Proceedings of the ICM2010 Satellite Conference
International Workshop on
Harmonic and Quasiconformal Mappings (HQM2010)
Editors: D. Minda, S. Ponnusamy, and N. Shanmugalingam

J. Analysis
Volume 18 (2010), 361–371

Generalized Matrix Spaces $c_0^2(X,\lambda,p)$, $c^2(X,\lambda,p)$ and $l_\infty^2(X,\lambda,p)$

J.K. Srivastava and R.K. Tiwari

Abstract. In this paper we introduce and study the spaces (matrix spaces) $c_0^2(X,\lambda,p)$, $c^2(X, \ \lambda,p)$ and $l_\infty^2(X,\lambda,p)$ of locally convex topological vector space $X-$valued double sequences which generalize several existing sequence spaces. Besides investigations of conditions connected with the comparison of these classes in terms of λ and p we study topological structure of these spaces when topologized through a paranorm or through a family of paranorms.

Keywords. Double sequence space, Matrix space, locally convex topological vector space, paranormed space.

2010 MSC. Primary 46A45, Secondary 46A15.

1. Introduction

Throughout this paper let $(X,\Im)$ be a Hausdorff locally convex topological vector space (lcTVS) over the field of complex numbers C and X^* be its topological dual. We denote by $\mathcal{U}$ the fundamental system of balanced, convex and absorbing neighbourhoods U of zero vector θ. We write g_U to denote the gauge (Minkowski functional) of $U\in\mathcal{U}$, i.e., $g_U(x)=\inf\{\alpha>0 : x\in\alpha U\}$. $D=\{g_U : U\in\mathcal{U}\}$ denotes the collection of continuous seminorms generating the topology $\Im$ of X. For details about lcTVS we refer [5, 16].

By a generalized matrix, a generalized double sequence or a vector double sequence we mean a double sequence $\overline{x}=(x_{mn})$ with elements from X. Let $p=(p_{mn})$ be a double sequence of strictly positive real numbers and $\lambda=(\lambda_{mn})$ be a double sequence of non-zero complex numbers. We introduce the following classes of vector double sequences and propose to study them in this paper.

This paper has been written while J.K. Srivastava held a project (No. F. 8-11/98 SR-1 dated Oct. 16, 1998) of the University Grants Commission and R.K. Tiwari worked in it as a Project Assistant. The support by Commission is very gratefully acknowledged.

ISSN 0971-3611

$$(1.1)\quad \begin{cases} c_0^2(X,\lambda,p)=\{\overline{x}=(x_{mn}):x_{mn}\in X, m,n\geq 1 \\ \text{and}\ \ (g_{_U}(\lambda_{mn}x_{mn}))^{p_{mn}}\to 0 \quad \text{as } m+n\to\infty \text{ for each } g_{_U}\in D\}; \end{cases}$$

$$(1.2)\quad \begin{cases} c^2(X,\lambda,p)=\{\overline{x}=(x_{mn}):x_{mn}\in X, m,n\geq 1 \text{ and there exists} \\ x\in X \text{ such that } (g_{_U}(\lambda_{mn}x_{mn}-x))^{p_{mn}}\to 0 \text{ as } m+n\to\infty \\ \text{for each } g_{_U}\in D\}; \end{cases}$$

and

$$(1.3)\quad \begin{cases} l_\infty^2(X,\lambda,p)=\{\overline{x}=(x_{mn}):x_{mn}\in X, m,n\geq 1 \text{ and} \\ \sup\limits_{m,n}(g_{_U}(\lambda_{mn}x_{mn}))^{p_{mn}}<\infty \text{ for each } g_{_U}\in D\}. \end{cases}$$

If $\lambda_{mn}=1$ for all m,n then $c_0^2(X,\lambda,p)$ will be denoted by $c_0^2(X,p)$ and when $p_{mn}=1$ for all m,n then $c_0^2(X,\lambda,p)$ will be denoted by $c_0^2(X,\lambda)$. Similarly we define $c^2(X,p)$, $c^2(X,\lambda)$, $l_\infty^2(X,p)$ and $l_\infty^2(X,\lambda)$.

We observe that the above defined classes are the generalizations of various sequence spaces, for instance,

(i) spaces of single scalar sequences $c_0(p)$, $c(p)$ and $l_\infty(p)$ [6, 7], $D_0^\wedge(p)$, $D_\infty^\wedge(p)$ [11], $c_0(\lambda)$, $c(\lambda)$ and $l_\infty(\lambda)$ [12];

(ii) spaces of Banach space X-valued single sequences $c_0(X,\lambda,p)$, $c(X,\lambda,p)$ and $l_\infty(X,\lambda),p)$ [13], $c_0(X)$, $c(X)$ and $l_\infty(X)$ [8]; spaces of lcTVS $X-$valued single sequences $c_0(X)$, $c(X)$ and $l_\infty(X)$ [2], $c_0(X,p)$, $c(X,p)$ and $l_\infty(X,p)$ [15]; $\Gamma(X)$ [14];

(iii) spaces of scalar double sequences $c_{0,2}^R$, c_2^R [9], C_{00}, C, $l_{\infty\infty}$ [3] and M_u, C_{uc} [1] can easily be obtained as special cases of the above introduced classes when $X, p=(p_{mn})$ and $\lambda=(\lambda_{mn})$ are suitably chosen.

