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# Harmonic Mapping Problem in the Plane

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**Abstract.** This article is an expanded version of lectures given by the first author at the International Workshop on Harmonic and Quasiconformal Mappings (Chennai, August 2010). They are based on several papers written by Tadeusz Iwaniec and the authors, some of which are also joint with Ngin-Tee Koh.

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## 1. Harmonic Mapping Problem and affine invariants

Let us consider differentiable mappings $f = u+iv : \Omega \to \mathbb{C}$ defined in a domain $\Omega$ of the complex plane $\mathbb{C} = \{z = x + iy : x, y \in \mathbb{R}\}$. The partial differentiation in $\Omega$ will be expressed by the Wirtinger operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{1.1}$$

Accordingly, we shall abbreviate the complex derivatives of $f$ to

$$f_z = \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \quad \text{and} \quad f_{\bar{z}} = \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right). \tag{1.2}$$

For the derivative matrix

$$Df(z) = \begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix} \tag{1.3}$$

we compute the operator norm and the Hilbert-Schmidt norm

$$\|Df\| = |f_z| + |f_{\bar{z}}|, \qquad |Df|^2 = 2(|f_z|^2 + |f_{\bar{z}}|^2) = u_x^2 + v_x^2 + u_y^2 + v_y^2 \tag{1.4}$$

and the Jacobian determinant

$$J_f(z) = \det Df(z) = u_x v_y \; - \; u_y v_x \; = |f_z|^2 - |f_{\bar{z}}|^2. \tag{1.5}$$

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We have $J_f \geqslant 0$ if the mapping is sense-preserving, which will usually be the case from now on.

The Laplace equation for $f$ is

$$\Delta u = \Delta v = 0, \quad \text{or, equivalently,} \quad f_{z\bar{z}} = 0. \tag{1.6}$$

Thus, for any harmonic mapping $f$ the functions $f_z$ and $\overline{f_{\bar{z}}}$ are holomorphic. Taking (1.5) into account, we can write

$$\overline{f_{\bar{z}}} = \nu f_z, \tag{1.7}$$

where $\nu$ is a holomorphic function, $|\nu| < 1$ if $f$ is sense-preserving. This is the Beltrami equation of second kind [7]. The coefficient $\nu$ is invariant under a conformal change of variable $z$.

The Harmonic Mapping Problem asks when there exists a harmonic homeomorphism between two given domains $\Omega$ and $\Omega^*$. One can ask the same question about two manifolds [20], or metric spaces [23, 25], but here we restrict the consideration to domains in the complex plane $\mathbb{C}$. Note that the inverse of a harmonic mapping is in general not harmonic. Thus the mapping problem must take the order of the pair $(\Omega, \Omega^*)$ into account. Since any conformal mapping is harmonic, the case of simply connected domains is covered by the Riemann mapping theorem with just one exception. Specifically, if neither of the domains $\Omega, \Omega^*$ is the entire plane $\mathbb{C}$, then there is a conformal mapping between them, and a conformal mapping is a harmonic homeomorphism.

The studies of the Harmonic Mapping Problem began with Radó's theorem (1927) which states that there is no harmonic homeomorphism $f\colon \Omega \xrightarrow{\text{onto}} \mathbb{C}$ for any proper domain $\Omega \subsetneq \mathbb{C}$. There is no harmonic homeomorphism $f\colon \mathbb{C} \to \Omega$ either, which can be proved as follows. Suppose such $f$ exists. Recall the function $\nu$ from (1.7) (the second Beltrami coefficient). Being bounded and holomorphic in $\mathbb{C}$, it must be constant by Liouville's theorem. It follows that $f$ is an *affine* mapping; that is, $f(z) = az + b\bar{z} + c$ with $|a|^2 - |b|^2 \neq 0$. But then, of course, $f(\mathbb{C}) = \mathbb{C}$.

Harmonic Mapping Problem for doubly connected domains originated from the work of Johannes C. C. Nitsche on minimal surfaces. In 1962 he conjectured [26] a precise condition that allows one circular annulus to be mapped onto another by a harmonic homeomorphism. The recent solution of this conjecture [15] will be the subject of Section 2.

Let $\mathbb{A} = A(r, R) := \{z \in \mathbb{C}\colon r < |z| < R\}$ denote a circular annulus in the complex plane. We allow $0 \leqslant r < R \leqslant \infty$. The quantity $\operatorname{Mod}\mathbb{A} := \log \frac{R}{r}$ is called the *conformal modulus* of an annulus. This notion extends to other doubly connected domains as follows: $\operatorname{Mod}\Omega = \operatorname{Mod}\mathbb{A}$ if there is a conformal mapping of $\Omega$ onto $\mathbb{A}$. Indeed, any doubly connected domain can be conformally mapped onto some circular annulus, and circular annuli with different values of the ratio $R/r$ are not conformally equivalent.

The reason why $\operatorname{Mod}\Omega$ is relevant to the Harmonic Mapping Problem is that harmonic functions remain harmonic upon conformal change of the independent variable $z \in \Omega$. The harmonicity of a mapping $f\colon \Omega \to \Omega^*$ is also preserved under affine transformations of the target $\Omega^*$. Thus it is natural to investigate necessary and sufficient conditions for the existence of a harmonic homeomorphism in terms of the conformal modulus of $\Omega$ and of some affine invariant of the target $\Omega^*$. This leads us to the concept of *affine modulus* introduced in [17].

