From Euclidean to Metric Spaces: Regularity of p-Harmonic Functions

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Abstract. This is a survey article about the regularity of solutions to the p-Laplace equation on Euclidean spaces. Such functions can be characterized as minimizers to certain non-linear energy functionals. The methods presented here, originally due to DeGiorgi, show that Harnack's inequality and Hölder continuity follow solely from this minimization property.

These methods also extend to a large class of metric spaces, specifically those supporting doubling measures and Poincaré inequalities. In this setting we discuss the basic theory and survey recent results.

Keywords. p-harmonic function, Harnack inequality, analysis on metric spaces.

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Contents

1. Introduction

This is an introduction to p-harmonic functions and their regularity theory. For smooth domains in $\mathbb{R}^n$ with $p > 1$, such functions correspond to solutions to a certain non-linear partial differential equation, called the p-Laplace equation:

$$(1.1) \qquad 0 \;=\; \Delta_p u \;:=\; \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \;=\; \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\right].$$

For $p = 2$, these are the usual harmonic functions.

Surprisingly there are well-defined analogues of these functions in the setting of *metric measure spaces*: that is, metric spaces (X, d) equipped with Borel regular measures μ. The generality of our perspective is motivated by its connections to analysis, partial differential equations, geometry (both smooth and non-smooth), and even subjects outside of mathematics.

As a first example, Coifman and Weiss [9], [10] discovered that many facts from harmonic analysis can be generalized from Euclidean spaces to metric spaces supporting doubling measures (or *spaces of homogeneous type*). This includes good analogues of Riesz potentials, the Lebesgue differentiation theorem, as well as the theory of Hardy spaces H^p and the well-known duality between H^1 and BMO, the class of functions of *bounded mean oscillation* [19].

Later, Grigor'yan [23] and Saloff-Coste [43] have separately shown that several hypotheses of a metric space nature — namely, the *volume doubling condition* and a *Poincaré inequality* — are crucial in developing tools for partial differential equations on Riemannian manifolds. Colding and Minicozzi [11] later used these tools in their solution of Yau's conjecture [47], a generalized Liouville theorem for harmonic functions on Riemannian manifolds with non-negative Ricci curvature.

Using similar ideas for p-harmonic functions, Cheeger [6] proved the existence of generalized differentiable structures for metric spaces satisfying analogues of the above hypotheses. In later work by Cheeger and Kleiner [7] and by Lee and Naor [35], such structures have led to counter-examples to the Goemans-Linial conjecture in theoretical computer science!

Regularity in our context refers only to "zeroth-order" behavior, such as continuity and growth properties of p-harmonic functions. Specifically, our approach

avoids the explicit geometry of Euclidean spaces, as well as special properties of functions defined on them. This includes the usual notion of *weak* or *distributional* derivative, as discussed in [18] or in [48].

One may ask further questions of $C^{1,\alpha}$-regularity (i.e. the modulus of continuity for derivatives) and the case for Euclidean spaces is well-established; see [36], [14] for details. As of now, however, there is no robust theory of higher-order derivatives for general metric measure spaces, except in certain special cases, such as *sub-Riemannian manifolds*. For more about the geometry of such spaces, see [1] and [24] and the corresponding regularity theory can be found in [4], [5], [17], and [39].

Remark 1.1. As another direction of interest, one could generalize the equations instead. For example, we may consider second-order partial differential equations similar to Equation (1.1), but where the terms $|\nabla u|^{p-2}\nabla u$ are replaced by more general vector fields $\mathbf{A}(x, u(x), \nabla u(x))$, where $\mathbf{A} : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies certain *ellipticity* conditions. We could also consider equations with additional *non-homogeneous* terms:

$$(1.2) \qquad \text{div}[\mathbf{A}(x, u(x), \nabla u(x))] + B(x, u(x), \nabla u(x)) \ = \ 0.$$

Here $B : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ also obeys certain *structure* conditions, which we will not specify here. For references, see the classical work of Ladyžhenskaya and Uralt'seva [34] as well as the book of Giusti [21].

The article is outlined as follows. We begin with the case of $\mathbb{R}^n$ and the remainder of Section §1 is devoted to identifying *p*-harmonic functions as minimizers to certain energy functionals. We then introduce Harnack's inequality, the main result of this survey, and discuss its immediate consequences for regularity.

Section §2 is about the celebrated De Giorgi approach to regularity. Our intent is to discuss the main ideas of the theory and the heuristics behind technical arguments of proof. Occasionally we will make simplifying assumptions or treat specific elementary cases in order to clarify the nature of these ideas. The most general cases will not be treated here. As the discussion progresses, we instead direct the reader to more substantial references in the literature.