By space of vector double sequences or vector matrix space $E(X)$ we mean a vector space of double sequences in X with respect to coordinatewise addition and scalar multiplication. We know that depending upon the mode of tending m and n to ∞ the double infinite summation $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}$ has several different meanings associated to it, however we shall denote $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}$ by $\sum\sum$ and shall take it in the sense $\lim_{N\to\infty}\sum\sum_{2\leq m+n\leq N}$.

Throughout the paper we shall denote $t_{mn}=|\lambda_{mn}/\mu_{mn}|^{p_{mn}}$, $H=H(p)=\sup_{m,n}p_{mn}$, $M=M(p)=\max(1,H)$ and $A[\alpha]=\max(1,|\alpha|)$. For $x\in X$, $\delta^{mn}(x)$ denotes the double sequence with x at $(m,n)^{\text{th}}-$position and remaining terms are θ; and $\delta(x)$ denotes the double sequence whose all terms are x.

Definition 1.4. A paranormed space (E, q) is a topological vector space whose pseudometric topology is given by the paranorm q (a real subadditive function with $q(\theta) = 0$, $q(x) = q(-x)$ for all $x \in E$ and with continuous scalar multiplication), see [16, p.15].

Now corresponding to K-, AK-, AD-, and C- properties of scalar sequence spaces [4] and its vector version for sequence spaces [10], we define :

Definition 1.5. A topological sequence space $(E(X), \Im)$ where $E(X)$ is a vector double sequence space in X is said to be

(i) a GK-space if the mapping $P_{mn} : E(X) \to X, P_{mn}(\overline{x}) = x_{mn}$ is continuous for each $m, n \geq 1$;
(ii) a GAD-space if it is a GK-space and $\Phi^2(X)$ is dense in $E(X)$ with respect to $\Im$, where $\Phi^2(X) = \{\overline{x} = (x_{mn}) : x_{mn} \in X$ and $x_{mn} = \theta$ for all but finitely many $m, n \geq 1\}$;
(iii) a GAK-space if it is a $GK-$space and for each $\overline{x} = (x_{mn})$ in $E(X)$, $S^N(\overline{x}) \to \overline{x}$ as $N \to \infty$ where $S^N(\overline{x}) = \sum\sum_{2 \leq m+n \leq N} \delta^{mn}(x_{mn})$;
(iv) a GC-space if $R_{mn} : X \to E(X)$, $R_{mn}(x) = \delta^{mn}(x)$ is continuous for each $m, n \geq 1$.

2. $c_0^2(X, \lambda, p)$, $c^2(X, \lambda, p)$ and $l_\infty^2(X, \lambda, p)$

Here we investigate conditions on $p = (p_{mn})$ and $\lambda = (\lambda_{mn})$ so that a class $c_0^2(X, \lambda, p)$, $c^2(X, \lambda, p)$ or $l_\infty^2(X, \lambda, p)$ is contained in or equal to a similar class.

Throughout this section, unless stated otherwise, we shall take $p = (p_{mn})$ and $q = (q_{mn})$ in l_∞^2, space of all bounded scalar double sequences.

Lemma 2.1. $c_0^2(X, \lambda, p) \subset c_0^2(X, \mu, p)$ *if and only if*

$$\liminf_{m+n\to\infty} t_{mn} > 0. \tag{2.2}$$

Proof. Sufficiency of the condition can easily be proved. For necessity, suppose that $\lim_{m+n\to\infty} \inf t_{mn} = 0$. Then there exist sequences $(m(k))$ and $(n(k)$ of integers such that for each $k \geq 1$

$$k \left|\lambda_{m(k)n(k)}\right|^{p_{m(k)n(k)}} < \left|\mu_{m(k)n(k)}\right|^{p_{m(k)n(k)}}.$$

We now choose $z \in X$ and $g_V \in D$ such that $g_V(z) = 1$ and define $\overline{x} = (x_{mn})$ by

$$\begin{aligned} x_{mn} &= \lambda_{mn}^{-1} k^{-1/p_{mn}} z, \quad \text{for } m = m(k),\ n = n(k),\ k \geq 1, \text{ and} \\ &= \theta, \ \text{otherwise.} \end{aligned}$$

