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# Introduction to $p$-modulus of Path-families and Newtonian Spaces

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**Abstract.** The notion of $p$-moduli of families of paths is a useful tool in the study of analysis in metric measure space setting. In this note we briefly describe this tool, and use it to construct the Newtonian spaces, which is an analog of Sobolev type spaces in the metric measure space setting. The results in this note assume only that the measure is Borel and that bounded sets have finite measure and non-empty open sets have positive measure.

**Keywords.** $p$-modulus, metric space, paths, Newtonian spaces, path open topology.

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## 1. Introduction

Recent studies of analysis in metric measure space setting has been successful because of the tools such as moduli of path-families. These tools were developed originally in the context of complex analysis, where this notion (with $p = 2$) was used to prove Caratheodory's extension theorem of conformal mappings between planar Jordan domains (see [Ah1] for a nice description of this use), and to consider quasiconformal mappings (see for example [Ah2], [Oh1], [Nä]). The The notion of $p$-modulus of path-families is now used in the study of potential theory in metric measure spaces and in the study of quasiconformal mappings. In this note we will give a brief description of $p$-moduli of curve families and their associated notion of Sobolev type spaces, called the Newtonian spaces.

The rest of this section is devoted to the basic notations and the definition of $p$-moduli of curve families. In Section 2 we will study some properties of the $p$-modulus, and in Section 3 we will use this notion to define the Newtonian class of functions on the metric measure space. This will be followed by a description of basic properties of the Newtonian class.

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The notion of 2-modulus (called the conformal modulus in the plane) of a path-family was extensively used in Complex Analysis. Indeed, some of the other papers in this proceedings collection, such as the ones by Kovalev–Onninen and Sugawa give a good survey of its uses in complex analytic theory. The following definition of $p$-Modulus of a curve family is a natural extension of this notion to the non-linear setting.

Throughout this note, $X = (X, d, \mu)$ is a metric measure space equipped with a metric $d$ and a Borel measure $\mu$ such that, whenever $B$ is a ball in $X$, we have $0 < \mu(B) < \infty$. Here, by a ball $B$ we mean the set $B = B(x, r) = \{y \in X : d(x, y) < r\}$ for some $x \in X$ and $r > 0$. We will also assume that $X$ is separable, complete, and connected, though in general some of these assumptions can be relaxed.

A path or a curve in $X$ is a continuous mapping $\gamma : I \to X$ for some interval $I \subset \mathbb{R}$. The path $\gamma$ is said to be rectifiable if the length of $\gamma$, defined by

$$\ell(\gamma) := \sup \left\{ \sum_{i=1}^{N} d(\gamma(t_i), \gamma(t_{i+1})) \,:\, t_1 < t_2 < \cdots < t_N \text{ with } t_1, t_2, \cdots, t_N \in I \right\}$$

is finite. We say that $\gamma$ is a compact rectifiable curve if $\gamma$ is rectifiable and the domain $I$ of the map $\gamma$ is a compact interval of $\mathbb{R}$. If $\gamma$ is a compact rectifiable curve in $X$, then there is an associated "length" mapping $s : I = [a, b] \to [0, \ell(\gamma)]$ given by $s(t) = \ell(\gamma|_{[a,t]})$; this map is monotone increasing map, and hence is differentiable almost everywhere on the interval $I = [a, b]$. We set $|\gamma'(t)| := s'(t)$, wherever $s'(t)$ exists. Using the map $s$ one can construct a re-parametrization of $\gamma$, called the arc-length parametrization and denoted $\gamma_0$, such that $\gamma_0 : [0, \ell(\gamma)] \to X$ with $|\gamma_0'| = 1$ almost everywhere on $[0, \ell(\gamma)]$. A good discussion of this can be found in [Vä]. While the discussion in [Vä] is set in the Euclidean context, the discussion there is robust enough to hold also in our metric measure space setting.

If $\gamma$ is a compact rectifiable path and $g : X \to [0, \infty]$ is a Borel-measurable function, it follows that $g \circ \gamma$ is a Lebesuge measurable function on $I$, the domain of $\gamma$. We define

$$\int_\gamma g\, ds \,:=\, \int_0^{\ell(\gamma)} g \circ \gamma_0(s)\, ds.$$

We now define the $p$-modulus of a family of compact rectifiable curves; but the reader should keep in mind that this definition and the notions described above can be extended to all locally rectifiable paths as well.

