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Hyperbolic Distortion for Holomorphic Maps

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To Alan and Toni Beardon for their friendship

Abstract. Suppose Ω_j is a hyperbolic region in the complex plane $\mathbb{C}$ with hyperbolic metric $\lambda_j(z)|dz|$ and associated hyperbolic distance function $h_j(z, w)$, $j = 1, 2$. Let $\mathcal{H} = \mathcal{H}(\Omega_1, \Omega_2)$ be the family of holomorphic maps $f : \Omega_1 \to \Omega_2$ and $\mathcal{C}$ the subfamily of holomorphic coverings of Ω_1 onto Ω_2. The general version of Pick's Theorem provides a universal constraint on $\mathcal{H}$; every function in $\mathcal{H}$ is a weak contraction relative to the hyperbolic metric and functions in $\mathcal{C}$ are local isometries. In particular, each $f \in \mathcal{H} \setminus \mathcal{C}$ is a strict contraction: $h_2(f(z), f(w)) < h_1(z, w)$ for all distinct $z, w \in \Omega_1$ and $|f^h(z)| < 1$ for all $z \in \Omega_1$, where $f^h(z) = \lambda_2(f(z))f'(z)/\lambda_1(z)$ denotes the hyperbolic derivative. The first part of the paper deals with the issue of quantifying the size of the strict contraction from above and below in terms of the hyperbolic derivative at a point. The second part of the paper is concerned with comparisons between $\mathcal{C}$ and $\mathcal{H}^* = \mathcal{H} \setminus \mathcal{C}$ as subsets of the space $\mathcal{H}$ endowed with the metric of uniform convergence on compact subsets. If Ω_1 is not conformally equivalent to either $\mathbb{D}$ or $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$, then the complementary sets $\mathcal{C}$ and $\mathcal{H}^*$ form a separation of $\mathcal{H}$ because both are closed. This separation property explains a number of rigidity results for $\mathcal{H}$ when Ω_1 is not conformally equivalent to either $\mathbb{D}$ or $\mathbb{D}^*$. If Ω_1 is conformally equivalent to either $\mathbb{D}$ or $\mathbb{D}^*$, then there is flexibilty: $\mathcal{C}$ is the boundary of $\mathcal{H}^*$.

Keywords. hyperbolic regions, Pick's Theorem, hyperbolic distortion.

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1. Introduction

In many ways this paper can be regarded as a follow-up to [7]. The paper [7] deals with holomorphic self-maps of a hyperbolic region while this paper is concerned with holomorphic maps of one hyperbolic region into another. In the case that both regions are the same, some theorems of this paper reduce to

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theorems in [7]. Not all results from [7] extend to the context of this paper; at the same time there are results here without an analog in [7].

A region Ω in the complex plane $\mathbb{C}$ is hyperbolic if $\mathbb{C} \setminus \Omega$ contains at least two points. Let $\lambda_\Omega(z)|dz|$ denote the hyperbolic metric on Ω and h_Ω the associated hyperbolic distance function. For the unit disk $\mathbb{D}$, the hyperbolic metric is $\lambda_\mathbb{D}(z)|dz| = 2|dz|/(1-|z|^2)$ and the hyperbolic distance function is $h_\mathbb{D}(a,b) = 2\tanh^{-1} p(a,b)$, where $p(a,b) = |a-b|/|1-\bar{a}b|$ is the pseudohyperbolic distance function on $\mathbb{D}$. If $g : \mathbb{D} \to \Omega$ is a holomorphic covering, then $\lambda_\Omega(g(z))|g'(z)| = \lambda_\mathbb{D}(z)$. For a simply connected hyperbolic region Ω, g is a conformal mapping and $h_\Omega(g(z), g(w)) = h_\mathbb{D}(z,w)$ for all $z, w \in \mathbb{D}$. If Ω is multiply connected, then the isometry property $h_\Omega(g(z), g(w)) = h_\mathbb{D}(z,w)$ only holds locally. For $a \in \Omega$ let $g_a : \mathbb{D} \to \Omega$ be the unique holomorphic covering that satisfies $g_a(0) = a$ and $g_a'(0) > 0$. In this situation $\lambda_\Omega(a) = 2/g_a'(0)$. For $a \in \Omega$ and $r > 0$, the hyperbolic disk with center a and hyperbolic radius r is $D_\Omega(a,r) = \{z \in \Omega : h_\Omega(a,z) < r\}$. In particular, $D_\mathbb{D}(0,r) = \{z : |z| < \tanh(r/2)\}$ and $g_a(D_\mathbb{D}(0,r)) = D_\Omega(a,r)$.

Typically we will be concerned with a pair of hyperbolic regions, say Ω_j, $j = 1, 2$. In this context we will use a subscript j to indicate a quantity associated with Ω_j. For example, $\lambda_j(z)|dz|$ denotes the hyperbolic metric on Ω_j and $h_j(z,w)$ the associated hyperbolic distance function. If $f : \Omega_1 \to \Omega_2$ is a holomorphic map, then the hyperbolic derivative of f at z is the complex number $f^h(z) = \lambda_2(f(z))f'(z)/\lambda_1(z)$ and the real number $|f^h(z)|$ is the hyperbolic change of scale at z. It is convenient to fix notation for several families of holomorphic maps. For hyperbolic regions Ω_j, $j = 1, 2$, let $\mathcal{H} = \mathcal{H}(\Omega_1, \Omega_2)$ be the set of all holomorphic maps $f : \Omega_1 \to \Omega_2$. The collection of holomorphic coverings of Ω_1 onto Ω_2 is denoted by $\mathcal{C}$. We suppose that $\mathcal{H}$ is equipped with the metric topology of uniform convergence on compact subsets (see [1, p.220]) derived from using the hyperbolic distance on both Ω_1 and Ω_2. Note that if $f_n \in \mathcal{H}$, and if $f_n \to \zeta$ uniformly on compact subsets Ω_1, where $\zeta \in \partial\Omega_2$, then f_n is not a convergent sequence in $\mathcal{H}$ (because the constant map with value ζ is not a map of Ω_1 to Ω_2). Set $\mathcal{H}^* = \mathcal{H} \setminus \mathcal{C}$. Pick's Theorem gives a universal constraint on functions in $\mathcal{H}$.

Pick's Theorem (general version). *Suppose that Ω_j is a hyperbolic region in $\mathbb{C}$, $j = 1, 2$, and $f \in \mathcal{H}$.*
(a) *For all $z \in \Omega_1$, $|f^h(z)| \le 1$. If equality holds at some point of Ω_1, then $f \in \mathcal{C}$. Conversely, if $f \in \mathcal{C}$, then equality holds at every point of Ω_1.*
(b) *For all $z, w \in \Omega_1$, $h_2(f(z), f(w)) \le h_1(z,w)$. If equality holds for a pair of distinct points $z, w \in \Omega_1$, then $f \in \mathcal{C}$. Conversely, if $f \in \mathcal{C}$, then each point of Ω_1 has a neighborhood in which equality holds. If Ω_2 is simply connected, then each $f \in \mathcal{C}$ is a conformal mapping and equality holds for all pairs of points.*

Parts (a) and (b) of the general form of Pick's Theorem are equivalent. Integration of the inequality $|f^h(\zeta)| \le 1$ along a hyperbolic geodesic joining z and w produces the inequality in (b). Conversely, the inequality in (b) implies the inequality in (a) since $h_2(f(z), f(w))/h_1(z, w) \to |f^h(z)|$ when $w \to z$. If $f \notin \mathcal{C}$, then the strict inequalities in the general version of the Pick's Theorem can be made quantitative in terms of the hyperbolic derivative at a point. The first part of the paper concerns two types of quantitative strengthenings of the general version of Pick's Theorem. Moreover, certain local lower bounds hold. Results of this type are known in certain contexts, see [3], [4], [5], and [7].