Lastly, in Section §3 we formulate *p*-harmonic functions in greater generality, and indicate how the tools of the previous section generalize to a large class of metric measure spaces. We then formulate the corresponding regularity results, as discussed in [32] and indicate a few open problems in this setting.

Notation and Conventions. For a set S and a function $u : S \to \mathbb{R}$, its truncations are denoted by $u_+ := \max(u, 0)$ and $u_- := -\min(u, 0)$.

On a metric space (X, d) equipped with a Borel measure, let $L^p(X, \mu)$ denote the class of *p*-integrable functions on X with respect to μ. For a ball $B \subset X$ the

mean value of a (locally) integrable function over B is

$$u_B := \fint_B u \, d\mu := \frac{1}{\mu(B)} \int_B u \, d\mu.$$

If $X = \Omega$ is a domain in $\mathbb{R}^n$, then we write $L^p(\Omega) = L^p(\Omega, dx)$, where dx is the Lebesgue measure, and we write $|A|$ for the Lebesgue measure of a measurable set $A \subset \mathbb{R}^n$. For a domain Ω in $\mathbb{R}^n$, we denote by $C_c^\infty(\Omega)$ the space of smooth, real-valued functions with compact support in Ω.

As a convention in this paper, the symbol C denotes a non-negative constant that depends only on given parameters. Its exact value may change, even from line to line within a single estimate.

We will always assume $1 < p < n$, where n is the dimension of the Euclidean space. Because the issues discussed here are local in nature, Hölder's inequality gives $W^{1,q}(B) \subset W^{1,p}(B)$ for higher exponents $q \geq n > p$ and for balls $B \subset \mathbb{R}^n$, so there is no loss of generality.

1.1. From p-harmonic functions to energy minimizers.

For partial differential equations and variational problems, it is crucial to work in a suitable class of functions with good properties, such as compactness, from which one can prove the existence and uniqueness of solutions. This leads us to study the Sobolev spaces $W^{1,p}(\Omega)$, where Ω is a fixed domain in $\mathbb{R}^n$.

We begin by defining a norm on the class of smooth, Lebesgue p-integrable functions $f : \Omega \to \mathbb{R}$ whose gradients are also p-integrable:

$$\|f\|_{1,p} := \left\{ \int_\Omega |f|^p \, dx \right\}^{\frac{1}{p}} + \left\{ \int_\Omega |\nabla f|^p \, dx \right\}^{\frac{1}{p}}$$

We define the Sobolev space $W^{1,p}(\Omega)$ as the completion of this class of functions under this norm: that is,

$$(1.3) \qquad W^{1,p}(\Omega) := \overline{\left(\left\{ f \in C^\infty(\Omega) : \|f\|_{1,p} < \infty \right\}, \ \| \cdot \|_{1,p} \right)}.$$

Moreover, $W_0^{1,p}(\Omega)$ is defined as the norm-closure of $C_0^\infty(\Omega)$, the class of compactly supported smooth functions on Ω.

Looking ahead, an arbitrary metric space has little a priori structure, so the notion of a derivative may not be well-defined. We will later consider, however, an analogue of the *norm of the gradient* which gives rise to energy functionals.

Motivated by this, we now show that p-harmonic functions, for $p > 1$, are equivalently local minimizers of the p-energy integral

$$(1.4) \qquad u \mapsto \int_{\Omega'} |\nabla u|^p \, dx.$$

where the integral is tested over all domains Ω' compactly contained in Ω. As a related notion we also consider quasi-minimizers, as first studied by Giaquinta and Giusti [20].

Definition 1.2. Let $u \in W^{1,p}(\Omega)$. We call u a *(p-)energy minimizer* if

$$\int_{\Omega' \cap \{u \neq v\}} |\nabla u|^p \, dx \ \leq \ \int_{\Omega' \cap \{u \neq v\}} |\nabla v|^p \, dx$$

holds for all bounded sub-domains $\Omega' \subset \Omega$ and all functions $v \in W^{1,p}(\Omega')$ with $u - v \in W_0^{1,p}(\Omega)$. If there exists $K \geq 1$ so that

$$\int_{\Omega \cap \{u \neq v\}} |\nabla u|^p \, dx \ \leq \ K\left\{ \int_{\Omega \cap \{u \neq v\}} |\nabla v|^p \, dx \right\}$$

holds for all Ω' and v as before, then we call u a *(K-)quasi-minimizer*.

