := \begin{cases} \gamma(t) & \text{for } t \in [0,1)\,, \\ \zeta & \text{for } t = 1\,; \end{cases}$$

moreover, $\xi \in \partial X$ if and only if $\zeta \in \partial_d X$, and then $\gamma \in \mathcal{P}_{\rm b}(X)$ and $\gamma_d \in \mathcal{P}_{\rm b}(X_d)$.

Proof. Assume (**a**) holds. Then for any sequence $(t_n)_1^\infty$ in $[0,1)$ with $t_n \to 1$, $(\gamma(t_n))_1^\infty$ is a Cauchy sequence in X and hence converges to some point $\xi \in \bar{X}$. It is straightforward to check that the point ξ does not depend on the choice of $(t_n)_1^\infty$; therefore, (**d**) holds.

Assume (**d**) holds. Setting $\gamma(1) := \xi$, we obtain a path $[0,1] \xrightarrow{\gamma} X \cup \{\xi\} \subset \bar{X}$. Condition (**c**) follows by using the uniform continuity of γ. Indeed, given $\varepsilon > 0$, pick $\delta \in (0,1)$ so that

$$\forall\, s,t \in [0,1]\,, \quad |s-t| < \delta \implies |\gamma(s) - \gamma(t)| < \varepsilon\,.$$

Then for all $u, v \in [s,t] \subset (1-\delta, 1)$, $|\gamma(u) - \gamma(v)| < \varepsilon$, so $\operatorname{diam}(\gamma|_{[s,t]}) \le \varepsilon$.

Clearly (**c**) $\implies$ (**b**) and (**e**) $\implies$ (**a**). That (**b**) $\implies$ (**e**) is just the above argument that (**a**) $\implies$ (**d**), but now applied to the metric space X_d.

The maps $[0,1] \xrightarrow{\gamma} X \cup \{\xi\}$ and $[0,1] \xrightarrow{\gamma_d} X \cup \{\zeta\}$, as defined, are continuous—so, are paths—provided (**d**) and (**e**) hold; cf. Lemma 3.7. To see that $\xi = i(\zeta)$, and hence that $\gamma = i \circ \gamma_d$, we use (**d**) and (**e**) (i.e., the continuity of γ and γ_d) as follows:

$$|\xi - i(\zeta)| \le |\xi - \gamma(t)| + |\gamma(t) - i(\zeta)| \le |\xi - \gamma(t)| + d(\gamma_d(t), \zeta) \to 0 \quad \text{as } t \to 1^-\,.$$

Finally, if $\zeta \in X_d = X$, then clearly $\xi = i(\zeta) = \zeta \in X$. Suppose that $\xi \in X$. Then γ is a path in X (with terminal endpoint ξ). According to Lemma 3.7, γ is also a path in X_d, so its terminal endpoint as such belongs to X_d. Condition (**e**) asserts that ζ is the terminal endpoint of γ as a path in X_d, so $\zeta \in X_d$. ∎

Here is some basic information pertaining to the map $\bar{X}_d \xrightarrow{i} \bar{X}$. Note that item (b) below provides an especially important connection between $\partial_d X$ and $\partial^{\mathrm{pa}} X$. This in turn reveals how to tell whether or not X_d is a complete metric space. A useful example to keep in mind is the space $\mathsf{DC}^* := \mathsf{DC} \setminus (\{0\} \times (0,1])$ that is non-complete but has complete diameter and length distances.

3.18. Proposition. *Let $X = (X, |\cdot|)$ be a path connected metric space with associated diameter distance space $X_d := (X, d)$ and let $i : \bar{X}_d \to \bar{X}$ be the extension of the identity map $\mathsf{id} : X_d \to X$.*

(a) *The map i is 1-Lipschitz and $i[\bar{\mathsf{B}}_d(\zeta; r)] \subset \bar{\mathsf{B}}(i(\zeta); r) \subset \bar{X}$.*

(b) *We always have $i(\partial_d X) = \partial^{\mathrm{pa}} X \subset \partial X$.*

(c) *Each point $\xi \in i(\bar{X}_d)$ has a pre-image $i^{-1}(\xi)$ that is totally disconnected.*

(d) *If X is uniformly locally path connected, then $\bar{X}_d \xrightarrow{i} \bar{X}$ is a homeomorphism; in particular, $\partial X = i(\partial_d X) = \partial^{\mathrm{pa}} X$.*

(e) *If X is bounded turning, then so is $\bar{X}_d$ and then $\bar{X}_d \xrightarrow{i} \bar{X}$ is bilipschitz.*

Proof. The basic inequalities (3.1) assert that the identity map $X_d \xrightarrow{\mathsf{id}} X$ is 1-Lipschitz, so its extension to the metric completions also has this property. Thus (**a**) holds.

Let $(x_n)_0^\infty$ be a Cauchy sequence in X_d. By extracting a subsequence, if necessary, we may assume that $d(x_n, x_{n-1}) < 1/2^n$. For each $n \in \mathsf{N}$, select a path γ_n in X that joins x_{n-1} to x_n with $\operatorname{diam}(\gamma_n) < 1/2^n$.

The (infinite) concatenation $\gamma := \gamma_1 \star \gamma_2 \star \dots$ is a well-defined continuous map $[0,1) \xrightarrow{\gamma} X$. Indeed, put $t_n = 1 - 1/2^n$ (so $t_0 = 0$, $t_1 = 1/2$, $t_2 = 3/4$, etc.) and assume γ_n is defined on $[t_{n-1}, t_n]$. Then $\gamma(t) = \gamma_n(t)$ for $t \in [t_{n-1}, t_n]$. Since $\gamma_n(t_n) = x_n = \gamma_{n+1}(t_n)$, γ is indeed well-defined and continuous on $[0,1)$.

Thanks to Lemma 3.7 we know that $[0,1) \xrightarrow{\gamma_d := \gamma} X_d$ is also continuous.

We claim that $\lim_{s,t\to 1^-} \operatorname{diam}(\gamma|_{[s,t]}) = 0$. To see this, let $\varepsilon > 0$ be given. Pick N so that $1/2^N = 1 - t_N < \varepsilon$. Suppose $s, t \in (t_N, 1)$. Then $\gamma[s,t] \subset \bigcup_{n>N} |\gamma_n|$, so

$$\operatorname{diam}(\gamma|_{[s,t]}) \le \operatorname{diam}(\bigcup_{n>N} |\gamma_n|) \le \sum_{n=N+1}^{\infty} \operatorname{diam}(\gamma_n) \le \sum_{n=N+1}^{\infty} \frac{1}{2^n} = \frac{1}{2^N} < \varepsilon\,.$$

An appeal to Lemma 3.17 now provides points $\xi \in \bar{X}$ and $\zeta \in \bar{X}_d$ such that by defining $\gamma(1) := \xi$ and $\gamma_d(1) := \zeta$ we obtain paths $\gamma : [0,1] \to X \cup \{\xi\}$ and $\gamma_d : [0,1] \to X_d \cup \{\zeta\}$ with $\gamma = i \circ \gamma_d$. Also, $\xi \in \partial X$ if and only if $\zeta \in \partial_d X$ (in which case $\gamma \in \mathcal{P}_{\mathrm{b}}(X)$ and $\gamma_d \in \mathcal{P}_{\mathrm{b}}(X_d)$).