**Definition 1.1.** Given a collection $\Gamma$ of paths in $X$, the set of all *admissible functions* of $\Gamma$, denoted $\mathcal{A}(\Gamma)$, is the set of all non-negative Borel-measurable functions $g$ on $X$ such that for all $\gamma \in \Gamma$ we have

$$\int_\gamma g\, ds \,\ge\, 1.$$

For $1 \le p < \infty$, the *p-modulus* of $\Gamma$ is the number

$$\mathrm{Mod}_p(\Gamma) = \inf_{g \in \mathcal{A}(\Gamma)} \int_X g^p\, d\mu.$$

If $\mathcal{A}(\Gamma)$ is empty (for example, if $\Gamma$ has a constant path), then $\mathrm{Mod}_p(\Gamma) = \infty$.

We point out here that there is an analogous notion of $\mathrm{Mod}_p$ for $p = \infty$; we direct the interested reader to the paper by Durand-Cartagena in this proceedings collection.

A motivation for the notion of $p$-modulus is as follows. Given a non-negative Borel measurable function $g$ on $X$, we can use $g$ to modify the metric on $X$ (just as in the Riemannian setting) by defining the new metric

$$d_g(x, y) = \inf_\gamma \int_\gamma g\, ds,$$

where the infimum is taken over all rectifiable paths $\gamma$ connecting $x$ to $y$. analogous to the Riemannian setting, given such a modification of the metric, we should modify the $p$-dimensional measure, or volume, of $X$ to $\int_X g^p\, d\mu$. So the idea behind the $p$-modulus is to find the smallest $p$-dimensional volume perturbation of $X$ amongst all the metric modifications that see the new length of each $\gamma \in \Gamma$ to be at least 1.

## 2. Properties of $p$-modulus

In this section we discuss some properties of the $p$-Modulus for $1 \le p < \infty$, and we show that the $p$-modulus is an outer measure on the set of all compact rectifiable curves on $X$. The notion of 2-modulus in the Euclidean planar domains and $n$-modulus in the $\mathbb{R}^n$ setting have been extensively studied in [Vä] and [Oh2]. The classical study of 2-moduli actually considered the reciprocal of the quantity, called the extremal length; see for example [Ah1], [Fu], and [AO].

Given two collections $\Gamma_1, \Gamma_2$ of compact rectifiable paths in $X$, we say that $\Gamma_1 < \Gamma_2$ if every path in $\Gamma_2$ has a sub-path in $\Gamma_1$.

**Lemma 2.1.** *If* $\Gamma_1 < \Gamma_2$*, then* $Mod_p(\Gamma_1) \ge Mod_p(\Gamma_2)$*.*

**Proof.** The lemma follows from seeing that $\mathcal{A}(\Gamma_1) \subset \mathcal{A}(\Gamma_2)$. ∎

Next, as an example we shall give estimates for the $p$-modulus of a family of paths in $\mathbb{R}^n$.

**Example 2.2.** Let $X$ be the Euclidean space, equipped with the Euclidean metric and $n$-dimensional Lebesgue measure. Let $E \subset S^{n-1}$, where $S^{n-1}$ is the unit sphere centered at the origin in $\mathbb{R}^n$, and let the $(n-1)$-dimensional measure of $E$ be denoted by $\mathcal{H}^{n-1}(E)$. Let $0 < r < R$, and let $\Gamma_0$ be the collection of the rays that connect the origin to $E$, and $\Gamma$ be the collection of segments of the paths from $\Gamma_0$ that have an end point in $S^{n-1}(r)$, the sphere centered at the origin with radius $r$, and an end point in $S^{n-1}(R)$.

Note that the function $g = (R-r)^{-1}\chi_{B(0,R)\setminus B(0,r)}$ belongs to $\mathcal{A}(\Gamma)$, and so

$$\mathrm{Mod}_p(\Gamma) \le \mathcal{H}^{n-1}(E)\,\frac{R^n - r^n}{(R-r)^p}.$$

On the other hand, if $g \in \mathcal{A}(\Gamma)$, then for each $\gamma \in \Gamma$ we know that $\int_\gamma g\,ds \ge 1$. For each $\theta \in E$ the path $\gamma_\theta : [r,R] \to \mathbb{R}^n$ given in polar coordinates by $\gamma_\theta(\rho) = (\rho,\theta)$ belongs to $\Gamma$, with $|\gamma_\theta'| \equiv 1$. Now by Hölder's inequality,