A word of caution is in order: solutions to (1.1) are a priori more regular than energy minimizers, because the *p*-Laplace equation requires second derivatives. To interpolate between the two notions, we recall the integration by parts formula: for a domain $\Omega \subset \mathbb{R}^n$, every smooth function $u : \Omega \to \mathbb{R}$ satisfies

$$(1.5) \qquad \int_\Omega \frac{\partial u}{\partial x_i} \varphi \, dx \ = \ -\int_\Omega u \frac{\partial \varphi}{\partial x_i} \, dx$$

for all $\varphi \in C_0^\infty(\Omega)$. In fact, one can define the classes $W^{1,p}(\Omega)$ as

$$W^{1,p}(\Omega) \ := \ \left\{ f \in L^p(\Omega) \, : \, \exists \, \{\partial_i f\}_{i=1}^n \subset L^p(\Omega), \forall \, \varphi \in C_c^\infty(\Omega), \text{ Eq. (1.5) holds} \right\}$$

and it is well-known that the notions are equivalent [38].

Using the integration by parts formalism, we now define *weak solutions* of (1.1) as those functions $u \in W^{1,p}(\Omega)$ so that the identity

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx \ = \ 0$$

holds for all $\varphi \in C_c^\infty(\Omega)$. In the case when u is C^2-smooth, this agrees with the usual (pointwise) notion of solution. For details, see [18] and [48].

We now turn to the correspondence between *p*-harmonic functions and *p*-energy minimizers.

Theorem 1.3. *Let Ω be a domain in $\mathbb{R}^n$. For every $u \in W^{1,p}(\Omega)$, the following conditions are equivalent:*

1. *u is p-harmonic: i.e. u is a weak solution of Equation (1.1);*
2. *u is a p-energy minimizer in the sense of Definition 1.2.*

So by studying p-energy minimizers on metric spaces, our conclusions will also hold in more regular settings, such as smooth manifolds, where the p-Laplace equation (1.1) is well-defined.

Proof. (1) $\Rightarrow$ (2). Let Ω' be any bounded domain in Ω and let $v \in W^{1,p}(\Omega')$ satisfy $u - v \in W_0^{1,p}(\Omega')$. By approximation, assume that $u - v$ is smooth and compactly supported in Ω, so we may extend it to a smooth, compactly supported function on $\mathbb{R}^n$. For simplicity, we denote the extension by $u - v$ as well.

Since u is a solution of (1.1), the integration by parts formula gives

$$0 \;=\; -\int_{\Omega'} |\nabla u|^{p-2} \nabla u \cdot \nabla(u - v)\, dx$$

$$\int_{\Omega'} |\nabla u|^p\, dx \;=\; \int_{\Omega'} |\nabla u|^{p-2} (\nabla u \cdot \nabla u)\, dx$$

from which it follows, by Hölder's inequality with $|\nabla u|^{p-1} \in L^{\frac{p}{p-1}}(\Omega)$ and by Young's inequality $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, that

$$\int_{\Omega'} |\nabla u|^p\, dx \;=\; \int_{\Omega'} |\nabla u|^{p-2} (\nabla u \cdot \nabla v)\, dx \;\leq\; \int_{\Omega'} |\nabla u|^{p-1} |\nabla v|\, dx$$

$$\leq\; \left(\int_{\Omega'} |\nabla u|^p\, dx \right)^{\frac{p-1}{p}} \left(\int_{\Omega'} |\nabla v|^p\, dx \right)^{\frac{1}{p}}$$

$$\leq\; \left(\frac{p-1}{p} \right) \int_{\Omega'} |\nabla u|^p\, dx \;+\; \frac{1}{p} \int_{\Omega'} |\nabla v|^p\, dx,$$

$$\text{so } \int_{\Omega'} |\nabla u|^p\, dx \;\leq\; \int_{\Omega'} |\nabla v|^p\, dx.$$

(2) $\Rightarrow$ (1). For a bounded sub-domain $\Omega' \subset \Omega$, let $\varphi \in C_c^\infty(\Omega')$ and define a function $E : \mathbb{R} \to \mathbb{R}$ by the formula

$$E(t) \;:=\; \int_\Omega |\nabla u + t \nabla \varphi|^p\, dx.$$

For $v := u - \varphi$ we have $u - v \in W_0^{1,p}(\Omega')$. It is easy to check that E is C^1-smooth, and it follows from (2) that E has a minimum at $t = 0$. By "differentiating under the integral sign" and integrating by parts, we obtain

$$0 \;=\; \frac{d}{dt} E(t) \Big|_{t=0} \;=\; \int_\Omega \frac{d}{dt} |\nabla u + t \nabla \varphi|^p \Big|_{t=0}\, dx \;=\; \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi\, dx,$$

so u is a weak solution of Equation (1.1). $\blacksquare$

Remark 1.4. For equations of the form (1.2) where **A** and B satisfy certain structure and growth conditions similar to (1.1), it is a general fact that solutions are quasi-minimizers of *modified* energy functionals

$$u \mapsto \mathcal{F}(x, u, \nabla u; \Omega) := \int_{\Omega} F\big(x, u(x), \nabla u(x)\big) \, dx$$

where F depends on the previous data **A** and B. For details, see [21].