Theorem 1.1. *Suppose that Ω_j is a hyperbolic region in $\mathbb{C}$, $j = 1, 2$, and $f \in \mathcal{H}$.*
(a) *Then for all $z, w \in \Omega_1$*

$$\frac{|f^h(w)| - \tanh h_1(w,z)}{1 - |f^h(w)| \tanh h_1(w,z)} \le |f^h(z)| \le \frac{|f^h(w)| + \tanh h_1(w,z)}{1 + |f^h(w)| \tanh h_1(w,z)}. \tag{1.1}$$

The lower bound is positive for $h_1(z, w) < \tanh^{-1} |f^h(w)|$.
(b) *For all $z, w \in \Omega_1$*

$$h_2(f(z), f(w)) \le \log\big(\cosh h_1(z,w) + |f^h(w)| \sinh h_1(z,w)\big). \tag{1.2}$$

The inequalities in Theorem 1.1(a) are weak versions of Pick's Theorem for hyperbolic derivatives

$$h_{\mathbb{D}}(|f^h(z)|, |f^h(w)|) \le 2h_1(z,w); \tag{1.3}$$

see [3], [5] and Theorem 5.1 of [7]. In the special case that $\Omega_1 = \Omega_2 = \mathbb{D}$, $w = 0$ and $f(0) = 0$, the upper bound (but in Euclidean terms) was established by Goluzin ([9], [10, p. 335]). Yamashita rediscovered Goluzin's result [19] and extended it [20]. The upper bound (1.2) is due to Beardon and Carne [4]. The simple identity

$$\begin{aligned} &\log\big(\cosh h_1(z,w) + |f^h(w)| \sinh h_1(z,w)\big) \\ &= h_1(z,w) + \log \frac{1}{2}\big(1 + |f^h(w)| + (1 - |f^h(w)|)e^{-2h_1(z,w)}\big) \end{aligned} \tag{1.4}$$

shows that inequality (1.2) improves the upper bound in part (b) of the general version of Pick's Theorem by roughly the negative additive factor $\log \frac{1}{2}(1 + |f^h(w)|)$ when z is far from w.

Given a hyperbolic region Ω and $a \in \Omega$, let $R_\Omega(a)$ denote the maximum value of $r > 0$ such that $D_\Omega(a, r)$ is simply connected. If $g : \mathbb{D} \to \Omega$ is a covering with $g(0) = a$, then g is injective on $D_{\mathbb{D}}(0, R_\Omega(a))$, maps this disk injectively onto $D_\Omega(a, R_\Omega(a))$ and $h_\Omega(a, g(z)) = h_{\mathbb{D}}(0, z)$ for all $z \in D_{\mathbb{D}}(0, R_\Omega(a))$. For this reason the quantity $R_\Omega(a)$ is sometimes called the *radius of injectivity* at a. Ω is *hyperbolically uniformly simply connected* if $R_\Omega = \inf\{R_\Omega(a) : a \in \Omega\} > 0$.

If Ω is simply connected, then $R(\Omega) = +\infty$. This concept is sometimes called *uniformly thick* by others [14].

Theorem 1.2. *Suppose that Ω_j is a hyperbolic region in $\mathbb{C}$, $j = 1, 2$, and $f \in \mathcal{H}$.*
(a) *Then for each $w \in \Omega_1$, f is injective on $D_1(w, r(w))$, where*

$$r(w) = \min\{R_2(f(w)), \tanh^{-1} |f^h(w)|\},$$

for all $z \in D_1(w, (1/2)r(w))$,

$$-\log\big(\cosh h_1(z,w) - |f^h(w)| \sinh h_1(z,w)\big) \le h_2(f(z), f(w)) \tag{1.5}$$

and for $r < r(w)$

$$D_2(f(w), \rho(r)) \subset f(D_1(w, r)), \tag{1.6}$$

where $\rho(r) = -\log(\cosh r - |f^h(w)| \sinh r)$.
(b) *If Ω_2 is hyperbolically uniformly simply connected, then these results are valid with $r(w)$ replaced by $r'(w) = \min\{R_2, \tanh^{-1} |f^h(w)|\}$.*
For Ω_j, $j = 1, 2$, simply connected these results are sharp.

The identity

$$\begin{aligned} &-\log\big(\cosh h_1(z,w) - |f^h(w)| \sinh h_1(z,w)\big) \\ &= h_1(z,w) - \log \frac{1}{2}\big(1 + |f^h(w)| + (1 - |f^h(w)|)e^{2h_1(w,z)}\big) \end{aligned}$$

helps to illustrate how (1.5) controls the amount of contraction that is possible near w. If Ω_2 is hyperbolically uniformly simply connected, then Theorem 1.2 holds with $r(w)$ replaced by $r'(w) = \min\{R_2, \tanh^{-1} |f^h(w)|\}$. If Ω_2 is simply connected, then $r'(w) = \tanh^{-1} |f^h(w)|$.

Example 1.3. In general, the inequality (1.5) does not hold in a neighborhood of w that depends only on $|f^h(w)|$. To see this, consider the punctured unit disk $\mathbb{D}^*$ which is not hyperbolically uniformly simply connected. For each positive integer n the function $f_n(z) = z^n$ is a holomorphic self-covering of $\mathbb{D}^*$, so $|f_n^h(w)| = 1$ for all $w \in \mathbb{D}^*$. If inequality (1.5) held in some fixed neighborhood of w for all f_n, then it would give

$$h_{\mathbb{D}^*}(w,z) \le h_{\mathbb{D}^*}(f_n(w), f_n(z))$$

for all z in this neighborhood of w. For $w \in \mathbb{D}^*$ this inequality is not valid in any fixed neighborhood of w since for n sufficiently large there exists z in this neighborhood of w with $z \neq w$ and $f_n(z) = f_n(w)$. Observe that $R_2(f_n(w)) \to 0$ as $n \to \infty$.

The preceding results deal with improvements to Pick's Theorem for a particular function based on its hyperbolic derivative at a point. Next, we consider refinements of Pick's Theorem that require a restriction on Ω_1 and sometimes a normalization on the function. These types of results are called *rigidity theorems*. The Aumann-Carathéodory Rigidity Theorem [2] concerns holomorphic self-maps of multiply connected hyperbolic regions that have a fixed point. This was extended to the context of holomorphic maps $f : \Omega_1 \to \Omega_2$ of hyperbolic regions when Ω_1 is not simply connected in [17]; the extension produces a rigidity form of the general version of Pick's Theorem for multiply connected hyperbolic regions.

Generalized Aumann-Carathéodory Rigidity Theorem. *Suppose that Ω_j is a hyperbolic region with Ω_1 multiply connected and $a_j \in \Omega_j$, $j = 1, 2$. There is a number α in $[0, 1)$, such that if $f \in \mathcal{H}$ and $f(a_1) = a_2$, then*

$$|f^h(a_1)| \quad \begin{cases} = 1, & \text{if } f \in \mathcal{C}; \\ \leq \alpha, & \text{if } f \in \mathcal{H}^*. \end{cases}$$

The constant α depends only on a_j and Ω_j, $j = 1, 2$. The smallest value of α for which the result holds is denoted by $\alpha = \alpha(a_1, \Omega_1, a_2, \Omega_2)$ and is called the *generalized Aumann–Carathéodory rigidity constant*. The exact value of the Aumann-Carathéodory rigidity constant is known only when $\Omega_1 = \Omega_2$ is an annulus [18] or a punctured disk [7]. The Generalized Aumann-Carathéodory Rigidity Theorem provides global control on the hyperbolic contraction for noncoverings.

Theorem 1.4. *Suppose that Ω_j is a hyperbolic region with Ω_1 multiply connected, $a_j \in \Omega_j$, $j = 1, 2$, and associated rigidity constant $\alpha = \alpha(a_1, \Omega_1, a_2, \Omega_2)$. If $f \in \mathcal{H}^*$ and $f(a_1) = a_2$, then for all $z \in \Omega_1$*

$$|f^h(z)| \leq \frac{\alpha + \tanh h_1(a_1, z)}{1 + \alpha \tanh h_1(a_1, z)}$$

and

$$h_2(f(z), f(a_1)) \leq \log\big(\cosh h_1(a_1, z) + \alpha \sinh h_1(a_1, z)\big).$$

The Generalized Aumann-Carathéodory Rigidity Theorem requires a choice of base points. Heins [11] established an extension of the Aumann-Carathéodory Rigidity Theorem that does not involve any specified points.

Heins' Rigidity Theorem. *Suppose Ω is a hyperbolic plane region that is neither simply connected nor conformally equivalent to $\mathbb{D}^*$. If f is a conformal automorphism of Ω, then there does not exist a sequence (f_n) of holomorphic self-maps of Ω that converges locally uniformly to f and $f_n \neq f$ for all n.*

The classical Aumann-Carathéodory Rigidity Theorem is an immediate corollary of Heins' result. We extend Heins' Rigidity Theorem by replacing the family $\mathcal{A}$ of conformal mappings with the larger family $\mathcal{C}$ of holomorphic coverings.