Thus for each $g_U \in D$

$$(g_U(\lambda_{m(k)n(k)} x_{m(k)n(k)}))^{p_{m(k)n(k)}} = \frac{1}{k}(g_U(z))^{p_{m(k)n(k)}} \leq \frac{1}{k} A[(g_U(z))^{M(p)}]$$

implies that $\overline{x} = (x_{mn}) \in c_0^2(X, \lambda, p)$, but for each $k \geq 1$

$$(g_V(\mu_{m(k)n(k)} x_{m(k)n(k)}))^{p_{m(k)n(k)}} = \left|\frac{\mu_{m(k)n(k)}}{\lambda_{m(k)n(k)}}\right|^{p_{m(k)n(k)}} \frac{1}{k}(g_V(z))^{p_{m(k)n(k)}} > 1$$

shows that $\overline{x} \notin c_0^2(X, \lambda, p)$. This completes the proof. ■

Similarly, we can prove:

Lemma 2.3. $c_0^2(X, \mu, p) \subset c_0^2(X, \lambda, p)$ *if and only if*

$$\lim_{m+n\to\infty} \sup t_{mn} < \infty. \tag{2.4}$$

Combining Lemmas 2.1 and 2.3, we easily get:

Theorem 2.5. $c_0^2(X, \lambda, p) = c_0^2(X, \mu, p)$ *if and only if*

$$0 < \lim_{m+n\to\infty} \inf t_{mn} \leq \lim_{m+n\to\infty} \sup t_{mn} < \infty. \tag{2.6}$$

Lemma 2.7. *If* $p = (p_{mn}) \in l_\infty^2$ *and* $q = (q_{mn})$, *not necessarily in* l_∞^2, *then* $c_0^2(X, \lambda, p) \subset c_0^2(X, \lambda, q)$ *if and only if*

$$\lim_{m+n\to\infty} \inf \frac{q_{mn}}{p_{mn}} > 0. \tag{2.8}$$

Proof. (2.8) is sufficient can easily be proved. For necessity of (2.8), suppose that $c_0^2(X, \lambda, p) \subset c_0^2(X, \lambda, q)$, but $\lim_{m+n\to\infty} \inf \frac{q_{mn}}{p_{mn}} = 0$. Then there exist sequence $(m(k))$ and $(n(k))$ of integers such that for each $k \geq 1$

$$k q_{m(k)n(k)} < p_{m(k)n(k)}.$$

We now choose $z \in X$ and $g_V \in D$ such that $g_V(z) = 1$ and define the sequence $\overline{x} = (x_{mn})$ by

$$\begin{aligned} x_{mn} &= \lambda_{mn}^{-1} k^{-1/p_{mn}} z, \qquad \text{for} \quad m = m(k)\, n = n(k),\ k \geq 1, \text{ and} \\ &= \theta, \text{ otherwise.} \end{aligned}$$

Thus we see that $\overline{x} \in c_0^2(X, \lambda, p)$, but for each $k \geq 1$

$$(g_V(\lambda_{m(k)n(k)}\, x_{m(k)n(k)}))^{q_{m(k)n(k)}} = k^{-q_{m(k)n(k)}/p_{m(k)n(k)}} > k^{-k} > e^{-1/2},$$

implies that $\overline{x} \notin c_0^2(X, \lambda, q)$, a contradiction. This completes the proof. ■

Similarly we can prove:

Lemma 2.9. *If* $q = (q_{mn}) \in l_\infty^2$, *and* $p = (p_{mn})$ *not necessarily in* l_∞^2, *then* $c_0^2(X, \lambda, q) \subset c_0^2(X, \lambda, p)$ *if and only if*

$$\lim_{m+n\to\infty} \sup \frac{q_{mn}}{p_{mn}} < \infty. \tag{2.10}$$

On combining Lemmas 2.7 and 2.9 we easily get :

Theorem 2.11. *For* $p = (p_{mn}),\ q = (q_{m,n}) \in l^2_\infty$, $c^2_0(X, \lambda, p) = c^2_0(X, \lambda, q)$ *if and only if*

$$0 < \lim_{m+n\to\infty} \inf \frac{q_{mn}}{p_{mn}} \le \lim_{m+n\to\infty} \sup \frac{q_{mn}}{p_{mn}} < \infty. \tag{2.12}$$

Theorem 2.13. (a) $c^2_0(X, \lambda, p) \subset c^2_0(X, \mu, q)$ *if and only if* (2.2) *and* (2.8) *hold.*
(b) $c^2_0(X, \mu, q) \subset c^2_0(X, \lambda, p)$ *if and only if* (2.4) *and* (2.10) *hold.*

Proof. Proof easily follows by Lemmas 2.1, 2.3, 2.7, and 2.9. ∎

Under the given conditions the containment in Theorem 2.13 (a) or (b) may be strict. We give below an example for the part (a), similar construction can be made for part (b).