1.2. Harnack's inequality: a first look. For the linear case ($p = 2$) it is well-known that harmonic functions are characterized by the mean value property, as stated below. Here σ denotes the surface area measure on a $(n-1)$-dimensional submanifold in $\mathbb{R}^n$.

Theorem 1.5. *For $u \in W^{1,2}(\Omega)$, we have $\Delta u = 0$ if and only if*

$$(1.6) \qquad u(x_0) = \fint_{\partial B} u(\omega) \, d\sigma(\omega) = \fint_B u(x) \, dx.$$

holds for all balls $B = B(x_0, r_0) \subset \Omega$.

Idea of the Proof. For *any C^2-smooth* function $u : \Omega \to \mathbb{R}$, the function

$$f(r) := \fint_{\partial B(x_0, r)} u(\omega) \, d\sigma(\omega) = \fint_{\partial B(0,1)} u(x_0 + r\theta) \, d\sigma(\theta)$$

is also C^2-smooth on $[0, r_0)$. "Differentiating under the integral sign" gives

$$\begin{aligned} f'(r) &= \fint_{\partial B(0,1)} Du(x_0 + r\theta) \cdot \theta \, d\sigma(\theta) \\ &= \fint_{\partial B(x_0, r)} Du(\omega) \cdot \frac{\omega - x_0}{r} \, d\sigma(\omega) = \fint_{\partial B} \frac{\partial u}{\partial \nu}(\omega) \, d\sigma(\omega) \end{aligned}$$

where $\frac{\partial u}{\partial \nu}$ is the derivative of u in the direction of the (outward) unit normal ν. Applying Green's theorem, we see that

$$(1.7) \quad f'(r) = \fint_{\partial B(x_0, r)} \frac{\partial u}{\partial \nu}(\omega) \, d\sigma(\omega) = \frac{1}{\text{Area}(\partial B(x_0, r))} \int_{B(x_0, r)} \Delta u(x) \, dx.$$

So if u is harmonic, then $f'(r)|_{(0, r_0)} = 0$, so f is constant on $[0, r_0]$ with

$$\fint_{\partial B(x_0, r_0)} u(\omega) \, d\sigma(\omega) = f(r_0) = \lim_{r \searrow 0} f(r) = u(x_0),$$

which is the mean value property.

On the contrary, suppose that $\Delta u(x_0) \neq 0$ for some $x_0 \in \Omega$; without loss, $\Delta u(x_0) > 0$. This means there is a $r > 0$ so that $B(x_0, 2r) \subset \Omega$ and $\Delta u|_{B(x_0, r)} > 0$, so the previous function f cannot be constant on $[0, 2r]$, and therefore the mean value property fails for u.

Now let $u \in W^{1,2}(\Omega)$. For each $\epsilon > 0$, let $\eta_\epsilon \in C_c^\infty(\Omega)$ be supported on a ball $B(x_0, \epsilon)$ and so that $x \mapsto \eta_\epsilon(x - x_0)$ is radially-symmetric on $B(0, \epsilon)$. We claim that if u is a weak solution to (1.1), then $\Delta[u * \eta_\epsilon] = 0$, for all $\epsilon > 0$. Indeed, a change of variables argument shows that

$$\int_\Omega \varphi \Delta[u * \eta_\epsilon] \, dx \;=\; -\int_\Omega [(\nabla u) * \eta_\epsilon] \cdot \nabla \varphi \, dx \;=\; -\int_\Omega \nabla u \cdot \nabla[\varphi * \eta_\epsilon] \, dx$$

for all $\varphi \in C_c^\infty(\Omega)$. Similarly, if u has the mean value property, then so does $u * \eta_\epsilon$ for all $\epsilon > 0$, so the argument reduces to the previous case of C^2-smoothness. ∎

Returning to the general case of $1 < p < n$, we now discuss the main result of this survey.

Theorem 1.6 (Harnack's inequality). *Let Ω be a domain in $\mathbb{R}^n$. There exists a constant $C = C(n, p, \Omega) \geq 1$ so that for every positive-valued p-energy minimizer $u \in W^{1,p}(\Omega)$ and for all balls $B(x, 2R) \subset \Omega$, we have*

$$\sup_{B(x,R)} u \;\leq\; C\Big\{ \inf_{B(x,R)} u \Big\}.$$

The proof for Theorem 1.6 is split into two steps.