**Definition 1.8.** The *affine modulus* of a doubly connected domain $\Omega \subset \mathbb{C}$ is defined by

$$\operatorname{Mod}_@ \Omega = \sup\{\operatorname{Mod}\phi(\Omega)\colon \quad \phi\colon \mathbb{C} \overset{\text{onto}}{\longrightarrow} \mathbb{C} \text{ affine}\}. \tag{1.9}$$

Obviously $\operatorname{Mod}_@ \Omega \geqslant \operatorname{Mod}\Omega$. We illustrate basic properties of the affine modulus with a few examples.

**Example 1.10.** There exists a doubly connected domain $\Omega$ such that $\operatorname{Mod}\Omega < \infty$ but $\operatorname{Mod}_@ \Omega = \infty$. For instance, let $\Omega$ be the upper half plane with the horizontal segment $[i, 1+i]$ removed. Under the affine transformation $x + iy \mapsto \epsilon x + iy$ the removed segment is mapped into $[i, \epsilon + i]$. As $\epsilon \to 0$, the conformal modulus of the affine image tends to infinity.

**Example 1.11.** There exists a doubly connected domain $\Omega$ for which $\operatorname{Mod}_@ \Omega < \infty$ but the supremum in (1.9) is not attained. Such a domain can be obtained by a modification of Example 1.10. Let $\Omega$ be the upper half plane with the square $\{x + iy\colon 0 \leqslant x \leqslant 1, 1 \leqslant y \leqslant 2\}$ removed. Under the affine transformation $x + iy \mapsto \epsilon x + iy$ the removed square is mapped into a rectangle with width $\epsilon$. As $\epsilon \to 0$, the conformal modulus of the affine image tends to the conformal modulus of the upper half plane minus the vertical segment $[i, 2i]$.

**Example 1.12.** A circular domain $\mathbb{A} = A(r, R)$ has $\operatorname{Mod}_@ \mathbb{A} = \operatorname{Mod}\mathbb{A} = \log \frac{R}{r}$. This follows from a classical theorem of Carleman [6] which asserts that $\mathbb{A}$ has the greatest conformal modulus among all doubly connected domains with the same area ratio. The area ratio of a doubly connected domain $\Omega$ is obtained by dividing the area of $\Omega$ by the area of the bounded component of its complement. Thus, Carleman's theorem can be stated as

$$\operatorname{Mod}(G \setminus K) \leqslant \frac{1}{2} \log \frac{|G|}{|K|} \tag{1.13}$$

where $G$ is a simply connected domain of finite area $|G|$ and $K$ is a compact connected subset of $G$ that does not separate $G$. Since the area ratio $|G|/|K|$ is an affine invariant, inequality (1.13) yields

$$\mathrm{Mod}_@(G\setminus K) \leqslant \frac{1}{2}\log\frac{|G|}{|K|}.$$

In particular, for a circular annulus $\mathbb{A}=A(r,R)$ we have

$$\log\frac{R}{r} = \mathrm{Mod}\,\mathbb{A} \leqslant \mathrm{Mod}_@\,\mathbb{A} \leqslant \frac{1}{2}\log\frac{\pi R^2}{\pi r^2} = \log\frac{R}{r}$$

with equality throughout.

In view of Example 1.12 one may ask if the area ratio of $\Omega^*$ can be used to detect the existence of harmonic homeomorphisms onto $\Omega^*$. The answer is negative: for any doubly connected domain $\Omega$ and any $\mu\in(0,\infty)$ one can find a domain $\Omega^*$ with $\mathrm{Mod}_@\,\Omega^*=\mu$ which does not receive any harmonic homeomorphism from $\Omega$.

However, the pair of invariants $\mathrm{Mod}\,\Omega$ and $\mathrm{Mod}_@\,\Omega$ allows us to give both a necessary condition (Theorem 1.16 below) and a sufficient condition (Theorem 1.14) for the existence of a harmonic homeomorphism $f\colon\Omega\xrightarrow{\text{onto}}\Omega^*$.

**Theorem 1.14.** [17] *Let $\Omega$ and $\Omega^*$ be doubly connected domains in $\mathbb{C}$ such that*

$$\mathrm{Mod}_@\,\Omega^* > \mathrm{Mod}\,\Omega. \tag{1.15}$$

*Then there exists a harmonic homeomorphism $f\colon\Omega\to\Omega^*$ unless $\mathbb{C}\setminus\Omega^*$ is bounded. In the latter case there is no such $f$.*

**Question 1.1.** Does equality in (1.15) (with both sides finite) suffice for the existence of $f$?

**Theorem 1.16.** [17] *If $f\colon\Omega\to\Omega^*$ is a harmonic bijection between doubly connected domains, and $\mathrm{Mod}\,\Omega<\infty$, then*

$$\mathrm{Mod}_@\,\Omega^* \geqslant \mathrm{Mod}\,\Omega\cdot\Phi(\mathrm{Mod}\,\Omega) \tag{1.17}$$

*where $\Phi\colon(0,\infty)\to(0,1)$ is an increasing function such that $\Phi(\tau)\to 1$ as $\tau\to\infty$. One can take*

$$\Phi(\tau)=\lambda\left(\coth\frac{\pi^2}{2\tau}\right),\quad \text{where } \lambda(t)=\frac{\log t-\log(1+\log t)}{2+\log t},\ t\geqslant 1. \tag{1.18}$$

When $\mathrm{Mod}\,\Omega\to\infty$, the comparison of inequalities (1.15) and (1.17) shows that both are asymptotically sharp. However, our function $\Phi$ can certainly be improved. Its best possible form is unknown.