$$\begin{aligned} 1 \le \left(\int_{\gamma_\theta} g\,d\rho\right)^p &\le \left(\int_{\gamma_\theta} g(\rho,\theta)^p\,\rho^{n-1}\,d\rho\right)\left(\int_{\gamma_\theta} \rho^{-(n-1)/(p-1)}\,d\rho\right)^{p-1} \\ &= \left(\int_r^R \rho^{-(n-1)/(p-1)}\,d\rho\right)^{p-1} \int_{\gamma_\theta} g(\rho,\theta)^p\,\rho^{n-1}\,d\rho. \end{aligned}$$

Setting $C_{r,R,p} = \left(\int_r^R \rho^{-(n-1)/(p-1)}\,d\rho\right)^{p-1}$, we have by Fubini's theorem,

$$\mathcal{H}^{n-1}(E) \le C_{r,R,p} \int_{R^n} g^p d\mathcal{L}^n,$$

where $\mathcal{L}^n$ is the $n$-dimensional Lebesgue measure on $\mathbb{R}^n$. It follows from taking the infimum over all $g \in \mathcal{A}(\Gamma)$ in the above estimate that

$$\mathrm{Mod}_p(\Gamma) \ge \frac{\mathcal{H}^{n-1}(E)}{C_{r,R,p}}.$$

Hence it follows that $\mathrm{Mod}_p(\Gamma)$ is positive if and only if $\mathcal{H}^{n-1}(E) > 0$.

The above example also demonstrates a way of computing estimates of $p$-moduli of path-families whenever there is a way of decomposing the measure as a product of two measures that is compatible with a sub-family of the family of paths in $\Gamma$. As a consequence of the above computation and Theorem 2.3 below, it follows that the $p$-modulus of the collection of *all* rectifiable curves connecting points in $S^{n-1}(r)$ to $S^{n-1}(R)$ is positive. For more examples of this nature, we refer the reader to [Vä], [Oh2], and [HeK].

**Theorem 2.3.** *The following hold true:*

1. $\mathit{Mod}_p(\emptyset) = 0$.

2. *If* $\Gamma_1 \subset \Gamma_2$ *are collections of compact rectifiable paths in* $X$*, then*
$$Mod_p(\Gamma_1) \le Mod_p(\Gamma_2).$$
3. *If* $\Gamma_i$*,* $i \in J \subset \mathbb{N}$*, is a countable collection of sets of compact rectifiable paths in* $X$*, then*
$$Mod_p\left(\bigcup_{i\in J}\Gamma_i\right) \le \sum_{i\in J} Mod_p(\Gamma_i).$$

**Proof.** The first assertion follows from the fact that $0 \in \mathcal{A}(\emptyset)$, while the second assertion follows from the fact that if $\Gamma_1 \subset \Gamma_2$ then $\mathcal{A}(\Gamma_1) \supset \mathcal{A}(\Gamma_2)$.

To prove the third assertion, we may assume without loss of generality that $\sum_{i\in J} \mathrm{Mod}_p(\Gamma_i)$ is finite. Fix $\varepsilon > 0$; then for each $i \in J$ we can choose $g_i \in \mathcal{A}(\Gamma_i)$ such that
$$\int_X g_i^p \, d\mu \le \mathrm{Mod}_p(\Gamma_i) + 2^{-i}\varepsilon.$$
Observe that we have $\int_\gamma g_i \, ds \ge 1$ for each $\gamma \in \Gamma_i$. For each $n \in \mathbb{N}$ we set
$$\rho_n := \sup_{i\in I\,:\,i\le n} g_i.$$
Note that
$$\int_X \rho_n^p \, d\mu \le \sum_{i\in I\,:\,i\le n} \int_X g_i^p \, d\mu \le \sum_{i\in I\,:\,i\le n} \big(\mathrm{Mod}_p(\Gamma_i) + 2^{-i}\varepsilon\big) \le \varepsilon + \sum_{i\in J} \mathrm{Mod}_p(\Gamma_i).$$
Since $\{\rho_n\}_n$ is a monotone increasing sequence in $L^p(X)$, it follows that it has a limit $\rho_\infty \in L^p(X)$ with
$$\int_X \rho_\infty^p \, d\mu \le \varepsilon + \sum_{i\in J} \mathrm{Mod}_p(\Gamma_i).$$
Since each $g_i$ is Borel measurable, so is $\rho_n$ and hence so is $\rho_\infty$. For each $\gamma \in \bigcup_{i\in J}\Gamma_i$ we have that $\gamma \in \Gamma_{i_0}$ for some $i_0 \in J$, and so
$$\int_\gamma \rho_\infty \, ds \ge \int_\gamma g_{i_0} \, ds \ge 1.$$
Therefore $\rho_\infty \in \mathcal{A}\left(\bigcup_{i\in J}\Gamma_i\right)$, and it follows that
$$\mathrm{mod}\,_p\left(\bigcup_{i\in J}\Gamma_i\right) \le \int_X \rho_\infty^p \, d\mu \le \varepsilon + \sum_{i\in J} \mathrm{Mod}_p(\Gamma_i).$$
Letting $\varepsilon \to 0$ now yields the desired result. ■