Theorem 1.7 (Local boundedness). *There exists $C = C(p, n, \Omega) > 0$ so that*

$$(1.8) \qquad \sup_{B(x_0, \frac{1}{2}R)} u \;\leq\; C\Big\{ \fint_{B(x_0, R)} |u|^p \, dx \Big\}^{\frac{1}{p}}$$

holds for all p-energy minimizers $u : \Omega \to \mathbb{R}$ and all balls $B(x_0, R) \subset \Omega$.

Theorem 1.8 (Weak Harnack). *There exists $C' = C'(p, n, \Omega) \geq 0$ so that*

$$(1.9) \qquad C'\Big\{ \fint_{B(x_0, R)} u^p \, dx \Big\}^{\frac{1}{p}} \;\leq\; \inf_{B(x_0, 2R)} u$$

holds, for all balls $B(x_0, 2R) \subset \Omega$ and all p-energy minimizers $u : \Omega \to \mathbb{R}$ that are positive on $B(x_0, 2R)$.

One can therefore interpret Harnack's inequality as a *generalized* mean value property: if u is a positive p-energy minimizer, then its values at $x_0 \in \Omega$ are comparable to its p-integral averages over balls centered at x_0:

$$u(x_0) \approx \Big\{ \fint_{B(x_0, R)} u^p \, dx \Big\}^{\frac{1}{p}}.$$

(Here $\approx$ refers to the previous uniform constants that are independent of x_0.)

Remark 1.9. To clarify, we interpret Theorems 1.7 and 1.8 to mean that *p*-energy minimizers have a.e. representatives that satisfy the above inequalities.

Indeed, in regularity theory we begin with a minimizer of the energy functional (1.4), which is known, a priori, to be *p*-integrable and have locally finite *p*-energy. The point is to show that it has good pointwise a.e. properties.

Related to this, we note that Harnack's inequality implies additional fine properties for energy minimizers. The first is the strong maximum principle for *p*-harmonic functions. We postpone the proof to §2.3.

Corollary 1.10 (Strong Maximum Principle). *Let Ω be a domain in $\mathbb{R}^n$, let $p > 1$, and let $u : \Omega \to \mathbb{R}$ be a p-energy minimizer. If u attains its maximum or minimum in Ω, then u is constant on Ω.*

The next consequence was first observed by Moser [40]. To fix terminology, we denote the *oscillation* of a function $u : \Omega \to \mathbb{R}$ on a subset $A \subset \Omega$ as

$$\operatorname*{osc}_{A} u := \sup_{A} u - \inf_{A} u.$$

Corollary 1.11 (Hölder continuity). *Let Ω be a domain in $\mathbb{R}^n$. There exist constants $C = C(n, p, \Omega) \geq 1$ and $\alpha = \alpha(n, p) > 0$ so that for every p-energy minimizer $u \in W^{1,p}(\Omega)$ and for all balls $B(x, r) \subset B(x, R) \subset \Omega$, we have*

$$\operatorname*{osc}_{B(x,r)} u \leq C \Big\{ \operatorname*{osc}_{B(x,R)} u \Big\} \Big(\frac{r}{R} \Big)^{\alpha}$$

In particular, every p-energy minimizer has an a.e. representative that is locally Hölder continuous, with exponent α.

Proof of Corollary 1.11. Assuming Theorem 1.7 through Corollary 1.10 for now, let $u \in W^{1,p}(\Omega)$ be a *p*-energy minimizer and let $B(x, 2r) \subset \Omega$. Putting

$$M := \sup_{B(x,2r)} u \quad \text{and} \quad m := \inf_{B(x,2r)} u,$$

note that $M - u$ and $u - m$ are non-negative *p*-energy minimizers. Since their infima are zero, Corollary 1.10 implies that both functions are strictly positive on Ω, so Harnack's inequality applies. Clearly we have

$$M - \inf_{B(x,r)} u = \sup_{B(x,r)} (M - u) \leq C \inf_{B(x,r)} (M - u) = C \Big\{ M - \sup_{B(x,r)} u \Big\}$$

$$\Big\{ \sup_{B(x,r)} u \Big\} - m = \sup_{B(x,r)} (u - m) \leq C \inf_{B(x,r)} (u - m) = C \Big(\Big\{ \inf_{B(x,r)} u \Big\} - m \Big)$$

and adding the inequalities gives

$$\operatorname*{osc}_{B(x,r)} u + \operatorname*{osc}_{B(x,2r)} u \;\leq\; C\Big\{ \operatorname*{osc}_{B(x,2r)} u - \operatorname*{osc}_{B(x,r)} u \Big\}$$

$$\text{so} \quad \operatorname*{osc}_{B(x,r)} u \;\leq\; \frac{C-1}{C+1} \operatorname*{osc}_{B(x,2r)} u$$

For arbitrary $R > r$, interpolating by powers of two — that is, $2^k r < R \leq 2^{k+1} r$ for some $k \in \mathbb{Z}$ — and iterating the previous inequality gives the result. ∎

With these motivations in mind, we now turn to the proof of Harnack's inequality.