**Conjecture 1.1.** [17] Suppose that $f\colon\Omega\to\Omega^*$ is a harmonic bijection between doubly connected domains. Then

$$\mathrm{Mod}_@\,\Omega^* \geqslant \log\cosh\mathrm{Mod}\,\Omega. \tag{1.19}$$

The example of a pair of circular annuli shows that the inequality (1.19) would be sharp, if true.

Let us write $\Omega_1 \overset{\sim}{\hookrightarrow} \Omega_2$ when $\Omega_1$ is contained in $\Omega_2$ in such a way that $\Omega_1$ separates the boundary components of $\Omega_2$. The monotonicity of modulus can be expressed by saying that $\Omega_1 \overset{\sim}{\hookrightarrow} \Omega_2$ implies $\operatorname{Mod}\Omega_1 \leqslant \operatorname{Mod}\Omega_2$ and $\operatorname{Mod}_{@}\Omega_1 \leqslant \operatorname{Mod}_{@}\Omega_2$. Observe that both conditions (1.15) and (1.17) are preserved if $\Omega$ is replaced by a domain with a smaller conformal modulus, or $\Omega^*$ is replaced by a domain with a greater affine modulus. Theorems 1.14 and 1.16 suggest the formulation of the following conjectural comparison principles.

**Problem 1.1.** (Domain Comparison Principle) Let $\Omega$ and $\Omega^*$ be doubly connected domains such that $\operatorname{Mod}\Omega < \infty$ and there exists a harmonic bijection $f\colon \Omega \xrightarrow{\text{onto}} \Omega^*$. If $\Omega_\circ \overset{\sim}{\hookrightarrow} \Omega$, then there exists a harmonic bijection $f_\circ\colon \Omega_\circ \xrightarrow{\text{onto}} \Omega^*$.

**Problem 1.2.** (Target Comparison Principle) Let $\Omega$ and $\Omega^*$ be doubly connected domains such that there exists a harmonic bijection $f\colon \Omega \xrightarrow{\text{onto}} \Omega^*$. If $\operatorname{Mod}\Omega^*_\circ < \infty$ and $\Omega^* \overset{\sim}{\hookrightarrow} \Omega^*_\circ$, then there exists a harmonic bijection $h_\circ\colon \Omega \xrightarrow{\text{onto}} \Omega^*_\circ$.

Let us now examine the properties of the affine modulus in more detail. The equality $\operatorname{Mod}_{@}\Omega = \operatorname{Mod}\Omega$ is attained, for example, if $\Omega$ is the *Teichmüller ring*

$$\mathcal{T}(s) := \mathbb{C} \setminus ([-1,0] \cup [s,+\infty)), \qquad s > 0. \tag{1.20}$$

Indeed, for any affine automorphism $\phi\colon \mathbb{C} \to \mathbb{C}$ there is a $\mathbb{C}$-affine automorphism $\psi(z) = \alpha z + \beta$ that agrees with $\phi$ on $\mathbb{R}$. Since $\phi(\mathcal{T}(s)) = \psi(\mathcal{T}(s))$ and $\psi$ is conformal, it follows that $\operatorname{Mod}\phi(\mathcal{T}(s)) = \operatorname{Mod}\mathcal{T}(s)$.

It is desirable to have an upper estimate for $\operatorname{Mod}_{@}\Omega$ in terms of some geometric properties of $\Omega$. Recall that the *width* of a compact set $E \subset \mathbb{C}$, denoted $w(E)$, is the smallest distance between two parallel lines that enclose the set. For connected sets this is also the length of the shortest 1-dimensional projection.

**Proposition 1.21.** [17] *Let $\Omega$ be a doubly connected domain such that* $\operatorname{Mod}\Omega < \infty$. *Denote by $d$ the distance between its boundary components, and by $w$ the width of the inner boundary component. If $w > 0$, then*

$$\operatorname{Mod}_{@}\Omega \leqslant \operatorname{Mod}\mathcal{T}(d/w). \tag{1.22}$$

**Proof.** Let $\phi\colon \mathbb{C} \to \mathbb{C}$ be an affine automorphism. Denote its Lipschitz constant by $L := |\phi_z| + |\phi_{\bar z}|$. For the annulus $\phi(\Omega)$ the distance between boundary components is at most $Ld$ and the diameter of the inner component is at least $Lw$. Now the inequality (1.22) follows from the extremal property of the Teichmüller ring: it has the greatest conformal modulus among all domains with given diameter of the bounded component and given distance between components [1]. ∎

Another canonical example of a doubly connected domain is the *Grötzsch ring*

$$\mathcal{G}(s) = \{z \in \mathbb{C} \colon |z| > 1\} \setminus [s, +\infty), \qquad s > 1. \tag{1.23}$$

We claim that

$$\operatorname{Mod}_{@} \mathcal{G}(s) = \operatorname{Mod} \mathcal{T}\big(\tfrac{s-1}{2}\big). \tag{1.24}$$

Proposition 1.21 implies one half of (1.24). To obtain the reverse inequality, consider the images of $\mathcal{G}(s)$ under mappings of the form $z + k\bar{z}$, $k \nearrow 1$. They converge to the domain $\mathbb{C} \setminus ([-2, 2] \cup [2s, \infty))$ which is a rescaled copy of $\mathcal{T}\big(\tfrac{s-1}{2}\big)$.