Generalized Heins Rigidity Theorem. *Suppose that Ω_j, $j = 1, 2$, is a hyperbolic region in $\mathbb{C}$ and Ω_1 is not conformally equivalent to either $\mathbb{D}$ or $\mathbb{D}^*$. Then there does not exist a sequence (f_n) in $\mathcal{H}^*$ that converges locally uniformly to a function $f \in \mathcal{C}$.*

The Generalized Heins Rigidity Theorem is not valid if Ω_1 is conformally equivalent to either $\mathbb{D}$ or $\mathbb{D}^*$. To see this, let $\Omega_1 = \Omega_2$ be either $\mathbb{D}$ or $\mathbb{D}^*$. Then $f_n(z) = \frac{n}{n+1} z$ is a sequence in $\mathcal{H}^*$ that converges locally uniformly to the identity function $f(z) = z$ which lies in $\mathcal{C}$. The Generalized Aumann-Carathéodory Rigidity Theorem for multiply connected hyperbolic regions that are not conformally equivalent to $\mathbb{D}^*$ is an immediate consequence of the Generalized Heins Rigidity Theorem. There is a general version of rigidity in terms of the metric topology on $\mathcal{H}$.

Theorem 1.5. *Suppose that Ω_j, $j = 1, 2$, is a hyperbolic region.*
(a) *The set $\mathcal{C}$ is closed in $\mathcal{H}$.*
(b) *If Ω_1 is simply connected or is conformally equivalent to $\mathbb{D}^*$, then $\mathcal{C} = \partial\mathcal{H}^*$.*
(c) *If Ω is neither simply connected nor conformally equivalent to $\mathbb{D}^*$, then $\mathcal{C}$ and $\mathcal{H}^*$ form a separation of $\mathcal{H}$.*

A result similar to Theorem 1.5 is established in [7]. The difference is that the related theorem in [7] is valid only when $\Omega_1 = \Omega_2$ and $\mathcal{C}$ is replaced by the smaller family $\mathcal{A}$ of conformal automorphisms of $\Omega_1 = \Omega_2$. There is an error in Theorem 1.5(a) in [7] where it is asserted that $\mathcal{A}$ is compact. In fact, it is only closed. Here is an example in which $\mathcal{A}$ is closed and not compact. Let $\Omega = \{z : |\operatorname{Im} z| < 1\}$. Then $f_n(z) = z + n$, $n = 1, 2, \ldots$, is a sequence in the group $\mathcal{A}$ of conformal automorphisms of Ω. Every subsequence of (f_n) converges locally uniformly to ∞; there is no subsequence which converges to a conformal automorphism of Ω, so $\mathcal{A}$ is not compact.

In the final portion of the paper we measure the hyperbolic distance between a holomorphic function and a related 'extremal' function when Ω_1 is simply connected. In this context there is flexibility rather than rigidity. For example, we can give bounds on the convergence of a sequence (f_n) in $\mathcal{H}^*$ to a covering $f \in \mathcal{C}$. For the remainder of this section Ω_1 denotes a simply connected hyperbolic region and for $a_j \in \Omega_j$, $g_j : \mathbb{D} \to \Omega_j$ is the unique covering with $g_j(0) = a_j$ and $g_j'(0) > 0$, $j = 1, 2$. Note that g_1 is a conformal mapping because Ω_1 is simply connected. In this context the goal is to compare a holomorphic map with an associated covering. In order to state these results, we introduce notation for canonical coverings. For $a_j \in \Omega_j$, $\varphi = \varphi_{a_1,a_2} = g_2 g_1^{-1}$ is the unique covering of Ω_1 onto Ω_2 that satisfies $\varphi(a_1) = a_2$ and $\varphi'(a_1) > 0$, or equivalently, $\varphi^h(a_1) = 1$. For $a_1 \in \Omega_1$ and $\theta \in \mathbb{R}$ let $\rho_{a_1,\theta}$ denote the hyperbolic rotation of Ω_1 about a_1

through angle θ. Then $\rho_{a_1,\theta} = g_1\rho_\theta g_1^{-1}$, where ρ_θ is the Euclidean rotation about the origin through angle θ, and $\rho'_{a_1,\theta}(c) = e^{i\theta}$. For a fixed $a_1 \in \Omega_1$, $\{\rho_{a_1,\theta} : \theta \in \mathbb{R}\}$ is the group of conformal automorphisms of Ω_1 that fix a_1. Then $\varphi_\theta := \varphi\rho_{a_1,\theta}$ is the unique covering of Ω_1 onto Ω_2 that sends a_1 to a_2 and satisfies $\varphi_\theta^h(a_1) = e^{i\theta}$. For $a_1 \in \Omega_1$ and $\eta \in \mathbb{R}$, let $\Gamma_{a_1,\eta}$ denotes the hyperbolic geodesic ray emanating from a_1 that has Euclidean unit tangent $e^{i\eta}$ at a_1. Clearly, $\rho_{a_1,\theta}(\Gamma_{a_1,\eta}) = \Gamma_{a_1,\eta+\theta}$.

Here we state one type of comparison theorem in which a holomorphic map is compared to the covering that satisfies initial conditions at a_1 determined from f; two other types of comparisons are established in Section 5. For $f \in \mathcal{H}$ with Ω_1 simply connected, $a_1 \in \Omega_1$ and $f'(a_1) \neq 0$, let $\varphi_f : \Omega_1 \to \Omega_2$ be the unique holomorphic covering that satisfies $\varphi_f(a_1) = f(a_1)$ and $\arg\varphi'_f(a_1) = \arg f'(a_1)$; if $f'(a_1) = 0$, choose any covering φ_f that maps a_1 to $f(a_1)$. If $\Omega_j = \mathbb{D}$, $a_j = 0$, $f(0) = 0$ and $f'(0) = \alpha e^{i\theta}$, where $\alpha \in [0,1]$, then $\varphi_f(z) = e^{i\theta}z$. When Ω_1 is simply connected, there is no gap as in the Generalized Aumann-Carathéodory Rigidity Theorem. Rather as $|f^h(a_1)| \to 1$ the function f approaches a covering and the rate of convergence to a covering can be made quantitative. The general version of Pick's Theorem gives

$$\begin{aligned} h_2(f(z),\varphi_f(z)) &\le h_2(f(a),\varphi_f(z)) + h_2(f(z),f(a)) \\ &\le 2h_1(a,z). \end{aligned} \tag{1.7}$$

This simple estimate can be improved.

Theorem 1.6. *Suppose that Ω_j, $j = 1,2$, is a hyperbolic region with Ω_1 simply connected, $f \in \mathcal{H}$, $a_1 \in \Omega_1$ and $f'(a_1) \neq 0$. Let $\varphi_f \in \mathcal{C}$ be the unique covering that satisfies $\varphi_f(a_1) = f(a_1)$ and $\arg\varphi'_f(a_1) = \arg f'(a_1)$; if $f'(a_1) = 0$, choose any covering φ_f that maps a_1 to $f(a_1)$.*
(a) *For all $z \in \Omega_1$*

$$h_2(f(z),\varphi_f(z)) \le h_1(a_1,z) + \log\left(\cosh h_1(a_1,z) - |f^h(a_1)|\sinh h_1(a_1,z)\right). \tag{1.8}$$

(b) *If Ω_2 is also simply connected, then for all $z \in \Omega_1$*

$$h_1(a_1,z) - \log\left(\cosh h_1(a_1,z) + |f^h(a_1)|\sinh h_1(a_1,z)\right) \le h_2(f(z),\varphi_f(z)). \tag{1.9}$$

These bounds are sharp.

In the special case that $\Omega_1 = \Omega_2$ is simply connected and $f(a_1) = a_1$ this result is given in [7]. The identity

$$\begin{aligned} &\log\left(\cosh h_1(a_1,z) - |f^h(a_1)|\sinh h_1(a_1,z)\right) \\ &= -h_1(a_1,z) + \log\frac{1}{2}\left(1 + |f^h(a_1)| + (1 - |f^h(a_1)|)e^{2h_1(a_1,z)}\right) \end{aligned} \tag{1.10}$$

shows that the inequality (1.8) can be written as

$$h_2(f(z),\varphi_f(z)) \le \log\frac{1}{2}\left(1+|f^h(a_1)|+(1-|f^h(a_1)|)e^{2h_1(a_1,z)}\right).$$

When $|f^h(a_1)|$ is close to 1, the upper bound is small when z is not too far from a_1. Likewise, the identity

$$\log\left(\cosh h_1(a_1,z)+|f^h(a_1)|\sinh h_1(a_1,z)\right)$$
$$= -h_1(a_1,z)+\log\frac{1}{2}\left(1+|f^h(a_1)|+(1-|f^h(a_1)|)e^{-2h_1(a_1,z)}\right)$$

gives the lower bound

$$-\log\frac{1}{2}\left(1+|f^h(a_1)|+(1-|f^h(a_1)|)e^{-2h_1(a_1,z)}\right) \le h_2(f(z),\varphi_f(z)),$$

when Ω_2 is simply connected. This provides a lower bound on the distance between f and φ_f.