Now we corroborate the assertion in (**b**). We start with some point $\zeta \in \partial_d X$, then—as above—we choose $(x_n)_0^\infty$ so that $d(x_n, \zeta) \to 0$, and then construct paths $\gamma \in \mathcal{P}_{\mathrm{b}}(X)$ and $\gamma_d \in \mathcal{P}_{\mathrm{b}}(X_d)$ with $\gamma_d(1) = \zeta$ and $\xi := \gamma(1) = i \circ \gamma_d(1) = i(\zeta)$. It follows that $\zeta \in \partial_d^{\mathrm{pa}} X$ (which, by the way, proves that $\partial_d^{\mathrm{pa}} X = \partial_d X$) and that $\xi \in \partial^{\mathrm{pa}} X$, so $i(\partial_d X) \subset \partial^{\mathrm{pa}} X$.

That $i(\partial_d X) \supset \partial^{\mathrm{pa}} X$ follows from Lemma 3.17. Indeed, given a path accessible boundary point ξ of X, there is a path γ in X with terminal endpoint ξ, and then by Lemma 3.17 there is $\zeta := \gamma_d(1) \in \partial_d X$ with $i(\zeta) = \xi$.

To facilitate its comprehension, we postpone the proof of (**c**) until after Proposition 3.26; see 3.27.

To check that (**d**) holds, suppose that X is uniformly locally path connected. Then by Lemma 3.10(d), $\mathsf{id}^{-1} : X \to X_d$ is uniformly continuous, so it has a uniformly continuous extension $k : \bar{X} \to \bar{X}_d$. We claim that $k \circ i$ and $i \circ k$ are just the identity maps on $\bar{X}_d$ and $\bar{X}$ respectively. Therefore, i is indeed a homeomorphism. Since $i(X_d) = X$, $i(\partial_d X) = \partial X$.

Finally, by using Lemma 3.10(e), it is straightforward to see that (**e**) holds. ∎

3.19. Remarks. (**a**) Note that the above proof of (b) reveals that $\partial_d^{\mathrm{pa}}X = \partial_d X$. (**b**) The fact that $i(\partial_d X) = \partial^{\mathrm{pa}}X$ tells us that X_d is complete if and only if $\partial^{\mathrm{pa}}X = \emptyset$. In particular, if X is complete, so is X_d. (**c**) It is not difficult to give a constructive proof for item (d) above; that is, to show that each point of ∂X is path accessible, provided X is uniformly locally path connected. Assume this property holds, and let $\xi \in \partial X$ be given. We mimic the reasoning at the beginning of the proof of Proposition 3.18.

There are $r_n > 0$ such that for all points $a \in X$, each point in $\mathsf{B}(a; r_n)$ can be joined to a via a path in $\mathsf{B}(a; 1/2^{n+1})$. Now let $(x_n)_0^\infty$ be a Cauchy sequence in X that converges to ξ. By extracting a subsequence, if necessary, we may assume that $|x_n - x_{n-1}| < r_n$. For each $n \in \mathsf{N}$, select a path γ_n in $\mathsf{B}(x_{n-1}; 1/2^{n+1})$ that joins x_{n-1} to x_n; so $\operatorname{diam}(\gamma_n) < 1/2^n$.

The (infinite) concatenation $\gamma := \gamma_1 \star \gamma_2 \star \dots$ is a well-defined continuous map $[0,1) \xrightarrow{\gamma} X$. Also, $\lim_{t\to 1^-} |\gamma(t) - \xi| = 0$. Thus by defining $\gamma(1) := \xi$ we obtain a path $\gamma : [0,1] \to X \cup \{\xi\}$ with $\gamma \in \mathcal{P}_{\mathrm{b}}(X)$.

Thus, whenever X is complete, so is X_d but—as illustrated by numerous examples (e.g., see Example 3.4)—the converse may not hold. However, there are three noteworthy situations in which completeness of X_d implies completeness of X. First, according to Proposition 3.18(c), if X is uniformly locally path connected, then $\partial X = i(\partial_d X)$ and so in this setting X and X_d are either both complete or both non-complete.

Next, we use compactness: evidently, if X_d is compact, so is X and hence X is complete. However, we can say a bit more.

3.20. Lemma. *Let $X = (X, |\cdot|)$ be a path connected metric space with associated diameter distance space $X_d = (X, d)$. Let $i : \bar{X}_d \to \bar{X}$ be the 1-Lipschitz extension of the identity map $\mathsf{id} : X_d \to X$. Suppose $\bar{X}_d$ is compact. Then i is surjective; in particular, $i(\partial_d X) = \partial X$.*

Proof. Let $\xi \in \partial X$. Choose a sequence $(x_n)_{n=1}^\infty$ in X with $|x_n - \xi| \to 0$ as $n \to \infty$. There exists a subsequence $(x_{n_k})_{k=1}^\infty$ and a point $\zeta \in \bar{X}_d$ such that $d(x_{n_k}, \zeta) \to 0$ as $k \to \infty$. As id is continuous, $\zeta \notin X$, so $\zeta \in \partial_d X$. Since

$$|\xi - i(\zeta)| \le |\xi - x_{n_k}| + |x_{n_k} - i(\zeta)| \le |\xi - x_{n_k}| + d(x_{n_k}, \zeta) \to 0 \quad \text{as } k \to \infty\,,$$

it follows that $\xi = i(\zeta)$. ∎

Finally, we consider local connectivity, either of X or of $\bar{X}$: we seek a connection between local connectivity and information about $\partial^{\mathrm{pa}}X$ (which may provide information about the completeness of $\partial_d X$). We make some initial comments regarding local connectivity. There are spaces X with both X and ∂X

locally connected, but $\bar{X}$ not locally connected. For example, both the space $\mathsf{DC}^* := \mathsf{DC} \setminus (\{0\} \times (0,1])$—the dyadic comb with its "limit tooth" removed—and its boundary $\partial\mathsf{DC}^* = \{0\} \times (0,1]$ are locally connected, but the metric completion of DC^* is DC and is not locally connected. There are spaces X with both X and $\bar{X}$ locally connected, but ∂X not locally connected. One such example is provided by the *Cantor tree* $\mathbb{CT}$; this binary tree is obtained by joining the point $(1/2, 1) \in \mathsf{R}^2$ to each point $(0,0), (1,0) \in \mathsf{R}^2$ with a line segment. Then the two points on each segment that are 'half-way' down—so with y-coordinate $1/2$—are joined, via line segments, to the points $(0, 1/3)$ and $(0, 2/3)$ respectively. We continue this going 'half-way' down and then joining the 2^n points on the segments to appropriate endpoints of the Cantor middle third dust $\mathbb{CD}$. Then $X := \mathbb{CT} \setminus \mathbb{CD}$ and $\bar{X} = \mathbb{CT}$ are both locally connected, but $\partial X = \mathbb{CD}$ is not locally connected.

Next we examine the path accessibility of boundary points. If $\xi := (0, y) \in \{0\} \times (0,1]$ and $X := \mathsf{DC} \setminus \{\xi\}$, then $\partial X = \partial^{\mathrm{pa}} X = \{\xi\}$ but $\bar{X} = \mathsf{DC}$ is not locally connected at ξ. On the other hand, for $X := \mathsf{DC}^*$, both X and ∂X are locally connected, but $\emptyset = \partial^{\mathrm{pa}} X \neq \partial X = \{0\} \times (0,1]$. An extreme example of this is $X := \{re^{i\theta} \mid r \geq 0\,, \theta \in [0, 2\pi) \cap \mathsf{Q}\}$ which is only locally connected at the origin with $\partial X = \mathsf{R}^2 \setminus X$ and $\partial^{\mathrm{pa}} X = \emptyset$. Finally, for $X := \mathbb{CT} \setminus \mathbb{CD}$ we have $\partial X = \partial^{\mathrm{pa}} X = \mathbb{CD}$, but ∂X is not locally connected.