The identity (1.24) somewhat resembles the well-known relation between conformal moduli of the Grötzsch and Teichmüller rings [1],

$$\operatorname{Mod} \mathcal{G}(s) = \frac{1}{2} \operatorname{Mod} \mathcal{T}(s^2 - 1). \tag{1.25}$$

**Remark 1.26.** Since equality holds in (1.24), Proposition 1.21 is sharp. The pair of domains $\Omega = \mathcal{T}\big(\tfrac{s-1}{2}\big)$ and $\Omega^* = \mathcal{G}(s)$ can serve as a test case for whether equality in (1.15) implies the existence of a harmonic bijection.

## 2. The Nitsche conjecture and beyond

As in the preceding section, we let $\mathbb{A} = A(r, R) := \{z \in \mathbb{C} \colon r < |z| < R\}$ denote a circular annulus in the complex plane. Schottky's theorem (1877) asserts that $A(r, R)$ can be mapped conformally onto another annulus $\mathbb{A}^* = A(r_*, R_*)$ if and only if

$$\frac{R}{r} = \frac{R_*}{r_*}, \tag{2.1}$$

which is the reason why the conformal modulus $\operatorname{Mod} \Omega$ of a doubly connected domain $\Omega$ is well-defined.

Since harmonic mappings are more flexible than conformal ones, one may ask: When does there exist a harmonic homeomorphism of $A(r, R)$ onto $A(r_*, R_*)$? Johannes C. C. Nitsche conjectured [26] that such a mapping exists if and only if

$$\frac{R_*}{r_*} \geqslant \frac{1}{2} \left( \frac{R}{r} + \frac{r}{R} \right). \tag{2.2}$$

To understand where (2.2) comes from, consider radial mappings, that is, $f \colon \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ of the form

$$f(\rho e^{i\theta}) = g(\rho) e^{i\theta} \tag{2.3}$$

where $g$ is a real-valued strictly monotone function. From Laplace's equation $\Delta f = 0$ one obtains that $g$ must be of the form $g(\rho) = A\rho + B\rho^{-1}$. The monotonicity of $g$ imposed a restriction on the coefficients $A$ and $B$, and thus yields (2.2).

But of course, harmonic homeomorphisms between annuli do not have to be radial. Let us give an explicit non-radial example. It is convenient to scale both annuli so that the inner radius of each is 1; that is, $r = r_* = 1$. Define

$$f(z) = \frac{z+a}{1+\bar{a}z} + c \log|z|, \qquad z \in \mathbb{A}, \tag{2.4}$$

where $a, c \in \mathbb{C}$ and $|a| < 1$. Clearly, $f$ maps the unit circle onto itself. The outer boundary of $\mathbb{A}$ is also mapped onto a circle, and we can choose $c$, depending on $a$ and $R$, so that this circle is centered at 0. We also want the mapping $f$ to have nonnegative Jacobian determinant

$$J_f := \left|\frac{\partial f}{\partial z}\right|^2 - \left|\frac{\partial f}{\partial \bar{z}}\right|^2 \tag{2.5}$$

everywhere in $\mathbb{A}$. This leads to an upper bound on $|a|$ which depends on $R$. Within this upper bound we have a one-parameter family of harmonic homeomorphisms.

The Nitsche conjecture was proved in [14, 15]. The following is an informal outline of the proof in which we assume that all mappings under consideration are smooth up to the boundary of their domain.

Suppose $f\colon A(1,R) \to A(1,R_*)$ is a sense-preserving harmonic homeomorphism that preserves the order of boundary components. Denote $\mathbb{T}_\rho = \{|z| = \rho\}$ and $\mathbb{T} = \mathbb{T}_1$. Introduce the quantity

$$U(\rho) := \fint_{\mathbb{T}_\rho} |f|^2, \qquad 1 \leqslant \rho \leqslant R,$$

where $\fint$ stands for the average value on the circle $\mathbb{T}_\rho$. The Nitsche conjecture will follow from $U(R) \geqslant (R + R^{-1})^2/4$. We would like to interpret $U$ as a subsolution of a second-order differential equation. The initial values are $U(1) = 1$ and $U'(1) \geqslant 0$, the latter being a consequence of $U(\rho) > 1$ for $\rho > 1$. Assume for a moment that $U$ satisfies the second-order differential inequality

$$\mathcal{L}[U] := \frac{d}{d\rho}\left[\rho^3 \frac{d}{d\rho}\left(\frac{U}{\rho^2+1}\right)\right] \geqslant 0. \tag{2.6}$$

Multiply (2.6) by $(R^2-\rho^2)/\rho^2$ and integrate by parts twice:

$$
\begin{aligned}
0 &\leqslant \int_1^R \frac{R^2-\rho^2}{\rho^2}\mathcal{L}[U]\,d\rho \\
&= -\frac{R^2-1}{2}(U'(1)-1) + \int_1^R 2R^2\frac{d}{d\rho}\left(\frac{U}{\rho^2+1}\right)d\rho \\
&\leqslant \frac{R^2-1}{2} + \frac{2R^2}{R^2+1}U(R) - R^2 \\
&= \frac{2R^2}{R^2+1}\left\{U(R) - \frac{(R^2+1)^2}{4R^2}\right\}
\end{aligned}
\tag{2.7}
$$

as desired.