The results of the above theorem tells us that $\mathrm{Mod}_p$ is an outer measure on the set of all compact rectifiable curves on $X$.

**Definition 2.4.** We say that a property applicable to paths in $X$ holds for $p$-*almost every* path in $X$ if the family of all paths for which the property fails has $p$-modulus zero.

The next result, in the Euclidean setting, is originally due to Fuglede.

**Proposition 2.5.** *Let $\{g_i\}_i$ be a sequence of non-negative Borel measurable functions on $X$ and $g$ be a non-negative Borel measurable function on $X$ such that $g_i - g \to 0$ in $L^p(X)$. Then there is a subsequence $\{g_{i_k}\}_k$ such that for $p$-almost every compact rectifiable path $\gamma$ in $X$,*

$$\lim_k \int_\gamma g_{i_k}\, ds = \int_\gamma g\, ds.$$

**Proof.** Since $g_i - g \to 0$ in $L^p(X)$, by passing to a subsequence if necessary we may assume that

$$\int_X |g_i - g|^p\, d\mu \le 2^{-i}.$$

Observe that if $\int_\gamma |g_{i_k} - g|\, ds \to 0$, then $\lim_k \int_\gamma g_{i_k}\, ds = \int_\gamma g\, ds$. Let

$$\Gamma = \left\{\gamma \text{ non-constant compact rectifiable path } : \limsup_i \int_\gamma |g_i - g|\, ds > 0\right\}.$$

Note also that if $\gamma$ is a constant curve, then $\int_\gamma h\, ds = 0$ whenever $h$ is a function on $X$. Thus the claim of the proposition follows if we show that $\mathrm{Mod}_p(\Gamma) = 0$. To this end, for $n \in \mathbb{N}$ let

$$\Gamma_n = \left\{\gamma \text{ non-constant compact rectifiable path } : \limsup_i \int_\gamma |g_i - g|\, ds \ge 1/n\right\}.$$

Because of Theorem 2.3, if for each $n$ we have $\mathrm{Mod}_p(\Gamma_n) = 0$, then it follows that $\mathrm{Mod}_p(\Gamma) = 0$. To prove that $\mathrm{Mod}_p(\Gamma_n) = 0$, for each $i \in \mathbb{N}$ we set

$$\Gamma_n^i = \left\{\gamma \text{ non-constant compact rectifiable path } : \int_\gamma |g_i - g|\, ds \ge 1/n\right\},$$

and note that

$$\Gamma_n = \bigcap_{k\in\mathbb{N}} \bigcup_{\mathbb{N}\ni i\ge k} \Gamma_n^i. \tag{2.6}$$

By the definition of $\Gamma_n^i$ we see that $n\,|g_i - g| \in \mathcal{A}(\Gamma_n^i)$. It follows that

$$\mathrm{Mod}_p(\Gamma_n^i) \le n^p \int_X |g_i - g|^p\, d\mu \le n^p\, 2^{-i}.$$

An application of Theorem 2.3 yields

$$\mathrm{Mod}_p\left(\bigcup_{\mathbb{N}\ni i\geq k}\Gamma_n^i\right)\leq n^p\sum_{\mathbb{N}\ni i\geq k}2^{-i}=2\,n^p\,2^{-k}$$

whenever $k\in\mathbb{N}$. The above estimate, together with Theorem 2.3 and (2.6), gives $\mathrm{Mod}_p(\Gamma_n)\leq n^p\,2^{-k}$ for every $k\in\mathbb{N}$, and it follows that $\mathrm{Mod}_p(\Gamma_n)=0$. ■

The above proposition asks that the limit function, $g$, also be Borel measurable. The next lemma allows us to dispense with this requirement, and is quite useful in allowing us to use more general functions in computing the $p$-modulus of a family of paths.