These examples suggest that there are minimal ties between path accessibility of boundary points and local connectivity. Nonetheless, we do have the following consequence of Fact 2.3.

3.21. Lemma. *Let $X = (X, |\cdot|)$ be a path connected metric space with associated diameter distance space $X_d = (X, d)$. Suppose that X is locally complete and that $\bar{X}$ is locally connected. Then $\partial^{\mathrm{pa}} X$ is dense in ∂X; thus X_d is complete if and only if X is complete.*

Here is the analogous basic information for the maps $\bar{X}_l \xrightarrow{j} \bar{X}$ and $\bar{X}_l \xrightarrow{h} \bar{X}_d$. Proofs for the following are similar to those for Proposition 3.18, but here we work with rectifiable paths. Note that item (b) provides an important connection between $\partial_l X$ and $\partial^{\mathrm{ra}} X$ which in turn reveals how to tell whether or not X_l is a complete metric space.

3.22. Proposition. *Let $X = (X, |\cdot|)$ be a rectifiably connected space with associated diameter and length spaces $X_d = (X, d)$ and $X_l = (X, l)$. Let $j : \bar{X}_l \to \bar{X}$ and $h : \bar{X}_l \to \bar{X}_d$ be the extensions of the identity maps* $\mathsf{id} : X_l \to X$ *and* $\mathsf{id} : X_l \to X_d$*, respectively.*

(a) *Both h, j are 1-Lipschitz and $j[\bar{\mathsf{B}}_l(\zeta; r)] \subset \bar{\mathsf{B}}(j(\zeta); r)$, $h[\bar{\mathsf{B}}_l(\zeta; r)] \subset \bar{\mathsf{B}}_d(h(\zeta); r)$.*

(b) *We always have $j(\partial_l X) = \partial^{\mathrm{ra}} X \subset \partial X$ and $h(\partial_l X) = \partial^{\mathrm{ra}}_d X \subset \partial_d X$.*

(c) *If X is uniformly uniformly locally rectifiably connected, then $\bar{X}_l \xrightarrow{j} \bar{X}$ is a homeomorphism; in particular, $\partial X = j(\partial_l X) = \partial^{\mathrm{ra}} X$.*

(d) *If X is quasiconvex, then so are both $\bar{X}_d, \bar{X}_l$ and each map h, i, j is bilipschitz.*

3.23. Remarks. (**a**) Note that a proof of (b) reveals that $\partial_l^{\mathrm{ra}} X = \partial_l X$. (**b**) The facts that $j(\partial_l X) = \partial^{\mathrm{ra}} X$ and $h(\partial_l X) = \partial_d^{\mathrm{ra}} X$ tells us that X_l is complete if and only if $\partial^{\mathrm{ra}} X = \emptyset$ if and only if $\partial_d^{\mathrm{ra}} X = \emptyset$. In particular, if X or X_d is complete, then so is X_l. (**c**) It is not difficult to give a constructive proof that $\partial X = \partial^{\mathrm{ra}} X$, provided X is uniformly uniformly locally rectifiably connected (meaning that the condition in Lemma 3.11(d) holds). (**d**) See the paragraph just preceding [HH08, Lemma 2.2, p.210] for a proof of item (d) above. (**e**) It would be interesting to know whether or not the analog of Proposition 3.18(c) holds for the map j.

3.C. Realizations of Boundaries. Our goal here is to demonstrate that there is a natural one-to-one correspondence between $\partial_d X$ and certain equivalence classes of paths in $\mathcal{P}_{\mathrm{b}}(X)$. Recall from §2.D that $\mathcal{P}_{\mathrm{b}}(X)$ is the set of paths in X with terminal endpoints in ∂X.

We start by noting that the 1-Lipschitz map $\bar{X}_d \xrightarrow{i} \bar{X}$ induces a natural one-to-one correspondence between $\mathcal{P}_{\mathrm{b}}(X)$ and $\mathcal{P}_{\mathrm{b}}(X_d)$.

3.24. Lemma. *The map $\mathcal{P}_{\mathrm{b}}(X_d) \xrightarrow{i_*} \mathcal{P}_{\mathrm{b}}(X)$, defined by $i_*[\gamma_d] := i \circ \gamma_d$, is a bijection.*

Proof. If $\gamma_d \in \mathcal{P}_{\mathrm{b}}(X_d)$ is a path in X_d with terminal endpoint $\zeta := \gamma_d(1) \in \partial_d X$, then $\gamma := i \circ \gamma_d$ is a path in X with terminal endpoint $\gamma(1) = i(\zeta) \in \partial X$, so $\gamma = i_*[\gamma_d] \in \mathcal{P}_{\mathrm{b}}(X)$. Clearly i_* is injective. That i_* is surjective follows from Lemma 3.17. ■

The natural terminal endpoint maps $\mathcal{P}_{\mathrm{b}}(X) \xrightarrow{p} \partial^{\mathrm{pa}} X$ and $\mathcal{P}_{\mathrm{b}}(X_d) \xrightarrow{p_d} \partial_d X$ are given by $p(\gamma) := \gamma(1)$ and $p_d(\gamma_d) := \gamma_d(1)$; recall that $\partial_d^{\mathrm{pa}} X = \partial_d X$. See §2.D and Remark 3.19(a). The following commutative diagram summarizes the interplay between the terminal endpoint maps p, p_d and the maps i, i_*. In particular, $i \circ p_d = p \circ i_*$.

$$
\begin{array}{ccccc}
\mathcal{P}_{\mathrm{b}}(X_d) & \xrightarrow{p_d} & \partial_d X & \hookrightarrow & \bar{X}_d \\
{\scriptstyle i_*}\big\downarrow & & & & \big\downarrow{\scriptstyle i} \\
\mathcal{P}_{\mathrm{b}}(X) & \xrightarrow{p} & \partial^{\mathrm{pa}} X & \hookrightarrow & \bar{X}
\end{array}
$$

As we have done so far, given $\gamma \in \mathcal{P}_{\mathrm{b}}(X)$, we let $\gamma_d \in \mathcal{P}_{\mathrm{b}}(X_d)$ be the path with $i_*[\gamma_d] = \gamma$.

3.25. Definition. We declare two paths α, β in $\mathcal{P}_{\rm b}(X)$ to be *diameter distance equivalent*, denoted by writing $\alpha \sim_d \beta$, provided $\lim_{t\to 1^-} d(\alpha(t), \beta(t)) = 0$.

Clearly, if β is a reparametrization of α, then $\alpha \sim_d \beta$. In fact, a straightforward application of Lemma 3.17 reveals that

$$\forall\, \alpha, \beta \in \mathcal{P}_{\rm b}(X): \quad \alpha \underset{d}{\sim} \beta \iff \alpha_d(1) = \beta_d(1)\,.$$

Finally, here is the promised connection between $\partial_d X$ and $\mathcal{P}_{\rm b}(X)$.