2. De Giorgi's approach to regularity

At the heart of the De Giorgi method [13] are ideas from geometric measure theory. As we will see, good pointwise properties of energy minimizers $u : \Omega \to \mathbb{R}$ will follow from estimating the density of their super-level sets $\{x : u(x) > \lambda\}$.

Similarly to Moser's approach [40], the basic tools here consist of iteration arguments that arise from Sobolev-type inequalities. On one hand, the Sobolev embedding theorem indicates that the growth of a function is controlled by its gradient, whereas energy minimizers actually satisfy a partial converse: when truncated, such functions control the growth of their gradients!

Combining these facts, this provides an initial "recurrence relation" that allows us to compare densities at different levels of the function and at different scales of space. This provides the starting point for the aforementioned iteration arguments. As a consequence, we will prove the local boundedness and the weak Harnack inequality for energy minimizers.

2.1. Preliminaries: Sobolev spaces on $\mathbb{R}^n$. Here we emphasize properties of $W^{1,p}(\Omega)$ that avoid the linear structure of Euclidean spaces, as well as the theory of distributions. Indeed, the integration by parts formula (as well as distributions) relies crucially on the Fundamental Theorem of Calculus, which is a property that is lacking in the metric space setting.

The first fact to discuss is the Sobolev embedding theorem; see [18] or [48] as references. To fix notation, we write the Sobolev conjugate exponent of $p \geq 1$ as

$$p^* \;=\; \frac{np}{n-p}.$$

Theorem 2.1 (Sobolev embedding). *Let $p \in [1, n)$. There exists $C = C(n, p, \Omega) \geq 0$ so that for all $f \in W^{1,p}(\Omega)$ with compact support in Ω, we have*

$$\left\{ \int_{\Omega} |f|^{p^*}\, dx \right\}^{1/p^*} \leq C \left\{ \int_{\Omega} |\nabla f|^p\, dx \right\}^{1/p}.$$

In short, the L^p-norm of a Sobolev function is controlled by the L^p-norm of its gradient. In the case of balls $\Omega = B(x_0, R)$, taking averages gives

$$(2.1) \qquad \left\{ \fint_{B(x_0,R)} |f|^{p^*}\, dx \right\}^{1/p^*} \leq CR \left\{ \fint_{B(x_0,R)} |\nabla f|^p\, dx \right\}^{1/p}.$$

Remark 2.2. Note that the function $x \mapsto |\nabla f(x)|$ does not require linearity from the gradient. In §3 we discuss "upper gradients," which are generalizations of such functions and which lead to analogues of the Sobolev spaces $W^{1,p}(\Omega)$.

We now recall three more facts about Sobolev functions. The first is a characterization of Sobolev functions in terms of a measurable modulus of continuity. For a proof, we refer the reader to [25, Thm 2.2].

Lemma 2.3 (Hajłasz). *Let Ω be a smooth domain in $\mathbb{R}^n$ and let $1 < p < \infty$. For each function $u \in L^p(\Omega)$, the following conditions are equivalent:*

1. *$u \in W^{1,p}(\Omega)$;*
2. *there exists $g \in L^p(\Omega)$ with $g \geq 0$ and so that for a.e. $x, y \in \Omega$, we have*
$$|u(y) - u(x)| \leq |x - y|\,\big(g(y) + g(x)\big)$$

Moreover, there exists $C = C(n, p; \Omega) \geq 1$ so that

$$\frac{1}{C} \inf_g \|g\|_{L^p(\Omega)} \leq \|\nabla u\|_{L^p(\Omega)} \leq C \inf_g \|g\|_{L^p(\Omega)}$$

where the infima are taken over all functions $g \in L^p(\Omega)$ that satisfy Condition (2) with respect to u.

The next two facts are used in the proof of Harnack's inequality. The next fact is a modified Sobolev embedding theorem: instead of compact support, we require that a Sobolev function satisfy an appropriate *density* condition on balls.