The problem with the above “solution” is that the inequality (2.6) fails for some harmonic homeomorphisms; counterexamples can be found in the form (2.4). To understand this further, consider the orthogonal expansion

$$
f = \sum_{n\in\mathbb{Z}} f_n \tag{2.8}
$$

where

$$
f_n(z) = a_n z^n + \frac{b_n}{\bar{z}^n}, \quad n \neq 0; \qquad f_0(z) = a_0 \log|z| + b_0.
$$

Then

$$
U(\rho) = \sum_{n\in\mathbb{Z}} U_n(\rho) \quad \text{where} \quad U_n(\rho) = \fint_{\mathbb{T}_\rho} |f_n|^2.
$$

Since (2.6) is linear it is enough to check it for each $U_n$ separately. They all satisfy it, except for $U_0$.

Although the pointwise inequality $\mathcal{L}[U] \geqslant 0$ fails, the computation (2.7) requires only the integral inequality

$$
\int_1^R \frac{R^2-\rho^2}{\rho^2}\,\mathcal{L}[U]\,d\rho \geqslant 0. \tag{2.9}
$$

The following identity [15] comes into play.

$$\begin{aligned}
&\frac{2R^2}{R^2+1}\fint_{\mathbb{T}_R}|f|^2-\frac{R^2+1}{2}\fint_{\mathbb{T}}|f|^2\\
&-(R^2-1)\fint_{\mathbb{T}}|f||f|_\rho-(R^2-1)\log R\fint_{\mathbb{T}}\operatorname{Im}\bar f\,(f_\theta-if)\\
&=\frac{1}{\pi}\iint_{\mathbb{A}}\left[(R^2-1)\log\frac{R}{\rho}+\frac{R^2-\rho^2}{\rho^2}\right]\cdot\left|\frac{\rho f_\rho-if_\theta}{1+\rho^2}-\frac{2\rho^2 f}{(1+\rho^2)^2}\right|^2\\
&+\frac{1}{\pi}\iint_{\mathbb{A}}\left[(R^2-\rho^2)-(R^2-1)\log\frac{R}{\rho}\right]\cdot\left|\frac{\rho f_\rho+if_\theta}{1+\rho^2}+\frac{2f}{(1+\rho^2)^2}\right|^2.
\end{aligned}\tag{2.10}$$

It should be emphasized that (2.10) holds for any complex-valued harmonic function that is continuously differentiable up to the boundary of $\mathbb{A}$. Now we must use the assumption that $f$ is a homeomorphism onto $A(1,R_*)$. In this case the left hand side of (2.10) is less than or equal to

$$\frac{2R^2}{R^2+1}\fint_{\mathbb{T}_R}|f|^2-\frac{R^2+1}{2}.\tag{2.11}$$

Thus, in order to show that (2.11) is nonnegative it suffices to check that the right hand side of (2.10) is nonnegative. This comes down to the signs of coefficients

$$(R^2-1)\log\frac{R}{\rho}+\frac{R^2-\rho^2}{\rho^2}\quad\text{and}\quad(R^2-\rho^2)-(R^2-1)\log\frac{R}{\rho}.$$

The first of them is indeed nonnegative. So is the second, when $R\leqslant e$. However, it takes on negative values when $R>e$.

The stubborn case $R>e$ forces us to abandon the differential operator $\mathcal{L}[U]$. Instead of the neat identity (2.10) we employ a less satisfying inequality

$$\begin{aligned}
&\fint_{\mathbb{T}_\rho}|f|^2-\left(\frac{\rho+\rho^{-1}}{2}\right)^2\fint_{\mathbb{T}}\operatorname{Im}(\bar f f_\theta)-2\fint_{\mathbb{T}}|f||f|_\rho\\
&-\frac{\rho^2-4-\rho^{-2}}{2}\left\{\fint_{\mathbb{T}}\det Df+\left[\fint_{\mathbb{T}}\det Dg-\frac{1}{2\pi}\iint_{\mathbb{D}}|Dg|^2\right]\right\}\geqslant 0
\end{aligned}\tag{2.12}$$

in which $g$ is the harmonic extension of $f_{|\mathbb{T}}$ to the unit disk $\mathbb{D}$. Inequality (2.12) holds for any complex-valued harmonic function $g$ provided that $\rho>e$.

Now recall that $f$ is also a sense-preserving homeomorphism. Its winding number on $\mathbb{T}$ amounts to

$$\fint_{\mathbb{T}}\operatorname{Im}(\bar f f_\theta)=1.$$

Also, the Jacobian det $Df$ is nonnegative, as is the integral

$$\fint_{\mathbb{T}} |f||f|_\rho.$$

It remains to verify that the expression in square brackets in (2.12) is nonnegative when $g$ is a sense-preserving harmonic homeomorphism of $\mathbb{D}$ onto itself. This is a nontrivial result in itself.

