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Pluripolarity of Graphs of Quasianalytic Functions of Several Variables in the Sense of Gonchar

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Abstract. In this paper we prove pluripolarity of graphs of quasianalytic functions of several variables in the sense of Gonchar.

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1. Introduction

Let f be a function, defined and continuous on a compact set K of complex space $\mathbb{C}^n$, and let $\rho_n(f)$ be the least deviation of K from the rational functions of degree less than or equal to n:

$$\rho_n(f) = \inf_{r_n} \|f - r_n\|_K,$$

where $\|\cdot\|_K$ is the uniform norm and the infimum is taken over all rational functions of the form

$$r_m(z) = \frac{\sum\limits_{|\alpha|\le m} a_\alpha z^\alpha}{\sum\limits_{|\alpha|\le m} b_\alpha z^\alpha}.$$

Here $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is a multi-index.

As usual, we denote by $e_m(f)$ the least deviation of on K from its polynomials approximation of degree less or equal to m. Obviously, $\rho_m(f) \le e_m(f)$ for every m=1,2,3,... In papers ([1, 2]) Gonchar proved in the one dimensional case that the class of functions

$$R(K) = \{f \in C(K) : \varliminf_{m\to\infty} \sqrt[m]{\rho_m(f)} < 1\}$$

possess one of the most important property of analytic functions. Namely, if

$$\varliminf_{m\to\infty} \sqrt[m]{\rho_m(f)} < 1$$

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and $f(z) = 0$ on the set $E \subset K \subset \mathbb{C}$ of positive logarithm capacity, then $f(z) \equiv 0$, $z \in K$.

By analogy with class,

$$B(K) = \{f \in C(K) : \varliminf_{m\to\infty} \sqrt[m]{e_m(f)} < 1\},$$

which is called the class of quasianalytic functions of Bernstein (see [3, 4, 5]), we call $R(K)$ the class of quasianalytic functions of Gonchar. It is known that functions that are analytic on K can be characterized by condition (Bernstein's theorem)

$$\varlimsup_{m\to\infty} \sqrt[m]{e_m(f)} < 1.$$

2. Main Theorem

In paper ([6]) K. Diederich and J.E. Fornaess constructed examples of smooth functions, whose graphs are not pluripolar in $\mathbb{C}^2$. In recent work ([4]) D. Coman, N. Levenberg and E.A. Poletsky have proved that if $f \in B(K)$, $K = [a, b] \subset \mathbb{R}$, then its graph Γ_f is not pluripolar in $\mathbb{C}^2$.

In this paper we prove following more general theorem.

Theorem 2.1. *If $f \in R(K)$, $K \subset \mathbb{C}^n$, then its graph Γ_f pluripolar in $\mathbb{C}^{n+1}$.*

Proof. According to the hypothesis of the theorem there exists a sequence of natural numbers m_k and a corresponding sequence of rational functions

$$r_{m_k} = \frac{p_{m_k}}{q_{m_k}}$$

such that

$$\rho_{m_k}(f) = \left\|f - r_{m_k}\right\|_K \leq \alpha^{m_k},$$

where $\alpha : 0 < \alpha < 1$ is some fixed number. Without loss of generality we can assume that

$$\|f\|_K \leq \frac{1}{2}, \ \left\|p_{m_k}\right\|_K \leq 1 \quad \left\|q_{m_k}\right\|_K = 1 \ .$$

According to Bernstein-Walsh inequality (see [8]),

$$\left|p_{m_k}(z)\right| \leq e^{m_k V^*(z,K)} \qquad \text{and} \qquad \left|q_{m_k}(z)\right| \leq e^{m_k V^*(z,K)}$$

for any $z \in \mathbb{C}^n$ and k $\in \mathbf{N}$. Put

$$V^*(z,\, K) = \varlimsup_{z'\to z} V(z, K),$$

where

$$V(z,\, K) = \sup\{\frac{1}{m} \ln|p_m(z')| : \|p_m\|_K \leq 1\}$$

is the extremal function of Green. We introduce the following auxiliary sequence of plurisubharmonic functions

$$u_k(z,w) = \frac{1}{m_k} \ln \left| q_{m_k}(z) \cdot w - p_{m_k}(z) \right|, \ (z,w) \in \mathbb{C}^{n+1}.$$

For $(z,w) \in \mathbb{C}^{n+1}$ we have

$$\begin{aligned} \frac{1}{m_k} \ln |q_{m_k}(z) \cdot w - p_{m_k}(z)| &\le \frac{1}{m_k} \ln(|q_{m_k}(z) \cdot w| + |p_{m_k}(z)|) \\ &\le \max\{\frac{1}{m_k} \ln 2|p_{m_k}(z)|, \ \frac{1}{m_k} \ln 2|q_{m_k}(z) \cdot w|\} \\ &= \max\{\frac{1}{m_k} \ln |p_{m_k}(z)|, \ \frac{1}{m_k} \ln |q_{m_k}(z)| + \frac{1}{m_k} \ln |w|\} + \frac{\ln 2}{m_k}. \end{aligned}$$

From here we obtain the following estimate

$$u_k(z,w) \le \max\{V^*(z, \ K), \ \ V^*(z, \ K) + \frac{1}{m_k} \ln |w|\} \ + \frac{\ln 2}{m_k}.$$

Consequently, the sequence of plurisubharmonic functions $u_k(z,w)$ is locally uniformly bounded from above.