Versions of the preceding results are valid for holomorphic maps of one hyperbolic Riemann surface into another hyperbolic Riemann surface. For simplicity we have stated our results only for the special case in which the two hyperbolic surfaces are regions in the plane. The reader should note that the hyperbolic derivative is not invariantly defined in the context of Riemann surfaces while the hyperbolic change of scale is an invariant quantity.

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2. Extremal functions

For simply connected hyperbolic regions Ω_1 and Ω_2 Theorems 1.1, 5.1, 1.6 and 5.2 are sharp and the extremal functions are two-sheeted branched coverings. We explicitly give the extremal function when the two hyperbolic regions are both the unit disk $\mathbb{D}$ and the holomorphic self-map fixes the origin. The sharpness in the general case follows by conformal invariance. For $\alpha \in [0,1)$ the function T_α given by

$$T_\alpha(z) = \frac{z(z+\alpha)}{1+\alpha z}$$

is a two-sheeted branched self-covering of $\mathbb{D}$ with $T_\alpha(0)=0$, $T'_\alpha(0)=\alpha$ and $T''_\alpha(0)=2(1-\alpha^2)$. Let $S_\alpha(z)=-T_\alpha(-z)$ and $r_\alpha = \alpha/\left(1+\sqrt{1-\alpha^2}\right)$. Note that

$$h_{\mathbb{D}}(0,-r_\alpha) = \frac{1}{2}h_{\mathbb{D}}(0,\alpha) = \tanh^{-1}\alpha.$$

The injectivity of T_α in $\{|z| < r_\alpha\} = D_{\mathbb{D}}(0, \tanh^{-1}\alpha)$ follows from

$$T'_\alpha(z) = \frac{\alpha z^2 + 2z + \alpha}{(1+\alpha z)^2}$$

and the fact that $\operatorname{Re} T'_\alpha(z) > 0$ for $|z| < r_\alpha$. Since $T'_\alpha(-r_\alpha) = 0$, the function T_α is not injective in any larger disk centered at the origin, so the injectivity radius in Theorem 1.1(c) is sharp. It is straightforward to verify that $T_\alpha(-r_\alpha) = -r_\alpha^2$ and $T_\alpha(|z| < r_\alpha) \supset \{|w| < r_\alpha^2\}$. From

$$h_{\mathbb{D}}(0, T_\alpha(r)) = \log\left(\cosh h_{\mathbb{D}}(0,r) + \alpha \sinh h_{\mathbb{D}}(0,r)\right), \quad 0 < r < 1, \tag{2.1}$$

the sharpness of inequality (1.2) follows. The formula

$$h_{\mathbb{D}}(0, S_\alpha(r)) = -\log\left(\cosh h_{\mathbb{D}}(0,r) - \alpha \sinh h_{\mathbb{D}}(0,r)\right), \tag{2.2}$$

which holds if $0 \le r \le \alpha$, establishes the sharpness of (1.5). The identities,

$$T^h_\alpha(r) = \frac{\alpha r^2 + 2r + \alpha}{r^2 + 2\alpha r + 1} = \frac{\alpha + \tanh h_{\mathbb{D}}(0,r)}{1 + \alpha \tanh h_{\mathbb{D}}(0,r)}$$

and

$$S^h_\alpha(r) = \frac{\alpha r^2 - 2r + \alpha}{r^2 - 2\alpha r + 1} = \frac{\alpha - \tanh h_{\mathbb{D}}(0,r)}{1 - \alpha \tanh h_{\mathbb{D}}(0,r)},$$

verify that inequalities (1.1) are best possible. The sharpness of Theorem 5.1 is a consequence of

$$h_{\mathbb{D}}(S_\alpha(r), \alpha r) = h_{\mathbb{D}}(0, \alpha r) + \log\left(\cosh h_{\mathbb{D}}(0,r) - \alpha \sinh h_{\mathbb{D}}(0,r)\right).$$

The exactness of Theorem 1.6 follows from

$$h_{\mathbb{D}}(S_\alpha(r), r) = h_{\mathbb{D}}(0, r) + \log\left(\cosh h_{\mathbb{D}}(0,r) - \alpha \sinh h_{\mathbb{D}}(0,r)\right)$$

and

$$\begin{aligned} h_{\mathbb{D}}(T_\alpha(r), r) &= h_{\mathbb{D}}(0,r) - h_{\mathbb{D}}(T_\alpha(r), 0) \\ &= h_{\mathbb{D}}(0,r) - \log\left(\cosh h_{\mathbb{D}}(0,r) + \alpha \sinh h_{\mathbb{D}}(0,r)\right). \end{aligned}$$

Finally,

$$h_{\mathbb{D}}(S_\alpha(r), T_\alpha(r)) = \log\left(\cosh^2 h_{\mathbb{D}}(0,r) - \alpha^2 \sinh^2 h_{\mathbb{D}}(0,r)\right)$$

illustrates the sharpness of Theorem 5.2.

3. Proofs of the strengthened versions of Pick's Theorem

Proof. (of Theorem 1.1) (a) Theorem 5.1 of [7] contains the upper bound in the special case that $\Omega_1 = \Omega_2$. The proof given there extends to the general case.

$$\begin{aligned}|f^h(z)| &= \tanh\left(\frac{1}{2}h_{\mathbb{D}}(0, f^h(z))\right) \\ &\le \tanh\left(\frac{1}{2}h_{\mathbb{D}}(0, |f^h(w)|) + \frac{1}{2}h_{\mathbb{D}}(|f^h(z)|, |f^h(w)|)\right) \\ &\le \tanh\left(\frac{1}{2}h_{\mathbb{D}}(0, |f^h(w)|) + h_1(z, w)\right) \\ &= \frac{|f^h(w)| + \tanh h_1(z, w)}{1 + |f^h(w)| \tanh h_1(z, w)}.\end{aligned}$$

The lower bound follows from the upper bound by interchanging z and w and then solving the inequality for $|f^h(z)|$.

(b) Inequality (1.2) is stated for the unit disk in [4] and the authors note it extends to general hyperbolic regions by making use of holomorphic coverings. We want to point out that (1.2) can be obtained by integrating the upper bound in (1.1). Let $\gamma(s), 0 \le s \le L = h_\Omega(z, w)$ be a hyperbolic geodesic from w to z that is parametrized by hyperbolic arc length. In particular, $h_\Omega(w, \gamma(s)) = s$ and $\lambda_\Omega(\gamma(s))|\gamma'(s)| = 1$ for all $s \in [0, L]$. Then

$$\begin{aligned}h_\Omega(f(z), f(w)) &\le \int_{f\circ\gamma} \lambda_\Omega(\omega)|d\omega| \\ &= \int_\gamma \lambda_\Omega(f(\zeta))|f'(\zeta)||d\zeta| \\ &= \int_0^L |f^h(\gamma(s))|\, ds \\ &\le \int_0^L \frac{|f^h(w)| + \tanh s}{1 + |f^h(w)| \tanh s}\, ds \\ &= \int_0^L \frac{|f^h(w)| \cosh s + \sinh s}{\cosh s + |f^h(w)| \sinh s}\, ds \\ &= \log\big(\cosh L + |f^h(w)| \sinh L\big),\end{aligned}$$

which is (1.2). ■

Proof. (of Theorem 1.2) Fix $w \in \Omega_1$ and a holomorphic function $f : \Omega_1 \to \Omega_2$. Let $g_j : \mathbb{D} \to \Omega_j$ be a covering $(j = 1, 2)$ with $g_1(0) = w$ and $g_2(0) = f(w)$. Take

$F : \mathbb{D} \to \mathbb{D}$ to be the lift of $fg_1 : \mathbb{D} \to \Omega_2$ relative to g_2 such that $F(0) = 0$. Then $g_2 F = fg_1$ and there is no harm in assuming that $\alpha := F'(0) = f^h(w) > 0$.