3.26. Proposition. *Let $X = (X, |\cdot|)$ be a path connected metric space with associated diameter distance space $X_d = (X, d)$. There is a natural one-to-one correspondence between $\partial_d X$ and $\mathcal{P}_{\rm b}(X)/\sim_d$.*

Proof. Let $\mathcal{P}_{\rm b}(X) \xrightarrow{q} \mathcal{Q} := \mathcal{P}_{\rm b}(X)/\sim_d$ be the quotient map. We know that for all paths $\alpha, \beta \in \mathcal{P}_{\rm b}(X)$,

$$\alpha \underset{d}{\sim} \beta \implies p_d \circ i_*^{-1}[\alpha] = p_d[\alpha_d] = \alpha_d(1) = \beta_d(1) = p_d[\beta_d] = p_d \circ i_*^{-1}[\beta]\,.$$

Thus the map $p_d \circ i_*^{-1} : \mathcal{P}_{\rm b}(X) \to \partial_d X$ respects the identifications of q. That is, if $q(\alpha) = q(\beta)$, then $p_d \circ i_*^{-1}[\alpha] = p_d \circ i_*^{-1}[\beta]$.

This means that we can factor $p_d \circ i_*^{-1}$ through q; i.e.,

there exists a map $\mathcal{Q} \xrightarrow{\tilde{p}} \partial_d X$

$$\begin{array}{ccc} \mathcal{P}_{\rm b}(X) & \xrightarrow{q} & \mathcal{Q} \\ {\scriptstyle i_*}\uparrow & & \downarrow{\scriptstyle \tilde{p}} \\ \mathcal{P}_{\rm b}(X_d) & \xrightarrow{p_d} & \partial_d X \end{array}$$

such that the pictured diagram commutes. Since $p_d = \tilde{p} \circ q \circ i^*$ is surjective, $\tilde{p}$ is also surjective. We claim that $\tilde{p}$ is injective and hence provides the asserted one-to-one correspondence.

We verify that $q(\alpha) \neq q(\beta) \implies \tilde{p}(q(\alpha)) \neq \tilde{p}(q(\beta))$. Let $\alpha, \beta \in \mathcal{P}_{\rm b}(X)$ and suppose $q(\alpha) \neq q(\beta)$. We show that $\alpha_d(1) \neq \beta_d(1)$. This is clear if

$$i(\alpha_d(1)) = \alpha(1) \neq \beta(1) = i(\beta_d(1))\,;$$

we assume that $\alpha(1) = \beta(1)$. Then $q(\alpha) \neq q(\beta)$ means $\lim_{t\to 1^-} d(\alpha(t), \beta(t)) \neq 0$, so there exists some $\varepsilon_0 > 0$ such that for each $\tau \in (0,1)$ there is some $t \in (\tau, 1)$ with $d(\alpha(t), \beta(t)) \geq \varepsilon_0$. Thus we can find $t_n \in (0,1)$ with $t_n \nearrow 1$ and so that $a_n := \alpha(t_n)$, $b_n := \beta(t_n)$ satisfy $d(a_n, b_n) \geq \varepsilon_0$. Then $(a_n)_1^\infty$ and $(b_n)_1^\infty$ are non-equivalent Cauchy sequences in X_d that represent the points $\alpha_d(1)$ and $\beta_d(1)$ in $\partial_d X$. Therefore, $\alpha_d(1) \neq \beta_d(1)$. ■

As an application of the above ideas, we now provide the following.

3.27. *Proof of Proposition 3.18(c).* Let $\xi \in i(\bar{X}_d)$. We must verify that the pre-image $i^{-1}(\xi)$ is totally disconnected. In fact, we show that $i^{-1}(\xi)$ has the following "small separated sets" property: For each $\varepsilon > 0$, $i^{-1}(\xi)$ can be expressed as a (possibly uncountable) union of sets S that are ε-separated and of diameter at most $4\,\varepsilon$; i.e., for two such distinct sets S, S', $\operatorname{dist}_d(S, S') \geq \varepsilon$ and $\operatorname{diam}_d(S) \leq 4\,\varepsilon$.

This property clearly implies that $i^{-1}(\xi)$ is totally disconnected, since each such set S is both open in $i^{-1}(\xi)$ and closed. To see that there may be uncountably many such sets S, consider the metric subspace of R^2 given by

$$X := \{r\,e^{i\theta} \mid 0 < r < 1\,, \theta \in [0, 2\pi) \setminus \mathsf{Q}\} \cup \mathsf{S}^1 \quad \text{with } |\cdot| := \|\cdot\|_{\mathsf{R}^2}\,.$$

Then $\partial X = \{r\,e^{i\theta} \mid 0 \leq r < 1\,, \theta \in [0, 2\pi) \cap \mathsf{Q}\}$ but $\partial^{\mathrm{pa}} X = \{0\}$. Also, for all distinct $\zeta, \eta \in i^{-1}(0)$, $1 \leq d(\zeta, \eta) \leq 2$; so $\partial_d X = i^{-1}(0)$ is bilipschitz equivalent to the discrete metric space with cardinality $\mathfrak{c} = \operatorname{card}(\mathsf{R})$.

Now we establish the "small separated sets" property for $i^{-1}(\xi)$. Clearly, if $\xi \in X$, then $i^{-1}(\xi) = \{\xi\}$ and there is nothing to prove. Assume $\xi \in i(\partial_d X) = \partial^{\mathrm{pa}} X$. Let $\varepsilon > 0$ be given.

Consider any path γ in $\mathcal{P}_{\mathrm{b}}(X; \xi)$, the subcollection of $\mathcal{P}_{\mathrm{b}}(X)$ consisting of paths in X that have terminal endpoint ξ. The "tail" of γ determines a unique path component P of $X \cap \bar{\mathsf{B}}(\xi; 2\varepsilon)$ with $\xi \in \bar{P}$. That is, there exists a $\tau \in (0, 1)$ such that $\gamma([\tau, 1)) \subset P$.

This can be used to define an equivalence relation on $\mathcal{P}_{\mathrm{b}}(X; \xi)$, two paths being "tail equivalent" if they determine the same such path component. (Of course this relation depends on ε.) Note that diameter distance equivalent paths are necessarily "tail equivalent" (but not conversely). This means that we can partition $i^{-1}(\xi)$ into a disjoint union of sets P_d where each P_d corresponds to a "tail equivalence class" of $\mathcal{P}_{\mathrm{b}}(X; \xi)$. That is, all points ζ in some $P_d \subset i^{-1}(\xi)$ have the property that for each $\gamma_d \in \mathcal{P}_{\mathrm{b}}(X_d; \zeta)$, the "tail" of $\gamma := i_*[\gamma_d] = i \circ \gamma_d$ determines the same such P (a path component of $X \cap \bar{\mathsf{B}}(\xi; 2\varepsilon)$ with $\xi \in \bar{P}$).

We claim that the sets P_d have the "small separated sets" property. Let P be any such path component of $X \cap \bar{\mathsf{B}}(\xi; 2\varepsilon)$ with $\xi \in \bar{P}$. Let $\zeta, \eta \in P_d$. Let $\alpha_d \in \mathcal{P}_{\mathrm{b}}(X_d; \zeta)$ and $\beta_d \in \mathcal{P}_{\mathrm{b}}(X_d; \eta)$. Since $\alpha := i_*[\alpha_d] = i \circ \alpha_d$ and $\beta := i_*[\beta_d] = i \circ \beta_d$ both determine P, there exists a $\tau \in (0, 1)$ such that $\alpha([\tau, 1)), \beta([\tau, 1)) \subset P$. Since P is path connected, for each $t \in [\tau, 1)$ there is a path γ in P that joins $\alpha(t), \beta(t)$. Thus

$$d(\alpha(t), \beta(t)) \leq \operatorname{diam}(\gamma) \leq \operatorname{diam}(P) \leq 4\,\varepsilon\,,$$

so we conclude that

$$d(\zeta, \eta) = \lim_{t \to 1^-} d(\alpha_d(t), \beta_d(t)) \leq \limsup_{t \to 1^-} d(\alpha(t), \beta(t)) \leq 4\,\varepsilon$$

and therefore $\operatorname{diam}_d(P_d) \leq 4\,\varepsilon$.