Example 2.14. Take $z \in X$ and $g_V \in D$ such that $g_V(z) = 1$ and define a sequence in X such that $x_{mn} = (m+n)^{-(m+n)}z$ for all m, $n \ge 1$. Now take $p_{mn} = (m+n)^{-1}$, if $m+n$ is odd integer, $p_{mn} = (m+n)^{-2}$, if $m+n$ is even integer, $q_{mn} = (m+n)^{-1}$, $\lambda_{mn} = 3^{(m+n)}$ and $\mu_{mn} = 2^{(m+n)}$ for all m and n. Thus for any $g_U \in D$ and $m, n \ge 1$

$$(g_U(\mu_{mn}x_{mn}))^{q_{mn}} \le 2(m+n)^{-1}A[(g_U(z))]$$

implies that $\overline{x} = (x_{mn}) \in c^2_0(X, \mu, q)$ but for even integers $m+n$

$$(g_V(\lambda_{mn}x_{mn}))^{p_{mn}} = 3^{1/(m+n)}(m+n)^{-1/(m+n)}$$

implies that $\overline{x} = (x_{mn}) \notin c^2_0(X, \lambda, p)$, however the conditions (i) and (ii) of Theorem 2.13 (a) are satisfied.

Lemma 2.15. *If for a sequence* $\overline{x} = (x_{mn})$ *there exits* $l \in X$ *such that*

$$(g_U(\lambda_{mn}x_{mn} - l))^{p_{mn}} \to 0 \text{ as } m+n \to \infty, \text{ for each } g_U \in D, \tag{2.16}$$

then l *is unique.*

Proof. If (2.16) holds for l_1, $l_2 \in X$ then from

$$(g_U(l_1 - l_2))^{p_{mn}/M} \le (g_U(\lambda_{mn}x_{mn} - l_1))^{p_{mn}/M} + (g_U(\lambda_{mn}x_{mn} - l_2))^{p_{mn}/M}$$

it follows that $(g_U(l_1 - l_2))^{p_{mn}/M} \to 0$ as $m+n \to \infty$. Thus, $g_U(l_1 - l_2) = 0$, for each $g_U \in D$ and X is Hausdorff therefore, $l_1 = l_2$. This completes the proof. ∎

Lemma 2.17. *If* $p = (p_{mn}) \in l^2_\infty$ *and* $q = (q_{mn})$, *not necessarily in* l^2_∞, *then* $c^2(X, \lambda, p) \subset c^2(X, \lambda, q)$ *if and only if (2.8) holds.*

Proof. Proof of sufficient part is straightforward hence omitted. For necessity suppose that $c^2(X,\lambda,p) \subset c^2(X,\lambda,q)$ but $\lim_{m+n\to\infty} \inf \frac{q_{mn}}{p_{mn}} = 0$. Then there exist sequences $(m(k))$ and $(n(k))$ of integers such that $kq_{m(k)n(k)} < p_{m(k)n(k)}$ for all $k \geq 1$. Now choose $z \in X$ and $g_V \in D$ such that $g_V(z) = 1$ and consider the sequence $\overline{x} = (x_{mn})$ defined by

$$\begin{aligned} x_{mn} &= \lambda_{mn}^{-1} k^{-1/p_{mn}} z + \lambda_{mn}^{-1} l, \qquad \text{for } m = m(k),\ n = n(k),\ k \geq 1, \text{ and} \\ &= \lambda_{mn}^{-1} l \quad \text{otherwise,} \end{aligned}$$

where $l \in X$. Then we easily see that $\overline{x} = (x_{mn}) \in c^2(X,\lambda,p)$. Now,

$$(g_V(\lambda_{m(k)n(k)} x_{m(k)n(k)} - l))^{q_{m(k)n(k)}} = (g_V(k^{-1/p_{m(k)n(k)}} z))^{q_{m(k)n(k)}} > k^{-k} > e^{1/2}$$

for each $k \geq 1$, shows that $(g_V(\lambda_{mn} x_{mn} - l))^{p_{mn}} \to 0$ as $m+n \to \infty$. Moreover, if possible, suppose that for $l_1 \neq l$, $(g_U(\lambda_{mn} x_{mn} - l_1))^{q_{mn}} \to 0$, as $m+n \to \infty$ for each $g_U \in D$. Then for $0 < \epsilon < 1$ and $g_U \in D$

$$(g_U(\lambda_{m(k)n(k)} x_{m(k)n(k)} - l_1))^{q_{m(k)n(k)}} < \epsilon^{M(p)}, \text{ for all sufficiently large } k.$$

and so for sufficiently large k

$$\begin{aligned} (g_U(\lambda_{m(k)n(k)} x_{m(k)n(k)} - l_1))^{p_{m(k)n(k)}} &< (g_U(\lambda_{m(k)n(k)} x_{m(k)n(k)} - l_1))^{kq_{m(k)n(k)}} \\ &< \epsilon^{kM(p)} < \epsilon^{M(p)}. \end{aligned}$$

Moreover, $(g_U(\lambda_{mn} x_{mn} - l))^{p_{mn}} \to 0$ as $m+n \to \infty$ and therefore

$$(g_U(\lambda_{m(k)n(k)} x_{m(k)n(k)} - l))^{p_{m(k)n(k)}} < \epsilon^{M(p)}$$

for all sufficiently large k. Thus,

$$(g_U(l - l_1))^{p_{m(k)n(k)/M(p)}} < 2\epsilon,$$

for all sufficiently large k, which leads to $g_U(l - l_1) = 0$ for each $g_U \in D$ or $l = l_1$, a contradiction. Hence $\overline{x} = (x_{mn}) \notin c^2(X,\lambda,q)$, a contradiction. This completes the proof. ■