Let

$$u(z,w) = \overline{\lim_{k \to \infty}} u_k(z,w).$$

The function $u(z,w)$ is also locally bounded from above, i.e.,

$$u(z,w) \le V^*(z,K).$$

We denote by

$$u^*(z,w) = \overline{\lim_{(z',w') \to (z,w)}} u(z',w')$$

the regularization of function $u(z,w)$. The set

$$E = \{(z,w) \in \mathbb{C}^{n+1} : \ u(z,w) < u^*(z,w)\}$$

is pluripolar in $\mathbb{C}^{n+1}$(see [7, 8]).

Let now $(z,w) \in \Gamma_f$ be a fixed point. (Note that $q_m(z) \neq 0$ for $z \in K$). Then

$$\begin{aligned} u(z,w) = \overline{\lim_{k \to \infty}} \ln |q_{m_k}(z)|^{\frac{1}{m_k}} \left| w - \tfrac{p_{m_k}(z)}{q_{m_k}(z)} \right|^{\frac{1}{m_k}} &\le \overline{\lim_{k \to \infty}} \ln \alpha |q_{m_k}(z)|^{\frac{1}{m_k}} \\ &= \ln \alpha + \overline{\lim_{k \to \infty}} |q_{m_k}(z)|^{\frac{1}{m_k}}. \end{aligned}$$

If $(z,w) \in (K \times \mathbb{C}) \backslash \Gamma_f$, then

$$u(z,w) = \overline{\lim_{k\to\infty}} \frac{1}{m_k} \ln |q_{m_k}(z)w - p_{m_k}(z)| = \overline{\lim_{k\to\infty}} \ln |q_{m_k}(z)|^{\frac{1}{m_k}} \left| w - \frac{p_{m_k}(z)}{q_{m_k}(z)} \right|^{\frac{1}{m_k}}$$
$$= \overline{\lim_{k\to\infty}} \ln |q_{m_k}(z)|^{\frac{1}{m_k}}.$$

It follows that

$$\overline{\lim_{k\to\infty}} |q_{m_k}(z)|^{\frac{1}{m_k}} \neq 0$$

at a point $z \in K$, then $(z, f(z))$ belongs to the pluripolar set E. Therefore, to complete the proof of the theorem it is enough to show that the set

$$A = \left\{ z \in K : \overline{\lim_{k\to\infty}} |q_{m_k}(z)|^{\frac{1}{m_k}} = 0 \right\} = \left\{ z \in K : \lim_{k\to\infty} |q_{m_k}(z)|^{\frac{1}{m_k}} = 0 \right\}$$

is pluripolar.

Assume that A is not pluripolar, i.e., plurisubharmonic capacity $cap(A) > 0$. We consider the following sequence of subharmonic functions

$$\vartheta_k^*(z) = \overline{\lim_{z'\to z}} \vartheta_k(z'), \ \ z \in \mathbb{C}^n,$$

where

$$\vartheta_k(z) = \sup_{s\geq k} |q_{m_s}(z)|^{1/m_s}.$$

It is clear that the sequence $\vartheta_k^*(z)$ is locally uniformly bounded, $0 \leq \vartheta_k^*(z) \leq e^{V^*(z,K)}$, and is not monotonically increasing. In addition, $\vartheta_k^*(z) \to 0$ on A except for the pluripolar set

$$F = \bigcup_{k=1}^{\infty} \{ z \in \mathbb{C}^n : \vartheta_k(z) < \vartheta_k^*(z) \}$$

since by definition the sequence $\vartheta_k(z)$ tends to zero on A.

Since $\vartheta_k^*(z)$ is monotonic, for every $\varepsilon : \ 0 < \varepsilon < cap(A)$ there exists an open set U_ε, $cap(U_\varepsilon) < \varepsilon$, such that the sequence $\vartheta_k^*(z)$ converges uniformly on the set $A_\varepsilon = A \backslash U_\varepsilon$ (see, for example, [7]). It follows that there exists a compact set $A_0 \subset A_\varepsilon$, $cap(A_0) > 0$ such that the sequence of subharmonic functions $\vartheta_k^*(z)$ converges uniformly to zero on the set A_0. Consequently, the sequence $|q_{m_k}(z)|^{1/m_k}$ also converges uniformly to zero on the compact set A_0.

Now, we use the so-called τ- capacity, introduced by A. Sadullaev (see [7]). Let K be compact set from some polydisk $U \subset \mathbb{C}^n$. We consider a polynomial $T_m(z)$, $\deg T_m(z) \leq m$, normalized by the condition $\|T_m\|_U = 1$, for which the norm $\|T_m\|_K$ is minimal among all such polynomials. Then the limit

$$\lim_{m\to\infty} \|T_m\|_K^{\frac{1}{m}} = \tau(K,U) \tag{2.2}$$

exists and, in fact,

$$\|T_m\|_K^{\frac{1}{m}} \geq \tau(K,U) \tag{2.3}$$

for every $m \in \mathbf{N}$. Moreover,

$$\tau(K,U) = \exp\{-\sup_{z\in U} V^*(z,K)\}. \tag{2.4}$$

It follows from (2.2), (2.3) and (2.4) that

$$\|q_{m_k}\|_{A_0}^{\frac{1}{m_k}} \geq \|T_{m_k}\|_K^{\frac{1}{m_k}} \geq \tau(K,U) > 0, \tag{2.5}$$

where $U \supset\supset K$, as $\|q_{m_k}\|_K = 1$ and $\|q_{m_k}\|_U \geq 1$. It follows from inequality (4) that the sequence $|q_{m_k}(z)|^{\frac{1}{m_k}}$ does not converges uniformly to zero on the set A_0. We arrived at a contradiction. Hence the set A is pluripolar, completing the proof. ■

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