Now suppose P and Q are distinct path components of $X \cap \bar{\mathsf{B}}(\xi; 2\varepsilon)$ with $\xi \in \bar{P} \cap \bar{Q}$. Let $\zeta \in P_d$ and $\eta \in Q_d$. Let $\alpha_d \in \mathcal{P}_{\mathrm{b}}(X_d; \zeta)$ and $\beta_d \in \mathcal{P}_{\mathrm{b}}(X_d; \eta)$.

Put $\alpha := i_*[\alpha_d] = i \circ \alpha_d$ and $\beta := i_*[\beta_d] = i \circ \beta_d$. Pick $\tau \in (0,1)$ such that $\alpha([\tau,1)), \beta([\tau,1)) \subset \bar{\mathsf{B}}(\xi;\varepsilon)$. Since $P \neq Q$, for each $t \in [\tau,1)$, each path that joins $\alpha(t)$ to $\beta(t)$ must leave $\bar{\mathsf{B}}(\xi;2\varepsilon)$ and therefore

$$d(\alpha(t),\beta(t)) \geq \varepsilon\,,$$

so we deduce that

$$d(\zeta,\eta) = \lim_{t\to 1^-} d(\alpha_d(t),\beta_d(t)) \geq \liminf_{t\to 1^-} d(\alpha(t),\beta(t)) \geq \varepsilon\,.$$

Thus $\mathrm{dist}_d(P_d,Q_d) \geq \varepsilon$. ∎

Here are the analogs of the above for $\partial^{\mathrm{ra}}X$, the rectifiably accessible boundary points of X, and $\mathcal{R}_{\mathrm{b}}(X)$, the rectifiable paths in X with terminal endpoints in ∂X. The 1-Lipschitz map $j : \bar{X}_l \to \bar{X}$ induces a natural one-to-one correspondence between $\mathcal{R}_{\mathrm{b}}(X)$ and $\mathcal{R}_{\mathrm{b}}(X_l)$.

3.28. Lemma. *The map $\mathcal{R}_{\mathrm{b}}(X_l) \xrightarrow{j_*} \mathcal{P}_{\mathrm{b}}(X)$, defined by $j_*[\gamma_l] := j \circ \gamma_l$, is a bijection.*

Next, we can identify $\partial_l X$ as the equivalence classes of paths in $\mathcal{R}_{\mathrm{b}}(X)$ where $\alpha \sim_l \beta$ if $\lim_{t\to 1^-} l(\alpha(t),\beta(t)) = 0$ (which holds if and only if $\alpha_l(1) = \beta_l(1)$).

3.29. Proposition. *Let $X = (X,|\cdot|)$ be a rectifiably connected metric space with associated length distance space $X_l = (X,l)$. There is a natural one-to-one correspondence between $\partial_l X$ and $\mathcal{R}_{\mathrm{b}}(X)/\sim_l$.*

4. Miscellaneous Properties

4.A. Diameters of Connected Sets. We have already seen that the diameter of a path in X is not changed when we consider it as a path in X_d; see Remark 3.9. Here we show that this fact continues to hold for arbitrary connected sets in $\bar{X}_d$. This is straightforward to check for a path connected set in X.

To handle the more general connected sets, we first establish the following approximation result; this should be useful in other contexts as well.

4.1. Proposition. *Let $X = (X,|\cdot|)$ be a path connected metric space with associated diameter distance space $X_d = (X,d)$, and let $i : \bar{X}_d \to \bar{X}$ be the 1-Lipschitz extension of the identity map $\mathsf{id} : X_d \to X$. Suppose C is a connected subspace of $\bar{X}_d$. Then for each $\varepsilon > 0$ there exists a path connected set Γ in X with*

$$\mathsf{dist}^d_{\mathcal{H}}(C,\Gamma) < \varepsilon \quad \textit{as well as} \quad \mathsf{dist}_{\mathcal{H}}(i(C),\Gamma) < \varepsilon\,.$$

Proof. As $\bar{X}_d \xrightarrow{i} \bar{X}$ is 1-Lipschitz, $\mathsf{dist}_{\mathcal{H}}(i(C),\Gamma) \le \mathsf{dist}^d_{\mathcal{H}}(C,\Gamma)$, and so it suffices to prove the first inequality.

Let $\varepsilon > 0$ be given. Put $\mathcal{B} := \{\bar{\mathsf{B}}_d(\xi;\varepsilon/5) \mid \xi \in C\}$, an open cover (by open balls in $\bar{X}_d$) of C. For each $\xi \in C$ we select one point $x \in X \cap \bar{\mathsf{B}}_d(\xi;\varepsilon/5)$; if $\xi \in X$, we take $x := \xi$. We call x the point *associated with* ξ.

If there is some $\xi \in C$ with $C \subset \bar{\mathsf{B}}_d(\xi;\varepsilon/5) \in \mathcal{B}$, we simply put $\Gamma := \{x\}$, where x is the point associated with ξ. Then $C \subset \mathsf{N}_d(\Gamma; 2\varepsilon/5)$ and so $\mathsf{dist}^d_H(C,\Gamma) < \varepsilon/2$.

Thus we may—and do—assume that for each $\xi \in C$, $C \setminus \bar{\mathsf{B}}_d(\xi;\varepsilon/5) \neq \emptyset$. Since C is connected, it now follows that for each $\xi \in C$, there exists at least one point $\eta \in C \setminus \bar{\mathsf{B}}_d(\xi;\varepsilon/5)$ with $\bar{\mathsf{B}}_d(\xi;\varepsilon/5) \cap \bar{\mathsf{B}}_d(\eta;\varepsilon/5) \neq \emptyset$. Fix two such points $\xi, \eta \in C$ and let $x, y \in X$ be the associated points (chosen as described in paragraph two of this proof). Evidently,

$$d(x,y) \le d(x,\xi) + d(\xi,\eta) + d(y,\eta) < 4\varepsilon/5\,,$$

so there is an injective path γ that joins x to y in X with $\operatorname{diam}(\gamma) < 4\varepsilon/5$. Note that for each $z \in |\gamma|$,

$$d(z,\xi) \le \operatorname{diam}(\gamma) + d(x,\xi) < \varepsilon\,,$$

and therefore $|\gamma| \subset \bar{\mathsf{B}}_d(\xi;\varepsilon) \cap \bar{\mathsf{B}}_d(\eta;\varepsilon)$.

Now define Γ to be the union of the trajectories $|\gamma|$ of all such paths γ. It follows directly from the above that

$$\Gamma \subset \mathsf{N}_d(C;\varepsilon) \ \text{ and } \ C \subset \mathsf{N}_d(\Gamma;\varepsilon/5)\,, \quad \text{whence } \ \mathsf{dist}^d_{\mathcal{H}}(C,\Gamma) < \varepsilon\,.$$

It remains to see that Γ is path connected.