Lemma 2.4. *Let Ω be a domain in $\mathbb{R}^n$ and let $p \in [1, n)$. If $u \in W^{1,p}(\Omega)$ satisfies*

$$|\{|u| > 0\} \cap B| \leq \gamma|B|$$

for some $\gamma \in (0, 1)$ and $B(x_0, R) \subset \Omega$, then there exists $C' = C'(n, p, \gamma) > 0$ so that for each $q \in (1, p]$ and each $t \in (1, q^*]$, we have

$$\left(\fint_{B(x_0,R)} |u|^t\, dx \right)^{\frac{1}{t}} \leq C'R \left(\fint_{B(x_0,R)} |\nabla u|^q\, dx \right)^{\frac{1}{q}}.$$

This lemma is proved in [32], where it is used as a substitute for De Giorgi's "discrete isoperimetric inequality" [13].

The last fact is a "local clustering" phenomenon for positive Sobolev functions as observed by DiBenedetto, Gianazza, and Vespri [16]. Roughly speaking, once a super-level set has sufficiently large density on a fixed ball, then all other density values are achieved by super-level sets on smaller balls, with lower levels.

To fix notation, we write $Q(x_0, R)$ for the cube centered at a point x_0, with edge length $2R$, and with sides parallel to the coordinate planes.

Lemma 2.5 (Local Clustering). *If $u \in W^{1,p}\big(Q(x_0, R)\big)$ satisfies*

$$\text{(2.2)} \qquad |\{u > 1\} \cap Q(x_0, R)| \; \geq \; \theta_0 |Q(x_0, R)|$$

$$\text{(2.3)} \qquad \int_{Q(x_0, R)} |\nabla u|^p \, dx \; \leq \; \gamma R^{n-p}$$

for some constants $\gamma > 0$ and $\theta_0 \in (0, 1)$, then for each $\lambda, \theta \in (0, 1)$, there exist $x \in Q(x_0, R)$ and $\epsilon = \epsilon(\theta_0, \theta, \gamma, n) > 0$ so that

$$\text{(2.4)} \qquad |\{u > \lambda\} \cap Q(x_0, \epsilon R)| \; \geq \; \theta |Q(x_0, \epsilon R)|.$$

Remark 2.6. (A) Lemma 2.5 also holds for balls. Clearly we have

$$Q\big(x_0, n^{-1/2}R\big) \; \subset \; B(x_0, R) \; \subset \; Q(x_0, R),$$

so the following density condition for balls

$$\theta_0 |B(x_0, R)| \; \leq \; |\{u > 1\} \cap B(x_0, R)|$$

implies hypothesis (2.2), with $n^{-n/2}\theta_0$ in place of θ_0. Lemma 2.5 then implies

$$|\{u > \lambda\} \cap B(x_0, \epsilon R)| \; \geq \; n^{-n/2}\theta |B(x_0, \epsilon R)|.$$

(B) In the lemma above, the energy estimate (2.3) ensures that the constant ϵ depends only on the parameters θ_0, θ, n, and p. If instead we allowed ϵ to depend on the function u, then (2.3) is unnecessary and the conclusion (2.4) would follow from the Lebesgue differentiation theorem.

For expository reasons we give an alternative proof to the ones found in [16] and [15]. Specifically, we use the previous characterization (Lemma 2.3) in place of polar coordinate integration over balls.

We proceed in two steps: one first divides $Q(x_0, R)$ into subsets and shows that the number of these subsets with initial density θ_0 is sufficiently large. One next shows, on such subsets, that (2.3) implies (2.4): if the density of a new super-level set is small, then the gradient of the function must be large.

Proof. *Step 1: Sub-cubes with good density.* For each $N \in \mathbb{N}$ we partition $Q = Q(x_0, R)$ into sub-cubes of edge length $\frac{R}{N}$ and with pairwise-disjoint interiors. We now sort the sub-cubes into two disjoint collections:

$$Q_j \in \mathcal{Q}_{\text{good}} \iff |\{u > 1\} \cap Q_j| > \frac{\theta_0}{2}|Q_j|$$

$$Q_j \in \mathcal{Q}_{\text{bad}} \iff |\{u > 1\} \cap Q_j| \leq \frac{\theta_0}{2}|Q_j|.$$

so $N^n = \#(\mathcal{Q}_{\text{good}}) + \#(\mathcal{Q}_{\text{bad}})$. It follows that

$$\left\{ \sum_{Q_j \in \mathcal{Q}_{\text{good}}} \frac{|\{u > 1\} \cap Q_j|}{|Q_j|} + \sum_{Q_i \in \mathcal{Q}_{\text{bad}}} \frac{|\{u > 1\} \cap Q_i|}{|Q_i|} \right\}$$

$$< (\#\mathcal{Q}_{\text{good}}) + \frac{\theta_0}{2}(\#\mathcal{Q}_{\text{bad}})$$

$$\leq (\#\mathcal{Q}_{\text{good}}) + \frac{\theta_0}{2}\big(m^n - \#\mathcal{Q}_{\text{good}}\big) = \frac{2 - \theta_0}{2}(\#\mathcal{Q}_{\text{good}}) + \frac{\theta_0 N^n}{2}$$