Similarly, we can prove :

Lemma 2.18. *If $q = (q_{mn}) \in l_\infty^2$ and $p = (p_{mn})$, not necessarily in l_∞^2, then $c^2(X,\lambda,q) \subset c^2(X,\lambda,p)$ if and only if (2.10) holds.*

On combining Lemmas 2.3.10 and 2.3.11 we have :

Theorem 2.19. *$c^2(X,\lambda,p) = c^2(X,\lambda,q)$ if and only if (2.12) holds.*

Proof of the following results (Lemma 2.20 to Theorem 2.26) connected with $l_\infty^2(X,\lambda,p)$ can easily be disposed of by proceeding along the case of $c_0^2(X,\lambda,p)$ discussed above :

Lemma 2.20. *$l_\infty^2(X,\lambda,p) \subset l_\infty^2(X,\mu,p)$ if and only if (2.2) holds.*

Lemma 2.21. *$l^2_\infty(X, \mu, p) \subset l^2_\infty(X, \lambda, p)$ if and only if (2.4) holds.*

On combining Lemmas 2.20 and 2.21 we get:

Theorem 2.22. *$l^2_\infty(X, \lambda, p) = l^2_\infty(X, \mu, p)$ if and only if (2.6) holds.*

Lemma 2.23. *If $p = (p_{mn}) \in l^2_\infty$ and $q = (q_{mn})$ not necessarily in l^2_∞, then $l^2_\infty(X, \lambda, p) \subset l^2_\infty(X, \lambda, q)$ if and only if (2.10) holds.*

Lemma 2.24. *If $q = (q_{mn}) \in l^2_\infty$ and $p = (p_{mn})$ is not necessarily in l^2_∞ then $l^2_\infty(X, \lambda, q) \subset l^2_\infty(X, \lambda, p)$ if and only if (2.8) holds.*

On combining Lemmas 2.23 and 2.24 we get the following theorem:

Theorem 2.25. *If $p = (p_{mn}),\ q = (q_{mn}) \in l^2_\infty$ then $l^2_\infty(X, \lambda, p) = l^2_\infty(X, \lambda, q)$ if and only if (2.12) holds.*

Theorem 2.26. *For $p = (p_{mn}) \in l^2_\infty$ and $q = (q_{mn}) \in l^2_\infty$ we have*

(a) *$l^2_\infty(X, \lambda, p) \subset l^2_\infty(X, \mu, q)$ if and only if (2.2) and (2.10) hold, and*
(b) *$l^2_\infty(X, \mu, q) \subset l^2_\infty(X, \lambda, p)$ if and only if (2.4) and (2.8) hold.*

Moreover examples (similar to Example 2.14) can be constructed to illustrate that under the mentioned conditions the containment in Theorem 2.26 may be strict.

3. Topological Structure

Throughout this section we take $p = (p_{mn}) \in l^2_\infty$. If $\overline{x} = (x_{mn})$ and $\overline{y} = (y_{mn}) \in c^2_0(X, \lambda, p)$ then $\overline{x} + \overline{y} = (x_{mn} + y_{mn}) \in c^2_0(X, \lambda, p)$ follows from

$$(g_U(\lambda_{mn}(x_{mn} + y_{mn})))^{p_{mn}/M} \leq (g_U(\lambda_{mn}x_{mn}))^{p_{mn}/M} + (g_U(\lambda_{mn}y_{mn}))^{p_{mn}/M},$$

$g_U \in D$. Moreover for scalar α

$$(g_U(\alpha\lambda_{mn}x_{mn}))^{p_{mn}/M} \leq A[\alpha](g_U(\lambda_{mn}x_{mn}))^{p_{mn}/M}, \qquad g_U \in D,$$

implies that $\alpha\overline{x} = (\alpha x_{mn}) \in c^2_0(X, \lambda, p)$. Thus $c^2_0(X, \lambda, p)$ is a linear space over the field of complex numbers with respect to coordinate-wise addition and scalar multiplication. $\overline{\theta} = (y_{mn})$, where $y_{mn} = \theta$ for all $m, n \geq 1$, is the zero vector of $c^2_0(X, \lambda, p)$. Similarly $c^2(X, \lambda, p)$ and $l^2_\infty(X, \lambda, p)$ are also linear spaces over the field of complex numbers. Further for each $U \in \mathcal{U}$, we define

$$G_U(\overline{x}) = \sup_{m,n}(g_U(\lambda_{mn}x_{mn}))^{p_{mn}/M} \text{ and } d_U(\overline{x}, \overline{y}) = G_U(\overline{x} - \overline{y}).$$

Theorem 3.1. (i) *$(c^2_0(X, \lambda, p), G_U)$ is a paranormed space, and*

(ii) $(c^2(X,\lambda,p), d_{_U})$ *and* $(l^2_\infty(X,\lambda,p), d_{_U})$ *are topological (pseudometric) groups, in which the mapping* $(\alpha,\overline{x}) \to \alpha\overline{x}$ *is continuous at* $\alpha = 0$ *and* $\overline{x} = \overline{\theta}$.