We first note that analogs of part (a) hold for F at the origin. By [7, Thm. 1.1(c)] the function F is injective on the disk $h_{\mathbb{D}}(0, Z) < \tanh^{-1} \alpha$. Next, we give a form of (1.5). From [7, Thm. 1.1(c)]

$$h_{\mathbb{D}}(F(Z), F(W)) \geq -\log\left(\cosh h_{\mathbb{D}}(Z, W) - \alpha \sinh h_{\mathbb{D}}(Z, W)\right)$$

for all $Z, W \in \mathbb{D}$. For $W = 0$ we get

$$h_{\mathbb{D}}(F(Z), 0) \geq -\log\left(\cosh h_{\mathbb{D}}(Z, 0) - \alpha \sinh h_{\mathbb{D}}(Z, 0)\right). \tag{3.1}$$

Third, $D_{\mathbb{D}}(0, \rho(r)) \subset F\big(D_{\mathbb{D}}(0, r)\big)$ follows from [5, Thm. 1.1(b)].

It remains to show that these results for F can be "pushed down" to results for $f : \Omega_1 \to \Omega_2$. We verify that g_1 and g_2 are injective on appropriate disks. The covering g_2 is injective on $D_{\mathbb{D}}\big(0, R_2(f(w))\big)$ and maps it onto $D_2\big(f(w), R_2(f(w))\big)$. The function F maps $D_{\mathbb{D}}\big(0, R_2(f(w))\big)$ into itself. Hence, if $r(w) = \min\{R_2(f(w)), \tanh^{-1} \alpha\}$, then $g_2 F = fg_1$ is injective on $D_{\mathbb{D}}(0, r(w))$. Therefore, g_1 is injective on $D_{\mathbb{D}}(0, r(w))$, maps it onto $D_1(w, r(w))$ and f is injective on $D_1(w, r(w))$. Also, for $r < r(w)$,

$$\begin{aligned} D_2(f(w), \rho(r)) &= g_2\big(D_{\mathbb{D}}(0, \rho(r))\big) \\ &\subset g_2 F\big(D_{\mathbb{D}}(0, r)\big) \\ &= fg_1\big(D_{\mathbb{D}}(0, r)\big) \\ &= f\big(D_1(w, r)\big), \end{aligned}$$

which establishes (1.6). Finally, (3.1) implies (1.5). The restriction to hyperbolic disk of radius $(1/2)r(w)$ is necessary to insure that both g_1 and g_2 are hyperbolic isometries on disks about the origin with this radius.

(2.2) establishes the sharpness of (1.5) for each $\alpha = |f^h(0)| \in (0, 1)$, when $\Omega_1 = \Omega_2 = \mathbb{D}$, $w = 0$ and $f(0) = 0$. ∎

4. Proofs of rigidity results

Proof. (of Theorem 1.4) If $f \in \mathcal{H}^*$ sends a_1 to a_2, then $|f^h(a_1)| \leq \alpha$ by the Generalized Aumann-Carathéodory Rigidity Theorem. The conclusions of Theorem 1.4 follow from Theorem 1.1 because the expressions

$$\frac{|f^h(a_1)| + \tanh h_1(a_1, z))}{1 + |f^h(a_1)| \tanh h_1(a_1, z)} \quad \text{and} \quad \cosh h_1(a_1, z) + |f^h(a_1)| \sinh h_1(a_1, z)$$

are increasing functions of $|f^h(a_1)|$. ∎

We recall a few basic facts about the hyperbolic length of a free homotopy class of closed paths in a hyperbolic region. For details the reader is referred to the work of [12], [16], [13] and [17]. For a path γ in a hyperbolic region Ω, let

$$L_\Omega(\gamma) = \int_\gamma \lambda_\Omega(z)|dz|$$

denote the hyperbolic length of γ, with the understanding that the length is taken to be infinity if γ is not rectifiable. A loop is a closed path. For a loop γ in Ω, let $\{\gamma\}$ denote the family of all loops δ in Ω that are freely homotopic to γ in Ω. The notation $\delta \approx \gamma$ means δ is freely homotopic to γ in Ω. The modulus of the free homotopy class $\{\gamma\}$ is

$$M_\Omega(\{\gamma\}) = \inf\left\{L_\Omega(\delta) : \delta \approx \gamma\right\}.$$

If γ is null homotopic, then $M_\Omega(\{\gamma\}) = 0$. It is possible that $M_\Omega(\{\gamma\}) = 0$ without γ being null homotopic. For instance, the hyperbolic metric for the punctured unit disk $\mathbb{D}^*$ is

$$\lambda_{\mathbb{D}^*}(z)|dz| = \frac{|dz|}{|z|\log(1/|z|)}.$$

If γ is the circle $|z| = r \in (0,1)$, then $L_{\mathbb{D}^*}(\gamma) = 2\pi/\log(1/r) \to 0$ as $r \to 0$. Therefore, every free homotopy class in $\mathbb{D}^*$ has modulus zero. A closed path γ in Ω is retractible to an isolated boundary point c if $c \in \mathbb{C}_\infty$ is an isolated boundary point of Ω, D^* is a disc punctured at c such that $D^* \subseteq \Omega$ and γ is freely homotopic to a closed path in D^*. γ is retractible to an isolated boundary point if and only if $M_\Omega(\gamma) = 0$. Let $\{\gamma\}$ denote the free homotopy class of γ in Ω. If γ is a closed path in Ω that is not retractible to an isolated boundary point, then $\{\gamma\}$ contains an essentially unique path γ_0 such that

$$\int_{\gamma_0} \lambda_\Omega(z)\,|dz| = M_\Omega(\gamma).$$

The extremal path γ_0 is unique up to initial point and parametrization.

The following result is part of the folklore, but the author has been unable to locate a reference, so a short proof is included.

Theorem 4.1. *Suppose Ω is a hyperbolic region. There exists a closed path γ in Ω with $M_\Omega(\gamma) > 0$ if and only if Ω is neither simply connected nor conformally equivalent to $\mathbb{D}^*$.*

Proof. If Ω is simply connected nor conformally equivalent to $\mathbb{D}^*$ it is elementary that $M_\Omega(\gamma) = 0$ for every closed path γ in Ω. Let $k : \mathbb{D} \to \Omega$ be a covering projection and Γ the associated group of cover transformations. We may assume Ω is not simply connected and prove that if $M_\Omega(\gamma) = 0$ for every closed path γ

in Ω, then Ω is conformally equivalent to $\mathbb{D}^*$. A closed path γ in Ω corresponds to a parabolic element of Γ if and only if $M_\Omega(\gamma) = 0$. We prove that if Γ consists of parabolic elements, then Ω is conformally equivalent to $\mathbb{D}^*$. If Γ contains only parabolic elements, then by [15, p. 92] Ω must be conformally equivalent to $\mathbb{D}^*$ since it cannot be conformally equivalent to a torus or the punctured plane. ∎

Proof. (of Generalized Heins Rigidity Theorem) Suppose $f_n \in \mathcal{H}$, $f \in \mathcal{C}$ and $f_n \to f$ locally uniformly. We show there exists N such that $f_n \in \mathcal{C}$ for all $n \geq N$. Because Ω_1 is not conformally equivalent to either $\mathbb{D}$ or $\mathbb{D}^*$, there is a free homotopy class $\{\gamma\}_1$ in Ω_1 with positive modulus. Let γ_0 denote the unique loop in $\{\gamma\}_1$ with minimal hyperbolic length. This means that if $\gamma \approx \gamma_0$, then $0 < L_1(\gamma_0) \leq L_1(\gamma)$ and equality holds if and only if γ is a reparametrization of γ_0. Because $f \in \mathcal{C}$ is a local isometry, $L_2(f \circ \gamma) = L_1(\gamma)$ for any path γ. Since f is a covering, $\{f \circ \gamma_0\}_2 = f \circ \{\gamma_0\}_1$. Therefore, $f \circ \gamma_0$ is the unique loop in $\{f \circ \gamma_0\}_2$ with minimal hyperbolic length. Pick's Theorem implies that $L_2(f_n \circ \gamma_0) \leq L_1(\gamma_0)$ with equality if and only if $f_n \in \mathcal{C}$. Because $f_n \to f$ locally uniformly, there exists N such that $f_n \circ \gamma_0 \approx f \circ \gamma_0$ for all $n \geq N$. Since $f_n \circ \gamma \in \{f \circ \gamma_0\}_2$ for $n \geq N$, we may conclude $L_1(\gamma_0) = L_2(f \circ \gamma_0) \leq L_2(f_n \circ \gamma_0)$ for all $n \geq N$. Hence, $L_2(f_n \circ \gamma_0) = L_1(\gamma_0)$ for $n \geq N$ and the equality statement in the general version of Pick's Theorem implies that $f_n \in \mathcal{C}$ for $n \geq N$. ∎

Proof. (of Theorem 1.5) (a) We begin by showing that $\mathcal{C}$ is closed. Suppose $f_n \in \mathcal{C}$, and $f_n \to f$, where $f \in \mathcal{H}$. We want to show that $f \in \mathcal{C}$. Since f_n is a local isometry of the hyperbolic metric, $|f_n^h(z)| = 1$ for all $z \in \Omega_1$. This implies that $|f^h(z)| = 1$ for all $z \in \Omega$, so the general version of Pick's Theorem implies $f \in \mathcal{C}$, or $\mathcal{C}$ is closed.