Let z, w be two points in Γ. According to the definition of Γ, z, w must be points on the trajectories of certain paths α, β in X. That is, if we let x, y each be an endpoint of α, β, respectively, then x, y are associated with certain points, say, ξ, η in C. Since C is connected and $\mathcal{B}$ is an open cover of C, an appeal to [HY88, Theorem 3-4, p.108] permits us to assert the existence of balls $B_1, \cdots, B_n$ selected from $\mathcal{B}$ (so each $B_i = \bar{\mathsf{B}}(\xi_i;\varepsilon/5)$) with the properties that $\xi = \xi_1$, $\eta = \xi_n$ and such that for each $1 \le i < n$, $B_i \cap B_{i+1} \neq \emptyset$. According to our construction, there are paths γ_i that join x_i, x_{i+1} (where x_i is the point associated with ξ_i). It is now evident that $|\alpha| \cup |\gamma_1| \cup \cdots \cup |\gamma_{n-1}| \cup |\beta|$ is the trajectory of a path in Γ that joins the points z, w. ■

An immediate consequence of the above is that, with the notation as above,

$$|\operatorname{diam}_d(C) - \operatorname{diam}_d(\Gamma)| \le 2\,\varepsilon \quad \text{and} \quad |\operatorname{diam}(i(C)) - \operatorname{diam}(\Gamma)| \le 2\,\varepsilon\,.$$

Note that in the special case where C is also compact, so a continuum, Γ can be chosen to be a finite graph.

An especially useful application is that the diameter of a connected set does not change when we pass to the diameter distance.

4.2. Proposition. *Let $X = (X, |\cdot|)$ be a path connected metric space with associated diameter distance space $X_d = (X, d)$. Let $i : \bar{X}_d \to \bar{X}$ be the 1-Lipschitz extension of the identity map* $\mathsf{id} : X_d \to X$*. Suppose C is a connected subspace of $\bar{X}_d$. Then* $\operatorname{diam}_d(C) = \operatorname{diam}(i(C))$.

Proof. Clearly, $\operatorname{diam}(i(C)) \le \operatorname{diam}_d(C)$. The opposite inequality is trivial if C is a path connected subspace of X, for then each pair of points in C can be joined by a path whose diameter is not larger than $\operatorname{diam}(C)$. For the general case, we proceed as follows. Let $\varepsilon > 0$ and select a path connected subspace Γ of X as provided by Proposition 4.1. Since Γ is path connected, the inequalities immediately proceeding this proposition reveal that

$$\operatorname{diam}_d(C) \le \operatorname{diam}_d(\Gamma) + 2\varepsilon = \operatorname{diam}(\Gamma) + 2\varepsilon \le \operatorname{diam}(i(C)) + 4\varepsilon\,.$$

Letting $\varepsilon \searrow 0$ yields $\operatorname{diam}_d(C) \le \operatorname{diam}(i(C))$. ■

4.3. Corollary. *Let $X = (X, |\cdot|)$ be a path connected metric space with associated diameter distance space $X_d = (X, d)$. Then $(\bar{X}_d)_d = \bar{X}_d$.*

The above Corollary is also easy to prove via a standard approximation argument. This method can readily be modified to verify that the analogous result holds for length distance; that is, for a rectifiably connected space X we always have $(\bar{X}_l)_l = \bar{X}_l$.

It would be worthwhile to determine length versions of Propositions 4.1 and 4.2. Mimicking the proof of Proposition 4.1, we should be able to approximate—with respect to Hausdorff one measure—a given rectifiable continuum in $\bar{X}_l$ by a finite rectifiable graph in X. This would then provide a means to establish the following.

Conjecture. *Let $X = (X, |\cdot|)$ be a rectifiably connected metric space with associated length space $X_l = (X, l)$, and let $j : \bar{X}_l \to \bar{X}$ be the 1-Lipschitz extension of the identity map* $\mathsf{id} : X_l \to X$*. Let L be a rectifiable continuum in $\bar{X}_l$. Then the Hausdorff one-measure of L with respect to the length distance is given by $\mathcal{H}^1_l(L) = \mathcal{H}^1(j(L))$.*

4.B. Metric Disks. Here we examine simply connected regions on metric surfaces whose associated diameter or length completions are topological disks or half-planes. Such spaces provide domains in which classical geometric function theory can be studied.

Throughout this subsection, D denotes the open *unit disk* and H is the open *upper half-plane*, both in the Euclidean plane R^2 (that we sometimes identify with the complex number field C and then use complex variables notation). We write U to denote either of these. More precisely, if Ω is a simply connected proper subdomain of C (aka, a *conformal disk*), then by a *natural* Riemann map $\mathsf{U} \to \Omega$ we mean a conformal map where U is D when Ω is bounded and otherwise U is H; in the latter case, when Ω is locally connected at infinity, we then require that 'infinity maps to infinity'. Also, $\mathsf{T} := \partial\mathsf{U}$ which is either $\mathsf{S}^1 := \partial\mathsf{D}$ (the unit circle) or $\mathsf{R} := \partial\mathsf{H}$ (the real line). In this setting, $\mathsf{D}(z;r)$ denotes the open disk centered at z with radius r and $\hat{A}$ is the closure of A with respect to the extended plane $\hat{\mathsf{C}} := \mathsf{C} \cup \{\infty\}$.

By a *metric circle* we mean a Jordan curve, so the homeomorphic image of either the circle S^1 or the line R, with a metric on it. Then a *metric disk* is a metric space that is homeomorphic to R^2. Such a metric disk Ω is called a *diameter disk* if

$$\bar{\Omega}_d \text{ is homeomorphic either to } \bar{\mathsf{D}}^2 \text{ or to } \bar{\mathsf{H}}^2\,;$$

similarly, a rectifiably connected metric disk Ω is a *length disk* provided

$$\bar{\Omega}_l \text{ is homeomorphic either to } \bar{\mathsf{D}}^2 \text{ or to } \bar{\mathsf{H}}^2\,.$$

The additional adjective *Euclidean* always indicates that Ω is a subspace of some Euclidean space and in addition that Euclidean distance is being used in Ω.

We note that when Ω is a length disk, $\bar{\Omega}_l$ is always proper (hence geodesic), and therefore $\partial_l\Omega$ is also proper. This is a consequence of the Hopf-Rinow Theorem (see [Gro99, p.9], [BBI01, p.51], or [BH99, p.35]) which says that every locally compact complete length space is proper; $\bar{\Omega}_l$ is locally compact, since it is homeomorphic to a subset of R^2, and $\partial_l\Omega$ is a closed subspace of $\bar{\Omega}_l$. On the other hand, typically we must require that the diameter boundary $\partial_d\Omega$ of a diameter disk be proper. Example 4.4 illustrates that this diameter boundary need not be proper.

Evidently, every Euclidean Jordan disk is a diameter disk, but not conversely. However, there are Euclidean Jordan disks that have finite length distance but are not length disks; see Example 4.5. In general, Euclidean diameter disks possess certain properties not shared by general diameter disks; see Remark 4.8.

Next we exhibit two elementary examples of metric disks to help the reader understand why we require some of our hypotheses; other more exotic examples are presented in [FH11, §2.D]. Typically we ask that $\partial_d\Omega$ be proper. There are Euclidean diameter disks whose diameter boundaries fail to be proper.