One may wonder about the underlying reason for separate consideration of annuli with very small modulus and annuli with very large modulus. In the expansion (2.8) the most difficult term to handle is $f_0$, as it grows slowly and lacks the convexity properties shared by $f_n$ for $n \neq 0$. When the radius $\rho$ is small, the logarithmic term $f_0$ is comparable to the powers of $\rho$, which allows us to take advantage of its contribution to the integral means $U(\rho)$. But when $\rho$ is large, $|f_0|$ contributes a negligible amount to $U(\rho)$ while its effect on $U(1)$ and $U'(1)$ can be substantial. This necessitates the introduction of integral averages other than $U(\rho)$, such as the average of the Jacobian of $f$.

The following area contraction result appeared as a follow-up to the investigation of the case $R > e$. Let $\mathbb{D}_r = \{z : |z| < r\}$; the area of a set $E$ will be denoted by $|E|$.

**Theorem 2.13.** [22] *Let $f : \mathbb{D} \xrightarrow{\text{onto}} \mathbb{D}$ be a bijective harmonic mapping. Then*

$$|f(\mathbb{D}_r)| \leqslant |\mathbb{D}_r|, \qquad 0 < r < 1. \tag{2.14}$$

The area contraction inequality (2.14) is also true for any holomorphic mapping $f : \mathbb{D} \to \mathbb{D}$, which is easy to prove. However, the following questions remains open.

**Question 2.1.** Does (2.14) hold for injective harmonic mappings $f : \mathbb{D} \to \mathbb{D}$? Or even for arbitrary harmonic mappings $f : \mathbb{D} \to \mathbb{D}$?

Let us now take another look at mappings between annuli, from the viewpoint of quasiconformality. Recall the distortion theorems of H. Grötzsch (1928) which presaged the development of the theory of quasiconformal mappings [1].

**Theorem 2.15.** *If $f : A(r, R) \xrightarrow{\text{onto}} A(r_*, R_*)$ is a $K$-quasiconformal mapping, then*

$$\left(\frac{R}{r}\right)^{1/K} \leqslant \frac{R_*}{r_*} \leqslant \left(\frac{R}{r}\right)^{K}. \tag{2.16}$$

*Equalities are attained, uniquely modulo conformal automorphisms of $\mathbb{A}$, for the multiples of the mappings $f(z) = |z|^{\frac{1}{K}-1} z$ and $h(z) = |z|^{K-1} z$, respectively.*

We notice at once that the extremal mappings in Theorem 2.15 fail to be harmonic except for $K = 1$. This naturally leads one to expect a better inequality for harmonic quasiconformal mappings. And indeed, we have the following sharp result.

**Theorem 2.17.** [16] *If* $f\colon A(r,R) \xrightarrow{\text{onto}} A(r_*,R_*)$ *is a* $K$*-quasiconformal harmonic mapping, then*

$$\frac{R_*}{r_*} \geqslant \frac{K+1}{2K}\frac{R}{r} + \frac{K-1}{2K}\frac{r}{R}. \tag{2.18}$$

*Equality is attained, uniquely modulo conformal automorphisms of* $\mathbb{A}$*, for*

$$f(z) = \frac{K+1}{2K}\frac{z}{r} + \frac{K-1}{2K}\frac{r}{\bar{z}}. \tag{2.19}$$

In contrast to (2.16), the inequality (2.18) is one-sided: it only gives the lower bound for the modulus of the image. There is a natural conjecture for the upper bound, but it remains open.

**Conjecture 2.1.** If $f\colon A(r,R) \xrightarrow{\text{onto}} A(r_*,R_*)$ is a $K$-quasiconformal harmonic mapping, then

$$\frac{R_*}{r_*} \leqslant \frac{K+1}{2}\frac{R}{r} - \frac{K-1}{2}\frac{r}{R}. \tag{2.20}$$

Equality is attained, uniquely modulo conformal automorphisms of $\mathbb{A}$, for $f(z) = \frac{K+1}{2}\frac{z}{r} - \frac{K-1}{2}\frac{r}{\bar{z}}$.

Theorem 2.17 and Conjecture 2.1 impose more constraints on $f$ than the original Nitsche conjecture did. In the opposite direction, one can try to remove the assumption that $f$ is a bijection.

**Conjecture 2.2.** (Generalized Nitsche bound) Let $\mathbb{A} = A(r,R)$ and $\mathbb{A}^* = A(r_*,R_*)$ be a pair of circular annuli. Suppose that $f\colon \mathbb{A} \to \mathbb{A}^*$ is a harmonic mapping not homotopic to a constant within the class of continuous mappings from $\mathbb{A}$ to $\mathbb{A}^*$. Then

$$\frac{R_*}{r_*} \geqslant \frac{1}{2}\left(\sqrt{\frac{R}{r}} + \sqrt{\frac{r}{R}}\right). \tag{2.21}$$

If $f$ is in addition injective, then

$$\frac{R_*}{r_*} \geqslant \frac{1}{2}\left(\frac{R}{r} + \frac{r}{R}\right). \tag{2.22}$$

The mapping $f(z) = z + \frac{1}{\bar{z}}$ and the domain $\mathbb{A} = A(1/R, R)$ turn (2.21) into an equality. The mapping $f$ also shows the sharpness of (2.22) when restricted to the annulus $A(1, R)$.