(b) Note that if φ is a conformal map of Ω_1 onto Ω_1', then the map $\Phi : \mathcal{H}(\Omega_1, \Omega_2) \to \mathcal{H}(\Omega_1', \Omega_2)$ given by $\Phi(f) = f \circ \varphi^{-1}$ is a homeomorphism of $\mathcal{H}(\Omega_1, \Omega_2)$ onto $\mathcal{H}(\Omega_1', \Omega_2)$ which, when restricted to $\mathcal{C}(\Omega_1, \Omega_2)$, is a homeomorphism of $\mathcal{C}(\Omega_1, \Omega_2)$ onto $\mathcal{C}(\Omega_1', \Omega_2)$. Therefore, we need only consider the cases in which Ω_1 is $\mathbb{D}$ or $\mathbb{D}^*$. For $\Omega_1 = \mathbb{D}$, $\mathcal{C}$ is the set of holomorphic coverings of $\mathbb{D}$ onto Ω_2. If $f \in \mathcal{C}$, and $f_n(z) = f(\frac{n}{n+1}z)$, then $f_n \to f$ so that $\mathcal{C} \subset \overline{\mathcal{H}^*}$. Note that if A is a closed subset of a metric space X, $H^* = X \setminus A$ and $A \subset \overline{H^*}$, then $\overline{H^*} = X$ and so $\partial H^* = \overline{H^*} \setminus \mathrm{Int}(H^*) = X \setminus H^* = A$. Hence, $\mathcal{C}(\mathbb{D}, \Omega_2) = \partial\mathcal{H}^*(\mathbb{D}, \Omega_2)$. Next, suppose that $\Omega = \mathbb{D}^*$. If $f \in \mathcal{C}(\mathbb{D}^*, \Omega_2)$, then $f(\frac{n}{n+1}z) \to f(z)$ in $\mathcal{H}(\mathbb{D}^*, \Omega_2)$, so that $\mathcal{C}(\mathbb{D}^*, \Omega_2) \subset \overline{\mathcal{H}^*(\mathbb{D}^*, \Omega_2)}$. As in the case when $\Omega_1 = \mathbb{D}$, we deduce that $\mathcal{C}(\mathbb{D}^*, \Omega_2) = \partial\mathcal{H}^*(\mathbb{D}^*, \Omega_2)$.

(c) We start by demonstrating that $\mathcal{C}$ is open if Ω_1 is not conformally equivalent to either $\mathbb{D}$ or $\mathbb{D}^*$. As the topology on $\mathcal{H}$ is a metric topology, a function $f \in \mathcal{H}$ lies in the closure $\overline{\mathcal{H}^*}$ of $\mathcal{H}^*$ if and only if it is the locally uniform limit of functions in $\mathcal{H}^*$. By the Generalized Heins Rigidity Theorem no function in $\mathcal{C}$ lies in the closure $\overline{\mathcal{H}^*}$ of $\mathcal{H}^*$. Consequently, $\mathcal{H}^*$ is closed, and so $\mathcal{C}$ is open, when Ω is not conformally equivalent to either $\mathbb{D}$ or $\mathbb{D}^*$. ∎

If Ω_1 is not conformally equivalent to either $\mathbb{D}$ or $\mathbb{D}^*$, then the disjoint sets, $\mathcal{C}$ and $\mathcal{H}^*$, are both closed, so the Generalized Heins' Rigidity Theorem is an immediate corollary.

We state without proof the analog of Theorem 1.5 when base points are utilized. The proof is analogous to that of Theorem 1.5. For $a_j \in \Omega_j$, $\mathcal{H}_{a_1,a_2}$ is the set of holomorphic maps that send a_1 to a_2. It is straightforward to show that $\mathcal{H}_{a_1,a_2}$ is closed in $\mathcal{H}$. Let $\mathcal{C}_{a_1,a_2}$ be the subset of coverings that map a_1 to a_2. Set $\mathcal{H}^*_{a_1,a_2} = \mathcal{H}_{a_1,a_2} \setminus \mathcal{C}_{a_1,a_2}$.

Theorem 4.2. *Suppose that Ω_j is a hyperbolic region and $a_j \in \Omega_j$, $j = 1, 2$.*
(a) *$\mathcal{C}_{a_1,a_2}$ and $\mathcal{H}_{a_1,a_2}$ are compact in $\mathcal{H}$.*
(b) *If Ω_1 is simply connected, then $\mathcal{C}_{a_1,a_2} = \partial\mathcal{H}^*_{a_1,a_2}$.*
(c) *If Ω is not simply connected, then $\mathcal{C}_{a_1,a_2}(\Omega)$ is both open and closed in $\mathcal{H}_{a_1,a_2}$.*

Let d denote the metric on $\mathcal{H}$. Since the disjoint sets, $\mathcal{H}^*_{a_1,a_2}$ and $\mathcal{C}^*_{a_1,a_2}$, are both compact, $d(\mathcal{H}^*_{a_1,a_2}, \mathcal{C}_{a_1,a_2}) > 0$. The Generalized Aumann-Carathéodory Rigidity Theorem is an immediate consequence of this.

5. Flexibility results

Initially we assume that Ω_2 also is simply connected. When both Ω_1 and Ω_2 are simply connected, conformal maps of Ω_1 onto hyperbolic disks in Ω_2 play a role. For $r > 0$ let $\sigma_{a_2,r}$ denote the conformal map of Ω_2 onto $D_2(a_2, r)$ that fixes a_2 and satisfies $\sigma'_{a_2,r}(a_2) > 0$. Note that $\sigma_{a_2,r} = g_2\sigma_r g_2^{-1}$, where σ_r is the Euclidean contraction $z \mapsto \tanh(r/2)z$ that maps $\mathbb{D}$ onto $D_{\mathbb{D}}(0, r)$, and $\sigma'_{a_2,r}(a_2) = \tanh(r/2)$. Observe that the hyperbolic rotation $\rho_{a_2,\theta}$ commutes with $\sigma_{a_2,r}$. For $a_2 \in \Omega_2$ and $D = D_2(a_2, r)$, we determine the density $\lambda_D(a_2)$ in terms of $\lambda_2(a_2)$ and r. Because $\sigma_{a_2,r}g_2(z)$ is a conformal map of $\mathbb{D}$ onto $D_\Omega(a_2, r)$ that maps the origin to a_2 with positive derivative at the origin,

$$\lambda_D(a_2) = \frac{2}{g_2'(0)\tanh(r/2)} = \frac{\lambda_2(a_2)}{\tanh(r/2)}\,.$$

Therefore, if $\psi : \Omega_1 \to D$ is a conformal mapping with $\psi(a_1) = a_2$, then

$$1 = \frac{\lambda_D(a_2)|\psi'(a_1)|}{\lambda_1(a_1)} = \frac{\lambda_2(a_2)|\psi'(a_1)|}{\lambda_1(a_1)\tanh(r/2)}$$

and so

$$|\psi^h(a_1)| = \tanh(r/2).$$

In fact, $\psi_\theta = \sigma_{a_2,r}\varphi_\theta$ is the unique conformal map of Ω_1 onto $D_2(a_2, r)$ that sends a_1 to a_2 and whose derivative at a_1 is a positive multiple of $e^{i\theta}$.