4.4. Example. There is a bounded Euclidean metric disk Ω in R^2 with

(a) $\bar{\Omega}_d \approx \bar{\mathsf{H}} \approx \bar{\Omega}_l$,
(b) $\partial_d\Omega$ is bounded and hence non-proper,

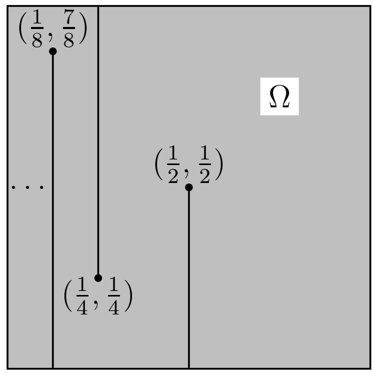

FIGURE 4. A 'bad' Euclidean metric disk

(c) Ω is not finitely connected along its boundary,
(d) $\partial\Omega$ is not locally connected.

Proof. See Figure 4. ■

We work with the diameter distance as opposed to the length distance because 'being a diameter disk' seems less restrictive than 'being a length disk'. For example, Theorem 4.7 tells us that every Euclidean Jordan disk is automatically an diameter disk, nonetheless, there are 'nice' Euclidean Jordan disks that fail to be length disks.

4.5. Example. There exist Jordan disks $\Omega \subset \mathsf{R}^2$ with finite length distance (so as sets, $\bar{\Omega}_l = \bar{\Omega}$ and $\partial^{\text{ra}}\Omega = \partial\Omega$), but Ω is not an length disk.

Proof. We start with $\Delta_0 := \mathsf{D} \cap \mathsf{H}$, the upper half of the unit disk. At each point 2^{-n} we attach decorations Δ_n which we now describe. Let $S_n \subset \mathsf{C} \setminus \mathsf{H}$ be a spiral in the lower half-plane having initial point 2^{-n}, terminal point ζ_n, $\operatorname{diam}(S_n) \simeq 2^{-n}$, and with $\ell(S_n) = 1$. To get Δ_n we slightly thicken S_n making sure that the decorations are mutually disjoint and so that $l(2^{-n}, \zeta_n) \simeq 1$, where l is the length distance in $\Omega := \cup_{n=0}^{\infty}\Delta_n$. Since $\partial_l\Omega$ is not compact, it is not homeomorphic to S. ■

The following fact was a crucial ingredient throughout the work [FH11]. As we did not provide a proof there, we do so here. Recall our convention that U denotes either D or H depending on the circumstances.

4.6. Proposition. *Let Ω be a diameter disk. Then $\partial_d\Omega$ is a metric circle.*

Proof. Using the identity map $\Omega_d \xrightarrow{\mathsf{id}} \Omega$ and homeomorphisms $\Omega \xrightarrow{h} \mathsf{U}$ and $\bar{\Omega}_d \xrightarrow{k} \bar{\mathsf{U}}$ (these exist by definition), we obtain a continuous injection $f : \mathsf{U} \to \bar{\mathsf{U}}$ with $f \circ h \circ \mathsf{id} = k\,|_{\Omega_d}$ as pictured. According to the Invariance of Domain property, [Mun00, §62], $f(\mathsf{U})$ is an open subset of R^2 (i.e., open in the Euclidean topology) and $f : \mathsf{U} \to f(\mathsf{U})$ is a homeomorphism. Thus, by way of the homeomorphisms $f \circ h$ and k, we may view Ω as an open subspace of $\bar{\mathsf{U}} = \bar{\Omega}_d$; needless to say, distance in Ω need not be Euclidean distance.

$$\begin{array}{ccccc} \Omega & \xleftarrow{\mathsf{id}} & \Omega_d & \hookrightarrow & \bar{\Omega}_d \\ {\scriptstyle h}\downarrow & & & & \downarrow{\scriptstyle k} \\ \mathsf{U} & & \xrightarrow{\quad f \quad} & & \bar{\mathsf{U}} \end{array}$$

In particular, $\partial_d\Omega = \bar{\Omega}_d \setminus \Omega = \bar{\mathsf{U}} \setminus \Omega \supset \partial\mathsf{U} =: \mathsf{T}$. We claim that in fact $\partial_d\Omega = \mathsf{T}$ (which is a metric circle ⌣̈). To corroborate this, we must confirm that $\partial_d\Omega \subset \mathsf{T}$; so, let $\zeta \in \partial_d\Omega$. Assume $\zeta \notin \mathsf{T}$. Then $\zeta \in \mathsf{U}$ and hence neither $\mathsf{U} \setminus \{\zeta\}$ nor $\bar{\mathsf{U}} \setminus \{\zeta\} = \bar{\Omega}_d \setminus \{\zeta\}$ is simply connected. We start with a loop λ in $\bar{\Omega}_d \setminus \{\zeta\}$ that generates the fundamental group of $\bar{\Omega}_d \setminus \{\zeta\}$. Then we construct a loop κ in Ω that is (freely) path-homotopic to λ in $\bar{\Omega}_d \setminus \{\zeta\}$. Since this contradicts the fact that Ω is simply connected, we must have $\zeta \in \mathsf{T}$. It remains to provide some details.

Assume $\zeta \in \mathsf{U}$. Choose $R > 0$ so that the closed Euclidean disk $\mathsf{D}[\zeta; 2R]$ lies in the open disk/half-plane U. Clearly the Euclidean circle $C := \mathsf{S}^1(\zeta; R)$ is a continuum in U and is the trajectory of a loop λ in $\mathsf{U} \setminus \{\zeta\}$ whose path-homotopy class generates the fundamental group of $\bar{\Omega}_d \setminus \{\zeta\}$ based at ζ_0, where $\zeta_0 := \zeta + R$. We may as well assume that $C \cap \partial_d\Omega \neq \emptyset$. Now we explain how to 'pull' λ into Ω to get a loop κ as described above.

Since the identity map $\bar{\Omega}_d \xrightarrow{\mathsf{id}} \bar{\mathsf{U}}$ is a homeomorphism, and C is compact, its restriction to C is uniformly continuous. Thus there exists an $r > 0$ such that for all points $\xi \in C$ we have $\bar{\mathsf{B}}_d(\xi; r) \subset \mathsf{D}(\xi; R/10)$. (Here $\bar{\mathsf{B}}_d(\xi; r)$ is an *open* ball in $\bar{\Omega}_d$ and $\mathsf{D}(\xi; t)$ is an open Euclidean disk.) Put

$$\mathcal{B} := \{\bar{\mathsf{B}}_d(\xi; r/10) \mid \ \xi \in C\}\,, \quad \text{which is an open cover of } C\,.$$

Since C is compact, there is a (Euclidean) Lebesgue number δ for $\mathcal{B}$; that is, $\delta > 0$ has the property that each subset of C with Euclidean diameter less than δ is contained in some $B \in \mathcal{B}$. In particular this holds for $\{\xi, \eta\} \subset C$ when the Euclidean distance $\|\xi - \eta\|_{\mathsf{R}^2}$ between ξ and η is less than δ.