The theorems and conjectures stated above have implications for the theory of minimal surfaces (see [7, 27] for background on minimal surfaces and their relation to planar harmonic mappings). Let $C$ be the interior of a right circular cylinder with inner radius 1 and outer radius $R > 1$. To fix ideas, suppose that the axis of symmetry of $C$ is vertical. Consider a doubly connected minimal surface $S$ which is contained in $C$, projects one-to-one on a horizontal plane, and has boundary curves lying on each component of $\partial C$. The Nitsche conjecture implies that under these conditions the conformal modulus of $S$ is greatest when $S$ is a one-sided slab of a catenoid. More precisely,

$$\operatorname{Mod} S \leqslant \log(R + \sqrt{R^2 - 1}). \tag{2.23}$$

If true, Conjecture 2.2 would imply that the same bound (2.23) holds for any minimal surface $S$ which is a graph contained in $C$ and not contractible within $C$. If the requirement that $S$ is a graph is dropped, then the conformal modulus of $S$ should be greatest when $S$ is a symmetric two-sided slab of a catenoid, namely

$$\operatorname{Mod} S \leqslant 2\log(R + \sqrt{R^2 - 1}). \tag{2.24}$$

We do not go into any details here and refer the reader to [15, 16] instead.

## 3. Hopf differentials and diffeomorphic approximation of Sobolev homeomorphisms

A quadratic differential on a domain $\Omega$ in the complex plane $\mathbb{C}$ takes the form $Q = F(z)\,\mathrm{d}z^2$, where $F$ is a complex function on $\Omega$. Given a complex harmonic function $h\colon \Omega \to \mathbb{C}$, the associated Hopf differential

$$Q_h = h_z \overline{h_{\bar{z}}}\,\mathrm{d}z^2$$

is holomorphic, meaning that

$$\frac{\partial}{\partial \bar{z}}\left(h_z \overline{h_{\bar{z}}}\right) = 0. \tag{3.1}$$

Naturally, the Sobolev space $W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{C})$ should be considered as the domain of definition of equation (3.1). This places $h_z \overline{h_{\bar{z}}}$ in $L^1_{\mathrm{loc}}(\Omega)$, so the complex Cauchy-Riemann derivative $\frac{\partial}{\partial \bar{z}}$ applies in the sense of distribution. By Weyl's lemma $h_z \overline{h_{\bar{z}}}$ is a holomorphic function.

Conversely, if a Hopf differential $Q_h = h_z \overline{h_{\bar{z}}}\,\mathrm{d}z^2$ is holomorphic for some $C^1$-mapping $h$, then $h$ is harmonic at the points where the Jacobian determinant

$J_h(z) = \det Dh = |h_z|^2 - |h_{\bar z}|^2 \neq 0$, see [8, 10.5]. Here the assumption that $J_h(z) \neq 0$ is critical. Let us illustrate it by the following.

**Example 3.2.** Consider a mapping $h \in C^{1,1}(\mathbb{C}_\circ)$ defined on the punctured plane $\mathbb{C}_\circ = \mathbb{C} \setminus \{0\}$ by the rule

$$h(z) = \begin{cases} \frac{z}{|z|} & \text{for } 0 < |z| \leqslant 1 \\ \frac{1}{2}\left(z + \frac{1}{\bar z}\right) & \text{for } 1 \leqslant |z| < \infty. \end{cases} \tag{3.3}$$

Direct computation shows that

$$h_z(z) = \begin{cases} \frac{1}{2}|z|^{-1} & \text{for } 0 < |z| \leqslant 1 \\ \frac{1}{2} & \text{for } 1 \leqslant |z| < \infty \end{cases}$$

and

$$h_{\bar z}(z) = \begin{cases} -\frac{1}{2}|z|\bar z^{-2} & \text{for } 0 < |z| \leqslant 1 \\ -\frac{1}{2}\bar z^{-2} & \text{for } 1 \leqslant |z| < \infty. \end{cases}$$

Thus

$$Q_h = -\frac{\mathrm{d}z^2}{4z^2} \qquad \text{in } \mathbb{C}_\circ. \tag{3.4}$$

It may be worth mentioning that the mapping $h$ in (3.3) is the unique (up to rotation of $z$) minimizer of the Dirichlet energy

$$\mathcal{E}[H] = \int_{\mathbb{A}} |DH|^2$$

over the annulus $\mathbb{A} = A(r,R) = \{z : r < |z| < R\}$, $0 < r < 1 < R$, subject to all weak limits of homeomorphisms $H : \mathbb{A} \xrightarrow{\text{onto}} A(1, R_*)$, where $R_* = \frac{1}{2}\left(R + \frac{1}{R}\right)$, see [3].

It is natural to ask whether a Sobolev homeomorphism $h \in W^{1,2}_{\text{loc}}(\Omega, \mathbb{C})$ with holomorphic Hopf differential is harmonic. This question originated in a series of papers by Eells, Lemaire and Sealey [9, 10, 28].

**Theorem 3.5.** [18] *Every homeomorphism $h$ of Sobolev class $W^{1,2}_{loc}(\Omega, \mathbb{C})$ that satisfies equation* (3.1) *is harmonic.*

The Eells-Lemaire problem under the additional assumption that $h$ is a quasiconformal homeomorphism was settled earlier by Hélein [12] in the affirmative. Theorem 3.5 dispenses with the quasiconformality condition and treats general planar $W^{1,2}$-homeomorphisms. Since the inverse of such a homeomorphism need not be in any Sobolev class, some difficulties arise. They were overcome with the aid of an approximation theorem which we will present next.