Given a holomorphic map $f : \Omega_1 \to \Omega_2$ of simply connected hyperbolic regions, we specify the first canonical self-map associated with f. Fix $a_1 \in \Omega_1$ and assume $f'(a_1) \neq 0$. We compare f to a conformal map of Ω onto a hyperbolic disk with center $a_2 := f(a_1)$ and radius determined by $f'(a_1) = |f'(a_1)|e^{i\theta}$. Precisely, if $r = 2\tanh^{-1}|f^h(a_1)|$ then $\psi_f := \sigma_{a_2,r}\varphi_\theta$ is the unique conformal mapping of Ω_1 onto $D_\Omega(a_2, r)$ with $f^h(a_1) = \psi_f^h(a_1)$. It is reasonable to compare $f(z)$ and $\psi_f(z)$. In the special case that $\Omega_j = \mathbb{D}$, $j = 1, 2$, $a_j = 0$ and $f'(0) = \alpha \in (0, 1)$, then $\psi_f(z) = \alpha z$. The general form of the Pick's Theorem implies that

$$\begin{aligned} h_2(f(z), \psi_f(z)) &\leq h_2(f(a_1), \psi_f(z)) + h_2(f(z), f(a_1)) \\ &\leq h_2(f(a_1), \psi_f(z)) + h_1(a_1, z) \\ &\leq 2h_1(a_1, z). \end{aligned} \tag{5.1}$$

The next result improves this elementary estimate.

Theorem 5.1. *Suppose that Ω_j is a simply connected hyperbolic region in $\mathbb{C}$ and $a_j \in \Omega$, $j = 1, 2$, $f \in \mathcal{H}$ with $f(a_1) = a_2$ and $f'(a_1) \neq 0$. Let ψ_f denote the conformal map of Ω_1 onto the hyperbolic disk $D_2(a_2, 2\tanh^{-1}|f^h(a_1)|)$ that maps a_1 to a_2 and satisfies $\psi_f^h(a_1) = f^h(a_1)$. Then, for all $z \in \Omega_1$,*

$$\begin{aligned} &h_2(f(z), \psi_f(z)) \\ &\qquad \leq h_2(f(a_1), \psi_f(z)) + \log\left(\cosh h_1(a_1, z) - |f^h(a_1)| \sinh h_1(a_1, z)\right). \end{aligned} \tag{5.2}$$

The bound is sharp.

Note that the term $\cosh h_1(a_1, z) - |f^h(a_1)| \sinh h_1(a_1, z) < 1$ for $h_1(a_1, z) < 2\tanh^{-1}|f^h(a_1)|$ and is larger than 1 when $h_1(a_1, z) > 2\tanh^{-1}|f^h(a_1)|$. The identity (1.10) and $h_2(f(a_1), \psi_f(z)) \leq h_1(a_1, z)$ together show that (5.2) implies

$$h_2(f(z), \psi_f(z)) \leq -\log\frac{1}{2}\left(1 + |f^h(a_1)| + (1 - |f^h(a_1)|)e^{2h_1(a_1,z)}\right).$$

Proof. (of Theorem 5.1) The idea of the proof is to reduce to the case of a holomorphic self-map of the $\mathbb{D}$ that fixes the origin and has non-vanishing derivative at the origin. Let $f \in \mathcal{H}$ with $f'(a_1) \neq 0$ and set $f(a_1) = a_2$. We may assume $f'(a_1) > 0$; if not, replace f by $\rho_{a_2,-\theta}f$. Then $F = g_2^{-1}fg_1$ is a holomorphic self-map of $\mathbb{D}$ that fixes the origin and $F'(0) = f^h(a_1) =: \alpha$. Moreover, $g_2^{-1}\psi_f g_1(z) = \alpha z$. Because g_j, $j = 1, 2$, is a hyperbolic isometry, inequality (5.2) is equivalent to

$$h_\mathbb{D}(F(z), \alpha z) \leq h_\mathbb{D}(0, \alpha z) + \log\big(\cosh h_\mathbb{D}(0, z) - \alpha \sinh h_\mathbb{D}(0, z)\big). \tag{5.3}$$

Note that $F \in \mathcal{B}_\alpha$, the family of holomorphic self-maps G of $\mathbb{D}$ that are normalized by $G(0) = 0$ and $G'(0) = \alpha$. If $G \in \mathcal{B}_\alpha$, then for each $\theta \in \mathbb{R}$ the 'rotation' $G_\theta(z) = e^{-i\theta}G(e^{i\theta}z)$ belongs to $\mathcal{B}_\alpha$. Let $H_\alpha(z) = (z+\alpha)/(1+\alpha z)$. Observe that $T_\alpha(z) = zH_\alpha(z)$. For $r \in (0,1)$ and $\alpha \in (0,1]$, the value region $\{G(r) : G \in \mathcal{B}_\alpha\}$ is the closed disk $r\Delta$ in $\mathbb{D}$ that is symmetric about the real axis and meets the real axis in the interval $[S_\alpha(r), T_\alpha(r)]$, where $\Delta = H_\alpha(|z| \le r)$; see [7].

The inequality (5.3) is trivial if $z = 0$, so we may assume $0 < |z| < 1$. Because of rotational invariance of both the hyperbolic distance and the family $\mathcal{B}_\alpha$, we may even assume $0 < z = r < 1$. It is straightforward to verify that $s := S_\alpha(r) < \alpha r < t := T_\alpha(r)$ and $h_\mathbb{D}(t, \alpha r) < h_\mathbb{D}(s, \alpha r)$, or equivalently,

$$p(t, \alpha r) = \frac{(1-\alpha^2)r^2}{1+\alpha r - \alpha^2 r^2 - \alpha r^3} < p(s, \alpha r) = \frac{(1-\alpha^2)r^2}{1-\alpha r - \alpha^2 r^2 + \alpha r^3}\,.$$

Consequently, if c is the hyperbolic center of the hyperbolic disk $r\Delta$, then $S_\alpha(r) < c < \alpha r < T_\alpha(r)$. Therefore, for any $w \in r\Delta$, $h_\mathbb{D}(w, \alpha r) \le h_\mathbb{D}(S_\alpha(r), \alpha r)$ with equality if and only if $w = S_\alpha(r)$. Direct calculation gives

$$\begin{aligned} h_\mathbb{D}(S_\alpha(r), \alpha r) &= \log \frac{(1+\alpha r)(1-2\alpha r + r^2)}{(1-\alpha r)(1-r^2)} \\ &= \log \frac{1+\alpha r}{1-\alpha r} + \log \frac{1-2\alpha r + r^2}{1-r^2}\,. \end{aligned}$$

From

$$\frac{1+|z|^2}{1-|z|^2} = \cosh h_\mathbb{D}(0,z)) \quad \text{and} \quad \frac{2|z|}{1-|z|^2} = \sinh h_\mathbb{D}(0,z),$$

and $h_\mathbb{D}(0, \alpha r) = \log(1+\alpha r)/(1-\alpha r)$, inequality (5.3) holds. ∎

In a similar manner one can show that the sharp Euclidean analog of (5.3) is

$$|F(z) - \alpha z| \le \alpha|z| - S_\alpha(|z|) = \frac{(1-\alpha^2)|z|^2}{1-\alpha|z|},$$

where F is a holomorphic self-map of $\mathbb{D}$ with $F(0) = 0$ and $F'(0) = \alpha \in [0,1]$. From this inequality we obtain

$$\begin{aligned} |F(z) - z| &\le |F(z) - \alpha z| + |\alpha z - z| \\ &\le \frac{(1-\alpha^2)|z|^2}{1-\alpha|z|} + (1-\alpha)|z| \\ &= \frac{(1-\alpha)|z|(1+|z|)}{1-\alpha|z|}, \end{aligned}$$

a sharp bounded noted in [7].

Proof. (of Theorem 1.6) (a) At first we assume that Ω_2 is simply connected; in this case Theorem 5.1 quickly gives the result. Observe that $\psi_f = \sigma_{a_2,r}\varphi_f$, where $r = 2\tanh^{-1}|f^h(a_1)|$. Then

$$\begin{aligned} h_2(f(z),\varphi_f(z)) &\le h_2(f(z),\psi_f(z)) + h_2(\psi_f(z),\varphi_f(z)) \\ &\le h_2(\psi_f(z),\varphi_f(z)) + h_2(f(a_1),\psi_f(z)) \\ &\qquad + \log\big(\cosh h_1(a_1,z) - |f^h(a_1)|\sinh h_1(a_1,z)\big) \\ &= h_2(f(a_1),\varphi_f(z))) + \log\big(\cosh h_1(a_1,z) - |f^h(a_1)|\sinh h_1(a_1,z)\big) \\ &= h_1(a_1,z) + \log\big(\cosh h_1(a_1,z) - |f^h(a_1)|\sinh h_1(a_1,z)\big) \end{aligned}$$

because the points $\psi_f(z)$ and $\varphi_f(z)$ lie on the same hyperbolic geodesic ray emanating from $f(a_1)$ and φ_f is a hyperbolic isometry.