Pick $n \in \mathsf{N}$ so that $\theta := 2\pi/n$ satisfies $R\|1 - e^{i\theta}\|_{\mathsf{R}^2} < \delta$. For each integer $m \in \{1, \ldots, n\}$, define $\zeta_m := \zeta + Re^{im\theta}$. So $\zeta_0, \zeta_1, \ldots, \zeta_n = \zeta_0$ are successive points equidistributed around the circle C. Let λ_m denote the injective path from ζ_{m-1} to ζ_m given by $\lambda_m(t) = \zeta + R\exp(i[m - 1 + t]\theta)$ with $t \in [0, 1]$. Thus $|\lambda_m|$ is the smaller subarc of C joining ζ_{m-1}, ζ_m. Evidently, $\lambda := \lambda_1 \star \cdots \star \lambda_m$ is a non-trivial (i.e., non-null homotopic) loop in $\mathsf{U} \setminus \{\zeta\} \subset \bar{\Omega}_d \setminus \{\zeta\}$ with $|\lambda| = C$.

Fix $m \in \{1,\dots,n\}$. Pick $z_m \in \Omega \cap \bar{\mathsf{B}}_d(\zeta_m; r/10)$ taking $z_m := \zeta_m$ if $\zeta_m \in \Omega$. There is a path β_m joining z_m to ζ_m in $\Omega \cup \{\zeta_m\}$ with $\mathrm{diam}_\Omega(\beta_m) < r/10$ (where diam_Ω denotes diameter measured using the distance in Ω). By construction, $\|\zeta_m - \zeta_{m-1}\|_{\mathsf{R}^2} = R\|1-e^{i\theta}\|_{\mathsf{R}^2} < \delta$, so there exists some $B_m := \bar{\mathsf{B}}_d(\xi_m; r/10) \in \mathcal{B}$ with $\zeta_{m-1}, \zeta_m \in B_m$. (We are not asserting that $\xi_m \in |\lambda_m|$.) Then

$$d(z_m, z_{m-1}) \le d(z_m,\zeta_m) + d(\zeta_m,\zeta_{m-1}) + d(\zeta_{m-1}, z_{m-1}) < \frac{r}{10} + \frac{r}{5} + \frac{r}{10} = \frac{2r}{5}\,,$$

so we can choose a path α_m joining z_{m-1} to z_m <u>in Ω</u> with $\mathrm{diam}_\Omega(\alpha_m) < 2r/5$.

Now $\kappa_m := \tilde{\beta}_{m-1} \star \alpha_m \star \beta_m$ is a path joining ζ_{m-1} to ζ_m in $\Omega \cup \{\zeta_{m-1},\zeta_m\}$. (Here $\beta_0 := \beta_n$ and $z_0 := z_n$). Also, for all points $x \in |\kappa_m|$,

$$\begin{aligned} d(x,\xi_m) &\le \mathrm{diam}_\Omega(\alpha_m) + [\mathrm{diam}_\Omega(\beta_{m-1}) \vee \mathrm{diam}_\Omega(\beta_m)] + [d(\zeta_{m-1},\xi_m) \vee d(\zeta_m,\xi_m)] \\ &< \frac{2r}{5} + \frac{r}{10} + \frac{r}{10} = \frac{3r}{5}\,, \end{aligned}$$

so by our initial choice of r, $|\kappa_m| \subset 6\,B_m = \bar{\mathsf{B}}_d(\xi_m; 6r/10) \subset \mathsf{D}(\xi_m; R/10) =: D_m$.

Since $D_m \subset \mathsf{U}\setminus\{\zeta\} \subset \bar{\Omega}_d\setminus\{\zeta\}$ and D_m is simply connected (being a Euclidean disk), it follows that λ_m and κ_m are path-homotopic in D_m (and so in $\bar{\Omega}_d \setminus \{\zeta\}$). Standard arguments then reveal that

$$\kappa := \kappa_1 \star \cdots \star \kappa_m = (\tilde{\beta}_0 \star \alpha_1 \star \beta_1) \star (\tilde{\beta}_1 \star \alpha_2 \star \beta_2) \star \cdots \star (\tilde{\beta}_{n-1} \star \alpha_n \star \beta_n)$$

is path-homotopic (in $\bar{\Omega}_d \setminus \{\zeta\} = \bar{\mathsf{U}} \setminus \{\zeta\}$) both to $\lambda = \lambda_1 \star \cdots \star \lambda_m$ and to

$$\tilde{\beta}_0 \star \alpha \star \beta_n \quad \text{where} \quad \alpha := \alpha_1 \star \cdots \star \alpha_n\,.$$

Now α is a loop <u>in Ω</u> (based at z_0), and since Ω is simply connected, it follows that α is null-homotopic in Ω. Since $\beta_n = \beta_0$, this implies that $\tilde{\beta}_0 \star \alpha \star \beta_n$, and hence κ and λ, are all null-homotopic in $\bar{\Omega}_d \setminus \{\zeta\}$, but this is nonsense. ■

We close this article by mentioning that there is an especially important class of examples of diameter disks. Similar types of examples can be obtained by looking at simply connected subspaces of metric surfaces.

4.7. Theorem. *For a conformal disk $\Omega \subset \mathsf{R}^2$, the following are equivalent.*

(a) Ω *is a diameter disk with $\partial_d\Omega$ proper.*
(b) Ω *is finitely connected along its boundary and locally connected at infinity.*
(c) $\partial\Omega$ *is locally connected and ∞ is not a cut point of $\hat{\partial}\Omega$.*

Idea of Proof. Any natural Riemann map $\mathsf{U} \xrightarrow{f} \Omega$ has a continuous surjective extension $\hat{\mathsf{U}} \xrightarrow{F} \hat{\Omega}$; see [Pom92, Theorem 2.1] or [Pal91, Theorem IX.4.7]. Each point $\zeta \in \partial\mathsf{U}$ has an associated endcut E_ζ: when $\mathsf{U} = \mathsf{D}$, $E_\zeta := [0,\zeta]$ and when

$\mathsf{U} = \mathsf{H}$, $E_\zeta := [\zeta + i, \zeta]$. Then $F(E_\zeta)$ is an endcut of Ω and so determines a point $\varphi(\zeta)$ of $\partial_d\Omega$. This yields a natural homeomorphic extension $\hat{\mathsf{U}} \xrightarrow{\varphi} \hat{\Omega}_d$ of f. ■

A complete detailed proof (for the case when Ω is bounded) can be found in [Her11]. We note that $F|_{\bar{\mathsf{U}}} = i \circ (\varphi|_{\bar{\mathsf{U}}})$ is a homeomorphism of $\bar{\mathsf{U}}$ onto $\bar{\Omega}$; here $i : \bar{\Omega}_d \to \bar{\Omega}$ is the extension of $\mathsf{id} : \Omega_d \to \Omega$. We also note that $\partial\Omega$ locally connected implies that $\hat{\partial}\Omega$ is also locally connected. Finally, we note that the above makes use of the special property of Euclidean diameter disks that $\partial_d\Omega$ proper implies that $\bar{\Omega}_d$ is also proper.

4.8. Remarks. (**a**) The domains Ω described in the preceding need not be length disks. And a length disk need not be finitely connected along its boundary. See Examples 4.4 and 4.5. (**b**) Every Euclidean diameter disk Ω has the property that $\partial^{\mathrm{pa}}\Omega = \partial\Omega$. This follows because the continuous extension $\hat{\mathsf{U}} \to \hat{\Omega}$ of any natural Riemann map $\mathsf{U} \to \Omega$ restricts to a continuous surjection $\bar{\mathsf{U}} \to \bar{\Omega}$. (**c**) The corresponding result for Euclidean length disks fails to hold as illustrated by Example 4.4. However, if $\bar{\Omega}_l \approx \bar{\mathsf{D}}$, then we do get $\partial^{\mathrm{ra}}\Omega = \partial\Omega$.

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