However, $(c^2(X,\lambda,p), G_{_U})$ *and* $(l^2_\infty(X,\lambda,p), G_{_U})$ *are paranormed spaces if and only if* $\inf\limits_{m,n} p_{mn} > 0$.

Proof. (i) For $G_{_U}$ to be a paranorm we prove the continuity of scalar multiplication only, other conditions are straight forward.

(a) Let $\alpha_k \to 0$ as $k \to \infty$ and $G_{_U}(\overline{x}^k - \overline{x}) \to 0$ as $k \to \infty$. Suppose $|\alpha_k| \le 1$ for all $k \ge 1$, then we have

$$G_{_U}(\alpha_k\,\overline{x}^k) \le G_{_U}(\overline{x}^k - \overline{x}) + G_{_U}(\alpha_k\,\overline{x}). \tag{3.2}$$

Let $\epsilon > 0$. Since $\overline{x} \in c^2_0(X,\lambda,p)$ so there exists N such that

$$(g_{_U}(\lambda_{mn}x_{mn}))^{p_{mn}/M} < \epsilon \quad \text{for all } m+n > N.$$

Also since $\alpha_k \to 0$ therefore there exists K such that

$$(g_{_U}(\alpha_k\lambda_{mn}x_{mn}))^{p_{mn}/M} = |\alpha_k|^{p_{mn}/M}\,(g_{_U}(\lambda_{mn}x_{mn}))^{p_{mn}/M} < \epsilon$$

for all $k \ge K$ and $2 \le m+n \le N$. Hence $(g_{_U}(\alpha_k\lambda_{mn}x_{mn}))^{p_{mn}/M} < \epsilon$ for all $m,n \ge 1$ and for all $k \ge K$ or $G_{_U}(\alpha_k\,\overline{x}) < \epsilon$ for all $k \ge K$, i.e., $G_{_U}(\alpha_k\,\overline{x}) \to 0$ as $k \to \infty$. Thus from (3.2) we get that $G_{_U}(\alpha_k\,\overline{x}^k) \to 0$ as $k \to \infty$.

(b) Let α be a scalar and $G_{_U}(\overline{x}^k) \to 0$ as $k \to \infty$. Then we have $G_{_U}(\alpha\,\overline{x}^k) \le A[\alpha]G_{_U}(\overline{x}^k)$, which implies that $G_{_U}(\alpha\,\overline{x}^k) \to 0$, as $k \to \infty$.

Thus, (a) and (b) together give continuity of scalar multiplication [16, see p. 17].

(ii) In view of definition of $d_{_U}$ it is straight forward to verify that $l^2_\infty(X,\lambda,p)$ is a pseudometric group. Further from $G_{_U}(\alpha\,\overline{x}) \le A[\alpha]\,G_{_U}(\overline{x})$ the continuity of $(\alpha,\overline{x}) \to \alpha\,\overline{x}$ at $\alpha = 0$ and $\overline{x} = \overline{\theta}$ follows easily.

Let $\inf_{m,n} p_{mn} = 0$, $0 < |\alpha| < 1$, $x \in X$ such that $g_{_U}(x) = 1$. Then $\overline{x} = (x_{mn})$ defined by $x_{mn} = \lambda^{-1}_{mn}x$, $m,n \ge 1$, is in $l^2_\infty(X,\lambda,p)$ but

$$G_{_U}(\alpha\,\overline{x}) = \sup_{m,n}\,|\alpha|^{p_{mn}/M}(g_{_U}(x))^{p_{mn}/M} = 1$$

shows that the mapping $\alpha \to \alpha\overline{x}$ is not continuous, hence $G_{_U}$ fails to be a paranorm on $l^2_\infty(X,\lambda,p)$.

Now suppose that $\inf_{m,n} p_{mn} = l > 0$ then for any $\overline{x} \in l^2_\infty(X,\lambda,p)$

$$\begin{aligned} G_{_U}(\alpha\overline{x}) &= \sup_{m,n}|\alpha|^{p_{mn}/M}(g_{_U}(\lambda_{mn}x_{mn}))^{p_{mn}/M} \\ &\le \max(|\alpha|, |\alpha|^{l/M})\,G_{_U}(\overline{x}) \end{aligned}$$

and hence the mapping $\alpha \to \alpha\overline{x}$ is continuous. Thus if $\inf_{m,n} p_{mn} > 0$ and we proceed along the lines of proof of (i) above we can easily show that G_U is a paranorm on $l^2_\infty(X, \lambda, p)$.