Next, we consider the case in which Ω_2 is multiply connected, so that φ_f is a covering of Ω_1 onto Ω_2. Let F be the lift of f relative to φ_f so that F is a holomorphic self-map of Ω_1 that fixes a_1 and $\varphi_f F = f$. Note that $F'(a_1) > 0$ so that the comparison conformal automorphism of Ω_1 for F is the identity. Then by using the facts that φ_f is distance decreasing, the first part of the proof and $|f^h(a_1)| = |F^h(a_1)|$ we have

$$\begin{aligned} h_2(f(z),\varphi_f(z)) &\le h_2(\varphi_f F(z)),\varphi_f(z)) \\ &\le h_2(F(z),z) \\ &\le h_1(a_1,z) + \log\big(\cosh h_1(a_1,z) - |F^h(a_1)|\sinh h_1(a_1,z)\big) \\ &= h_1(a_1,z) + \log\big(\cosh h_1(a_1,z) - |f^h(a_1)|\sinh h_1(a_1,z)\big) \end{aligned}$$

(b) If Ω_2 is simply connected, then φ_f is a conformal mapping and the lower bound comes from

$$h_1(a_1,z) = h_2(f(a_1),\varphi_f(z)) \le h_2(f(a_1),f(z)) + h_2(f(z),\varphi_f(z))$$

together with the upper bound in (1.2). ■

The final comparison is with a two-sheeted branched covering of Ω_1 onto Ω_2 when both regions are simply connected. In fact, these coverings are extremal for all of our theorems, so a comparison with these coverings is natural. Given $a_j \in \Omega_j$, we construct a canonical two-sheeted branched self-covering that sends a_1 to a_2. For $\alpha \in [0,1)$ the function $\tau_\alpha = g_2 T_\alpha g_1^{-1}$, where $T_\alpha(z) = z(z+\alpha)/(1+\alpha z)$, is a two-sheeted branched covering of Ω_1 onto Ω_2 with $\tau_\alpha(a_1) = a_2$ and $\tau_\alpha^h(a_1) = \alpha$. These two conditions do not uniquely determine a two-sheeted branched covering that sends a_1 to a_2. For instance, the functions $T_{\alpha,\theta}(z) = e^{-i\theta}T_\alpha(e^{i\theta}z)$ are all two-sheeted branched coverings of $\mathbb{D}$ onto itself that fix the origin and satisfy $T'_{\alpha,\theta}(0) = \alpha$, so $g_2 T_{\alpha,\theta} g_1^{-1}$ is a two-sheeted branched self-covering that satisfies the same two initial conditions at a_1 as τ_α. The function $T_{\alpha,\theta}$ satisfies $T''_{\alpha,\theta}(0) = 2(1-\alpha^2)e^{i\theta}$; so T_α has the special property that its second derivative at the origin is positive. The second derivative is not conformally invariant so it need not be true that $\tau''_\alpha(a_1) > 0$. A tedious calculation shows that

$$4\frac{\partial \log|\tau^h_{\alpha,\theta}(a_1)|}{\partial z} = \lambda_\Omega(a_1)\frac{T''_\alpha(0)}{T'_\alpha(0)}.$$

The canonical two-sheeted branched covering $\tau_\alpha = g_2 T_\alpha g_1^{-1}$ is uniquely determined by requiring the third initial condition that $\frac{\partial \log|\tau^h_\alpha(a_1)|}{\partial z} > 0$ in addition to the first two initial conditions, $\tau_\alpha(a_1) = a_2$ and $\tau^h_\alpha(a_1) = \alpha$. We note the important property that τ_α maps the geodesic ray $\Gamma_{a_1,0}$ onto $\Gamma_{a_2,0}$. For any $\theta \in \mathbb{R}$, $\tau_{\alpha,\theta} = \rho_{a_2,\theta}\tau_\alpha\rho_{a_1,-\theta}$ is the two-sheeted branched covering that sends a_1 to a_2, has hyperbolic derivative α at a_1 and maps $\Gamma_{a_1,\theta}$ onto $\Gamma_{a_2,\theta}$.

Theorem 5.2. *Suppose that Ω_j is a simply connected hyperbolic region in $\mathbb{C}$, $a_j \in \Omega_j$, $j = 1, 2$, $f \in \mathcal{H}$, $f(a_1) = a_2$ and $\alpha := f^h(a_1) \in [0, 1)$. Let $\tau_f = \tau_\alpha \in \mathcal{C}$ be the canonical holomorphic two-sheeted branched covering of Ω_1 onto Ω_2 that maps a_1 to a_2 and has hyperbolic derivative α at a_1. Then, for all $z \in \Gamma_{a_1,\theta}$,*

$$h_2(f(z), \tau_{f,\theta}(z)) \le \log\big(\cosh^2 h_1(a_1, z) - f^h(a_1)^2 \sinh^2 h_1(a_1, z)\big), \tag{5.4}$$

where $\tau_{f,\theta} = \rho_{f(a),\theta}\tau_f\rho_{a,-\theta}$. The bound is sharp.

If $f^h(a_1) = |f^h(a_1)|e^{i\eta}$, where $|f^h(a_1)| \in [0, 1)$, replace f by $\rho_{f(a),-\eta}f$ in Theorem 5.2. In the special case that $\Omega = \mathbb{D}$, $a = 0$, $f(0) = 0$ and $f'(0) = \alpha \in [0, 1)$, inequality (5.4) can be written as

$$h_\mathbb{D}(f(re^{i\theta}), e^{i\theta}T_\alpha(r)) \le \log\big(\cosh^2 h_\mathbb{D}(0, re^{i\theta}) - \alpha^2 \sinh^2 h_\mathbb{D}(0, re^{i\theta})\big).$$

When α is close to 1, the right-hand side is small when r is not too big.

Proof. (of Theorem 5.2) Initially we consider $z_0 \in \Gamma_{a_1,0}$. By using inequality (5.2) of Theorem 5.1 and the fact that $\psi_f(z_0)$ lies between $f(a)$ and $\tau_f(z_0)$ on $\Gamma_{f(a),0}$, we have

$$\begin{aligned}
h_2(f(z_0), \tau_f(z_0)) &\le h_2(f(z_0), \psi_f(z_0)) + h_2(\psi_f(z_0), \tau_f(z_0)) \\
&\le h_2(f(a_1), \psi_f(z_0)) + \log\big(\cosh h_1(a_1, z_0) - |f^h(a)| \sinh h_1(a_1, z_0)\big) \\
&\quad + h_2(f(a_1), \tau_f(z_0)) - h_2(f(a_1), \psi_f(z_0)) \\
&= \log\big(\cosh^2 h_1(a_1, z_0) - \alpha^2 \sinh^2 h_1(a_1, z_0)\big),
\end{aligned}$$

where we have also used

$$h_2(f(a_1), \tau_f(z_0)) = \log\big(\cosh h_1(a_1, z_0) + |f^h(a_1)| \sinh h_1(a_1, z_0)\big),$$

see (2.1). In order to obtain the general case when $z \in \Gamma_{a_1,\theta}$, apply the preceding to $\rho_{a_2,-\theta} f \rho_{a_1,\theta}$ which has nonnegative derivative at a_1 and the point $z_0 = \rho_{a_1,-\theta}(z) \in \Gamma_{a_1,0}$ and note that $h_\Omega(a_1, z) = h_\Omega(a_2, z_0)$. ■

For the unit disk the sharp Euclidean analog of Theorem 5.2 is

$$\left| F(r) - \frac{r(\alpha + r)}{1 + \alpha r} \right| \leq T_\alpha(r) - S_\alpha(r) = \frac{2(1 - \alpha^2) r^2}{1 - \alpha^2 r^2},$$

where F is a holomorphic self-map of $\mathbb{D}$ with $F(0) = 0$ and $F'(0) = \alpha \in [0, 1)$.

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