**Proposition 2.7.** *Let $g\in L^p(X)$ be a non-negative function. Then there is a Borel measurable function $g_0$ on $X$ such that $g_0=g$ $\mu$-almost everywhere in $X$ and for p-almost every compact rectifiable path $\gamma$ in $X$ we have $g\circ\gamma=g_0\circ\gamma$ almost everywhere in the domain of $\gamma$, and hence the path integral $\int_\gamma g\,ds$ makes sense for p-almost every compact rectifiable path in $X$.*

**Proof.** Given that $\mu$ is a Borel measure and bounded sets have finite measure, it follows from basic measure theory that there is a non-negative Borel measurable function $g_0$ such that $g=g_0$ $\mu$-almost everywhere in $X$. Let $E=\{x\in X\,:\,g_0(x)\neq g(x)\}$. Then $\mu(E)=0$. If $\gamma$ is a compact rectifiable arc-length parametrized curve such that $\mathcal{H}^1(\gamma^{-1}(E))=0$, then $g\circ\gamma=g_0\circ\gamma$ almost everywhere in the domain of $\gamma$, and so the claim of measurability and integrability follows for such $\gamma$. The proposition now follows from Lemma 2.9 below. ■

**Lemma 2.8.** *If $g\in L^p(X)$ is a non-negative function, then for p-almost every compact rectifiable path $\gamma$ in $X$ we have $\int_\gamma g\,ds<\infty$.*

**Proof.** By Proposition 2.7, we may assume without loss of generality that $g$ is Borel measurable. Now the lemma follows from the fact that if $\Gamma$ is the collection of all non-constant compact rectifiable paths $\gamma$ in $X$ for which $\int_\gamma g\,ds=\infty$, then for every $\varepsilon>0$ we have $\varepsilon g\in\mathcal{A}(\Gamma)$ and hence $\mathrm{Mod}_p(\Gamma)=0$. ■

Recall that given a rectifiable path $\gamma$ we denote by $\gamma_0$ its arc-length parametrization.

**Lemma 2.9.** *Let $E\subset X$ such that $\mu(E)=0$, and let*

$$\Gamma_E^+=\{\gamma \text{ compact rectifiable path }\,:\,\mathcal{H}^1(\gamma_0^{-1}(E))>0\}.$$

*Then $Mod_p(\Gamma_E^+)=0$.*

**Proof.** Since $\mu$ is Borel, we can find a Borel set $F \subset X$ such that $E \subset F$ and $\mu(F) = 0$. Now by the definition of $\Gamma_E^+$ we see that $\infty\,\chi_F \in \mathcal{A}(\Gamma_E^+)$. The claim now follows from the fact that $\int_X (\infty\chi_F)^p \, d\mu = 0$. ■

## 3. Newtonian spaces of functions on $X$

Sobolev spaces are useful in the study of potential theory and partial differential equations, see for example [Maz] and [EG].

There are many notions of Sobolev type spaces of functions in the metric setting; for an overview we refer the reader to [HaK], [He], [Sh3] and [Sh4]. The notion of Sobolev type space we consider here is the one based on the notion of upper gradients, a notion that was first formulated by Heinonen and Koskela in [HeK]. Much of the results given here can also be found in [Sh1] and [Sh2].

**Definition 3.1.** Let $f$ be a function on $X$, and $g$ a non-negative Borel measurable function on $X$. We say that $g$ is an *upper gradient* of $f$ if whenever $\gamma$ is a non-constant compact rectifiable path in $X$, we have

$$|f(y) - f(x)| \le \int_\gamma g \, ds$$

when both $f(x)$ and $f(y)$ are finite-valued, and $\int_\gamma g\, ds = \infty$ if at least one of $f(x)$, $f(y)$ is not finite-valued. Here $x$ and $y$ denote the end points of $\gamma$.

We say that $g$ is a *$p$-weak upper gradient* of $f$ if the above condition holds for $p$-almost every non-constant compact recitifiable path in $X$.

From the absolute continuity of integrals on intervals in $\mathbb{R}$, the following lemma immediately follows.