Similarly the case of $c^2(X, \lambda, p)$ can be disposed of. This completes the proof of the theorem. ■

We now observe that the collection $\mathcal{G} = \{G_U : U \in \mathcal{U}\}$ of all paranorms G_U on $c^2_0(X, \lambda, p)$ defines a linear topology $\sigma\mathcal{G}$ on $c^2_0(X, \lambda, p)$ where we say that the net $\overline{x} \to \overline{\theta}$ in $\sigma\mathcal{G}$ if and only if the net $\overline{x} \to \overline{\theta}$ with respect to each $G_U \in \mathcal{G}$ [16, p. 38]. We shall denote this topological vector space by $(c^2_0(X, \lambda, p), \sigma\mathcal{G})$. Similarly if we take $\inf_{m,n} p_{mn} > 0$ then $(c^2(X, \lambda, p), \sigma\mathcal{G})$ and $(l^2_\infty(X, \lambda, p), \sigma\mathcal{G})$ are also topological vector spaces.

Moreover X is Hausdorff therefore these spaces are also Hausdorff. For instance if $\overline{x} = (x_{mn}) \in c^2_0(X, \lambda, p)$ and $\overline{x} \neq \overline{\theta}$ then there exists some $x_{mn} \neq \theta$. Since X is Hausdorff therefore there exists a $g_U \in D$ such that $g_U(x_{mn}) \neq 0$ and so $G_U(\overline{x}) \neq 0$, i.e., $(c^2_0(X, \lambda, p), \sigma\mathcal{G})$ is Hausdorff.

In the following we denote by σd the topology such that a net $\overline{x}^i \to \overline{x}$ in σd if and only if $d_U(\overline{x}^i, \overline{x}) \to 0$ for each $U \in \mathcal{U}$.

Theorem 3.3. *If X is complete then*

(i) $(c^2_0(X, \lambda, p), \sigma\mathcal{G})$ *is complete (topological vector space), and*
(ii) $(c^2(X, \lambda, p), \sigma d)$ *and* $(l^2_\infty(X, \lambda, p), \sigma d)$ *are complete (topological groups).*

Proof. (i) For the completeness of $(c^2_0(X, \lambda, p), \sigma\mathcal{G})$, let $\{\overline{x}^i\}$ be a Cauchy net in $(c^2_0(X, \lambda, p), \sigma\mathcal{G})$ over the directed set I and $\epsilon > 0$. Then for each G_U there exists $i_0 \in I$ such that

$$G_U(\overline{x}^i - \overline{x}^j) < \epsilon \text{ for all } \quad i, j \geq i_0. \tag{3.4}$$

This implies that $\sup_{m,n}(g_U(\lambda_{mn}(x^i_{mn} - x^j_{mn}))^{p_{mn}/M} < \epsilon$. Thus for each $m, n \geq 1$, (x^i_{mn}) is a Cauchy net in X. Since X is complete so for each $m, n \geq 1$ there exists x_{mn} in X and hence $\overline{x} = (x_{mn})$ such that when the limit is taken over j in (3.4) we get $G_U(\overline{x}^i - \overline{x}) \leq \epsilon$ for $i \geq i_0$ and for each $U \in \mathcal{U}$. Thus $\overline{x}^i \to \overline{x}$ in $\sigma\mathcal{G}$. Now if we take an $i \geq i_0$ then we have

$$(g_U(\lambda_{mn} x_{mn}))^{p_{mn}/M} \leq G_U(\overline{x}^i - \overline{x}) + (g_U(\lambda_{mn} x^i_{mn}))^{p_{mn}/M} < 2\epsilon$$

for all sufficiently large m and n, whence $(g_U(\lambda_{mn} x_{mn}))^{p_{mn}/M} \to 0$ as $m, n \to \infty$. Thus $\overline{x} \in c^2_0(X, \lambda, p)$.

(ii) Completeness of $c^2(X, \lambda, p)$ and $l^2_\infty(X, \lambda, p)$ can easily be proved on the lines of (i).

This completes the proof. ■

Theorem 3.5. (i) $(c_0^2(X,\lambda,p),\sigma\mathcal{G})$ *is a GK-, GAD-, GAK- and GC-space; and*

(ii) *if* $\inf_{m,n} p_{mn} > 0$ *then* $(c^2(X,\lambda,p),\sigma\mathcal{G})$ *and* $(l_\infty^2(X,\lambda,p),\sigma\mathcal{G})$ *are GK- and GC-spaces.*

Proof. (i) For each $m,n \geq 1$, the continuity of linear map $F_{mn} : c_0^2(X,\lambda,p) \to X$ where $F_{mn}(\overline{x}) = x_{mn}$ follows from

$$g_U(F_{mn}(\overline{x})) = g_U(x_{mn}) \leq |\lambda_{mn}|^{-1}\left[G_U(\overline{x})\right]^{M/p_{mn}}.$$

Thus $c_0^2(X,\lambda,p)$ is a GK-space.