The Sobolev space $W^{1,p}(\Omega,\mathbb{R}^n)$, $1 \leqslant p < \infty$, is the completion of $C^\infty$-smooth mappings having finite Sobolev norm

$$\|f\|_{W^{1,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \|Df\|_{L^p(\Omega)} < \infty.$$

By the definition, any mapping $f \in W^{1,p}$ can be approximated by smooth mappings in the $W^{1,p}$ norm. If $f$ happens to be invertible, one may want the approximating smooth mappings to be invertible as well.

**Question 3.1.** Suppose that $h\colon \Omega \xrightarrow{\text{onto}} \Omega^*$ is a homeomorphism in $W^{1,p}(\Omega,\mathbb{R}^n)$. Can $h$ be approximated by diffeomorphisms $h_j\colon \Omega \xrightarrow{\text{onto}} \Omega^*$ in $W^{1,p}(\Omega,\mathbb{R}^n)$?

A different (but equivalent in dimensions $n \leqslant 3$) version of Question 3.1 asks for $h_j$ to be piecewise affine invertible mappings. In this form the approximation problem was put forward by J. M. Ball [4, 5] who attributed it to L.C. Evans. The paper [19] provides an affirmative solution of the Ball-Evans problem in the planar case when $1 < p < \infty$.

**Problem 3.1.** Does the approximation result in [19] hold when $p = 1$? Can it be extended to $n = 3$?

Our construction of an approximating diffeomorphism heavily relies on the following $p$-harmonic replacement argument. Let $U \subset \mathbb{C}$ be a bounded simply connected domain. For any $h_\circ \in W^{1,p}(U,\mathbb{C}) \cap C(\overline{U})$, $1 < p < \infty$, there exists a unique *coordinate-wise $p$-harmonic mapping* $h\colon U \to \mathbb{C}$; that is,

$$\begin{cases} \operatorname{div}|\nabla u|^{p-2}\nabla u = 0 \\ \operatorname{div}|\nabla v|^{p-2}\nabla v = 0 \end{cases}, \qquad 1 < p < \infty, \quad h = u + iv$$

such that $h_{|\partial U} = h_{\circ|\partial U}$.

The Radó-Kneser-Choquet Theorem ($p = 2$) and the Alessandrini-Sigalotti extension [2] of the Radó-Kneser-Choquet Theorem ($1 < p < \infty$) give a great tool for constructing coordinate-wise $p$-harmonic homeomorphisms. Given a sense-preserving homeomorphism $h_\circ\colon \partial U \to \partial\Gamma$ onto a convex Jordan curve $\Gamma$, the $p$-harmonic replacement produces a $C^\infty$-diffeomorphism from $U$ onto the bounded component of $\mathbb{C}\setminus\Gamma$ such that $h$ has boundary values $h_\circ$. In particular, $J_h(z) > 0$ in $U$.

We end this section with a strengthened version of the Radó-Kneser-Choquet Theorem found in [13]. We denote by $\mathfrak{P}_U f$ the harmonic replacement of $f$ in a domain $U$, as described above (with $p = 2$).

**Theorem 3.6.** *Let $U$ and $D$ be bounded simply connected domains in $\mathbb{C}$ with $D$ convex. Suppose that $f\colon \partial U \to \partial D$ is a mapping that can be continuously extended to a homeomorphism of $U$ onto $D$. Then $\mathfrak{P}_U f$ is a harmonic homeomorphism of $U$ onto $D$.*

In other words, if there is *some* homeomorphic extension of $f$, then there is a harmonic homeomorphism extension (necessarily unique). The difference between Theorem 3.6 and the versions of the Radó-Kneser-Choquet theorem commonly found in the literature (e.g., [7]) is that $U$ is not assumed to be a Jordan domain. Let us derive Theorem 3.6 from the Jordan case.

**Proof.** Let $F\colon U \xrightarrow{\text{onto}} D$ be some homeomorphism that extends $f$. Denote $g = \mathfrak{P}_U f$. It is not difficult to show that $g(U) = D$; the issue is the injectivity of $g$.

Let $\{D_n\}$ be an exhaustion of $D$ by convex domains. Define $U_n = F^{-1}(D_n)$ and note that $U_n$ is a Jordan domain. By the Radó-Kneser-Choquet Theorem the mapping $g_n := \mathfrak{P}_{U_n} F$ is a harmonic homeomorphism of $U_n$ onto $D_n$. As $n \to \infty$, $g_n \to \mathfrak{P}_U f$ uniformly on compact subsets of $U$. This can be seen by harmonic measure estimates, or directly from the Wiener solution of the Dirichlet problem presented in [11].

The convergence of harmonic functions implies the convergence of their derivatives. Therefore $J_{g_n} \to J_g$ pointwise, in particular $J_g \geqslant 0$. This means that the holomorphic functions $g_z$ and $\overline{g_{\bar z}}$ satisfy the inequality $|\overline{g_{\bar z}}| \leqslant |g_z|$. The latter is only possible when either $|\overline{g_{\bar z}}| < |g_z|$ in $U$ or $|\overline{g_{\bar z}}| \equiv |g_z|$ in $U$. The second case cannot occur, for it would yield $J_g \equiv 0$, contradicting $g(U) = D$. Therefore $J_h > 0$, so the mapping $g$ is locally invertible. But it is also a uniform limit of homeomorphisms $g_n$ (locally), which implies that $h$ is injective in $U$. ∎

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