**Lemma 3.2.** *Let $f$ be a function on $X$ and $\gamma$ be a non-constant, compact, rectifiable path on $X$. Suppose that $g$ is an upper gradient of $f$ such that $\int_\gamma g\, ds < \infty$. Then $f \circ \gamma$ is absolutely continuous.*

The next lemma is due to Koskela and McManus [KMc].

**Lemma 3.3.** *If $g$ is a $p$-weak upper gradient of $f$, then there is a sequence of upper gradients $\{g_i\}_i$ of upper gradients of $f$ such that $|g_i - g| \to 0$ in $L^p(X)$.*

**Proof.** Let $\Gamma$ be the collection of non-constant compact rectifiable paths in $X$ for which $g$ fails the upper gradient condition for $f$. Then $\operatorname{Mod}_p(\Gamma) = 0$ because $g$ is a $p$-weak upper gradient of $f$. So for each positive integer $k$ we can find $\rho_k \in \mathcal{A}(\Gamma)$ such that $\int_X \rho_k^p \, d\mu \le 1/k^p$. The choice of $g_k = g + \rho_k$ satisfies the claim of the lemma. ■

The next lemma follows immediately from Proposition 2.5 and Proposition 2.7.

**Lemma 3.4.** *Let $\{g_i\}$ be a sequence of upper gradients of $f$ and $g$ be a function on $X$ such that $|g_i - g| \to 0$ in $L^p(X)$. Then $g$ is a $p$-weak upper gradient of $f$.*

Note that if $g$ is an upper gradient of $f$ and $h$ isa non-negative Borel measurable function, then $g+h$ is also an upper gradient of $f$; that is, upper gradients are not unique. The next lemma gives a useful tool to computing minimal, and hence unique, $p$-weak upper gradient of $f$ in reasonable circumstances.

**Lemma 3.5.** *Let $Df$ be the set of all $p$-weak upper gradients of $f$. Then $Df \cap L^p(X)$ is a closed convex subset of $L^p(X)$, and hence if $1 < p < \infty$ and $Df \cap L^p(X)$ is non-empty, then there is a unique $g_f \in L^p(X) \cap Df$ such that $\|g_f\|_{L^p(X)} \le \|g\|_{L^p(X)}$ for all $g \in Df$, and furthermore, whenever $g \in L^p(X) \cap Df$, we have $g_f \le g$ $\mu$-almost everywhere in $X$.*

**Proof.** We leave it to the reader to verify that $Df$ is a convex set; it follows then that $Df \cap L^p(X)$ is also a convex set. By Lemma 3.4, it follows that $Df \cap L^p(X)$ is also a closed subset of $L^p(X)$. The remaining parts of the lemma are verified by the fact that when $1 < p < \infty$ the space $L^p(X)$ is a uniformly convex Banach space. Indeed, the $\mu$-almost everywhere minimizing property of $g_f$ is verified by the fact that if $g \in Df \cap L^p(X)$ and $E = \{x \in X : g(x) < g_f(x)\}$, then $g\chi_E + g_f\chi_{X\setminus E} \in Df \cap L^p(X)$. This last fact can be directly deduced from the definition of weak upper gradients; we leave it up to the reader to do so (see also [Sh2]). ∎

The function $g_f$ can be thought of as the metric space analog of the modulus of the gradient of an Euclidean Sobolev function.

**Definition 3.6.** Given a measurable function $f$ on $X$, we consider the following associated number

$$\|f\|_{N^{1,p}(X)} := \|f\|_{L^p(X)} + \inf_{g \in Df} \|g\|_{L^p(X)}.$$

We say that $f \in \widetilde{N}^{1,p}(X)$ if $\|f\|_{N^{1,p}(X)}$ is finite.

By Lemms 3.2 and Lemma 2.8, if $f \in \widetilde{N}^{1,p}(X)$ then $f$ is absolutely continuous on $p$-almost every non-constant compact rectifiable path in $X$. We leave the next lemma as an exercise for the reader.