Let $\overline{x} = (x_{mn}) \in c_0^2(X,\lambda,p)$, $\epsilon > 0$ and $G_U \in \mathcal{G}$. Then there exists N such that $(g_U(\lambda_{mn}x_{mn}))^{p_{mn}/M} < \epsilon$ for all $m+n > N$. Clearly $\overline{y} = (y_{mn})$ defined by

$$y_{mn} = \begin{cases} x_{mn}, & m+n \leq N \\ \theta, & \text{otherwise,} \end{cases}$$

is in $\Phi^2(X)$ and $G_U(\overline{x}-\overline{y}) < \epsilon$. Hence $\Phi^2(X)$ is dense in $c_0^2(X,\lambda,p)$. This proves that $c_0^2(X,\lambda,p)$ is a GAD- space.

Now, $(g_U(\lambda_{mn}x_{mn}))^{p_{mn}/M} < \epsilon$ for all $m+n > N$ implies that $G_U(\overline{x} - S^N(\overline{x})) < \epsilon$. Thus $S^N(\overline{x}) \to \overline{x}$ as $N \to \infty$ with respect to each G_U, and hence $c_0^2(X,\lambda,p)$ is a GAK-space.

Further for each $m,n \geq 1$ the continuity of $R_{mn} : X \to c_0^2(X,\lambda,p)$, $R_{mn}(x) = \delta^{mn}(x)$, follows from

$$G_U(R_{mn}(x)) = |\lambda_{mn}|^{p_{mn}/M}(g_U(x))^{p_{mn}/M}, \qquad G_U \in \mathcal{G}.$$

Hence $(c_0^2(X,\lambda,p),\sigma\mathcal{G})$ is a GC-space.

Proof of (ii) is similar hence omitted. This completes the proof. ∎

Acknowledgments. We are grateful to the referee for his valuable comments which, by incorporating them, improved the presentation of the paper.

References

[1] J. Boos, T. Leiger, and K. Zeller, Consistency theory of SM-methods, *Acta Math. Hungar* **76 (1-2)**(1997), 109–142.

[2] M. Gupta, P.K. Kamthan and K. L. N. Rao, The generalized sequence spaces c, c_0, l^1 and l^∞ and their Köthe duals, *Tamkang. J. Math.*, **7 (2)**(1976), 175–178.

[3] M. Gupta and P.K. Kamthan, Infinite matrices and tensorial transformations, *Acta Mathematica Vietnamica*, **Tom 5**, No. 1, (1980), 33–41.

[4] P.K. Kamthan, and M. Gupta, *Sequence spaces and series*, Lecture notes 65, Marcel Dekker Inc., New York, 1980.

[5] G. Köthe, *Topological vector spaces I*, Springer-Verlag, New York, 1969.

[6] C.G. Lascarides, A study of certain sequence spaces of Maddox and a generalization of a theorem of Iyer, *Pacific J. Math.* **38**(1971), 487–500.

[7] I.J. Maddox, Spaces of strongly summable sequences, *Quarterly J. Math. (Oxford)*, **18(2)**(1967), 345-355.

[8] I.J. Maddox, *Infinite matrices of operators*, Lecture notes in Mathematics 786, Springer-Verlag, New York, 1980.

[9] F. Morics, Extensions of the spaces c and c_0 from single to double sequences, *Acta Math. Hungar,* **57(1-2)**(1991), 129–136.

[10] J. Patterson, *Generalized Sequence Spaces and Matrix Transformations*, Dissertation, I.I.T., Kanpur, 1980.

[11] A. Rath and P.D. Srivastava, Matrix transformations between some classes of generalized analytic sequence spaces, *Journal of Faculty of Education,* **1**, No. 1, 1994, 29–45.

[12] J. Sikk, The rate spaces $m(\lambda)$, $c(\lambda)$, $c_0(\lambda)$ and $l^p(\lambda)$ of sequences, *Acta et Comment Univ. Tartuensis,* **970**(1994), 87–96.

[13] B.K. Srivastava, *On certain generalized sequence spaces*, Dissertation, Gorakhpur University, Gorakhpur, 1996.

[14] J.K. Srivastava, Köthe-Toeplitz and topological duals of generalized entire sequence spaces, *Tamkang J. Math.* **17**(3), (1986), 115–126.

[15] J.K. Srivastava, Köthe-Toeplitz and topological duals of spaces of strongly summable vector sequences, *J. Indian Math. Soc.*, **71**(2004), 141–152.

[16] A. Wilansky, *Modern methods in topological vector spaces*, McGraw Hill, New York, 1978.

J.K. Srivastava
Address:
Department of Mathematics and Statistics,
D.D.U. Gorakhpur University,
Gorakhpur–273009,
INDIA
E-mail: jks_ddugu@yahoo.com

R.K. Tiwari
Address:
Department of Mathematics and Statistics,
D.D.U. Gorakhpur University,
Gorakhpur–273009,
INDIA
E-mail: rkt_ddugu@rediffmail.com