**Lemma 3.7.** *If $f_1, f_2$ are functions on $X$, then*

$$Df_1 + Df_2 \subset D(f_1 + f_2) \cap D(\max\{f_1, f_2\}).$$

*Hence $\widetilde{N}^{1,p}(X)$ is a vector space and a lattice; that is, if $f_1, f_2 \in \widetilde{N}^{1,p}(X)$ and $\lambda \ge 0$, then $\max\{f_1, f_2\}, \min\{f_1, f_2\}, \min\{f_1, \lambda\}$ are in $\widetilde{N}^{1,p}(X)$.*

**Definition 3.8.** For $f, h \in \widetilde{N}^{1,p}(X)$, we say that $f \sim h$ if $\|f - h\|_{N^{1,p}(X)} = 0$. It is easy to see that $\sim$ is an equivalence relation on $\widetilde{N}^{1,p}(X)$. The *Newtonian space* is the quotient space $N^{1,p}(X) = \widetilde{N}^{1,p}(X)/\sim$.

It is directly verifiable that $N^{1,p}(X)$ is a normed vector space equipped with the norm $\|\cdot\|_{N^{1,p}(X)}$.

One should not get caught up too much in the notation of quotient space formalism in using Newtonian spaces. Instead, one should just keep in mind that these are merely functions that are well-defined up to an *exceptional* set (see Proposition 3.10 below). This is quite analogous to thinking of $L^p$-functions as those that are well-defined up to sets of $\mu$-measure zero.

The following definition, in the Euclidean setting, is due originally to [Oh2]. Recall that by Lemma 2.9 we know that if $\mu(E) = 0$ then $\mathrm{Mod}_p(\Gamma_E^+) = 0$. The following stronger condition is not necessarily satisfied by all sets of measure zero.

**Definition 3.9.** Let $E \subset X$. We say that $E$ is *p-exceptional* if $\mu(E) = 0$ and $\mathrm{Mod}_p(\Gamma_E) = 0$, where $\Gamma_E$ is the collection of all non-constant compact rectifiable paths in $X$ that intersect $E$.

**Proposition 3.10.** *Let $f, h \in \widetilde{N}^{1,p}(X)$. Then $f \sim h$ if and only if the set $E = \{x \in X : f(x) \neq h(x)\}$ is p-exceptional.*

**Proof.** The fact that $f \sim h$ if $E$ is $p$-exceptional is quite direct; we leave it to the reader to verify this.

Now suppose that $f \sim h$, and let $E$ be as above. Since

$$\|f - h\|_{L^p(X)} \leq \|f - h\|_{N^{1,p}(X)} = 0,$$

we see that $\mu(E) = 0$ and that (by Lemma 3.5) 0 is a $p$-weak upper gradient of $f - h$. Now the result follows from the fact that if $\gamma \in \Gamma_E$, the function 0 cannot satisfy the upper gradient condition on $\gamma$ for $f - h$. ∎

Under certain additional assumptions on the metric measure space the corresponding Newtonian functions exhibit the fine properties of Euclidean Sobolev functions, as shown in [Sh1]. For the fine properties of Sobolev functions in the Euclidean setting we refer the interested reader to [EG].

It was shown in [Sh1] that $N^{1,p}(X)$ is always a Banach space. We wrap up this note by considering the following refinement of the topology on $X$.

**Definition 3.11.** We say that a set $U \subset X$ is *p-path open* if for $p$-almost every non-constant compact rectifiable path $\gamma : I \to X$, the set $\gamma^{-1}(U)$ is relatively open in $I$. Let $\mathcal{T}_p$ be the collection of $p$-path open subsets of $X$.

It is a direct exercise to see that $\mathcal{T}_p$ is a topology on $X$ and that this topology is a refinement of the metric topology on $X$. The following is left to the reader as an exercise.

**Exercise 3.1.** Show that every function in $N^{1,p}(X)$ is continuous with respect to the $p$-path open topology $\mathcal{T}_p$.

The discussion in this note is an introduction to the use of the tool of $p$-moduli of path-families, and described basic results that are used in the theory of analysis in metric measure spaces. The discussion is far from being exhaustive, but an interested reader can find more information by consulting the references cited here and the references cited in those. In the general setting of this note, there are still many fundamental open problems. One of them is the following.

**Problem 3.1.** Under certain additional hypotheses on the metric measure space (such as doubling and Poincaré type inequalities), it is known that Lipschitz functions are dense in $N^{1,p}(X)$ and that functions in $N^{1,p}(X)$ are quasicontinuous (see for example [Sh1] and [BBS]). Are there metric spaces where locally Lipschitz continuous functions, or even merely continuous functions, are *not* dense in $N^{1,p}(X)$? If continuous functions form a dense subclass of $N^{1,p}(X)$ then functions in $N^{1,p}(X)$ would have quasicontinuous representatives.

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