On the other hand, each Q_j has measure $|Q_j| = N^{-n}|Q|$, so the density hypothesis (2.2) and sub-additivity gives

$$\sum_{Q_j \in \mathcal{Q}_{\text{good}}} \frac{|\{u > 1\} \cap Q_j|}{|Q_j|} + \sum_{Q_i \in \mathcal{Q}_{\text{bad}}} \frac{|\{u > 1\} \cap Q_i|}{|Q_i|} \geq \theta_0 \frac{|Q|}{N^{-n}|Q|} = \theta_0 N^n.$$

Combining the above inequalities, we obtain an upper bound of

$$(2.5) \qquad \#(\mathcal{Q}_{\text{good}}) \geq \frac{\theta_0}{2 - \theta_0} N^n.$$

Step 2: Persistence of good density. For fixed $\theta, \lambda \in (0,1)$, we now claim that there is a choice of $N = N(\theta_0, \theta, \lambda, n, \gamma) \in \mathbb{N}$ so that

$$(2.6) \qquad |\{u > \lambda\} \cap Q_j| \geq \theta |Q_j|$$

holds for some $Q_j \in \mathcal{Q}_{\text{good}}$; by taking $B(x, R')$ in Q_j with $R' \approx \operatorname{diam} Q_j \approx \frac{R}{N}$, the lemma follows with $\epsilon = \frac{1}{N}$.

Arguing by contradiction, if $Q_j \in \mathcal{Q}_{\text{good}}$ does not satisfy (2.6), then

$$|\{u \leq \lambda\} \cap Q_j| \geq (1 - \theta)|Q_j|.$$

On the other hand, by definition Q_j satisfies

$$\left|\left\{u > \frac{1 + \lambda}{2}\right\} \cap Q_j\right| \geq |\{u > 1\} \cap Q_j| \geq \frac{\theta_0}{2}|Q_j|.$$

Since $u \geq 0$ by hypothesis, we apply Lemma 2.3 to points $x \in \{u \leq \lambda\} \cap Q_j$ and $y \in \{u > \frac{1+\lambda}{2}\} \cap Q_j$, which gives

$$\frac{1-\lambda}{2} \leq u(y) - u(x) \leq |x-y|\,(g(x)+g(y)) \leq \frac{2R}{N}\Big(g(x)+g(y)\Big).$$

Integrating in x gives

$$\begin{aligned}
\frac{1-\lambda}{2}(1-\theta) &\leq \frac{1-\lambda}{2}\frac{|\{u<\lambda\}\cap Q_j|}{|Q_j|}\\
&\leq \frac{2R}{N}\frac{1}{|Q_j|}\int_{\{u<\lambda\}\cap Q_j}\Big(g(y)+g(x)\Big)\,dx\\
&\leq \frac{2R}{N}\Big[g(x)+\fint_{Q_j}g(y)\,dy\Big] \leq \frac{2R}{N}\Big[g(y)+\Big(\fint_{Q_j}g^p\,dx\Big)^{\frac{1}{p}}\Big]\\
&\leq \frac{2CR}{N}\Big[g(y)+\Big(\frac{R}{N}\Big)^{-\frac{n}{p}}\Big(\int_{Q_j}|\nabla u|^p\,dx\Big)^{\frac{1}{p}}\Big].
\end{aligned}$$

Similarly, integrating further in y gives

$$\begin{aligned}
\frac{\theta_0(1-\lambda)(1-\theta)}{4} &\leq \frac{2CR}{N}\Big(\frac{R}{N}\Big)^{-\frac{n}{p}}\Big[\Big(\int_{Q_j}|\nabla u|^p\,dy\Big)^{\frac{1}{p}}+\Big(\int_{Q_j}|\nabla u|^p\,dx\Big)^{\frac{1}{p}}\Big]\\
&\leq 4C\Big(\frac{R}{N}\Big)^{\frac{p-n}{p}}\Big(\int_{Q_j}|\nabla u|^p\,dx\Big)^{\frac{1}{p}}
\end{aligned}$$

from which it follows that

$$(2.7) \qquad c_0\Big(\frac{R}{N}\Big)^{n-p} := \Big[\frac{\theta_0(1-\lambda)(1-\theta)}{16C}\Big]^p\Big(\frac{R}{N}\Big)^{n-p} \leq \int_{Q_j}|\nabla u|^p\,dx.$$

In particular, we have $c_0 = c_0(\theta_0, \theta, \lambda, n, p) > 0$.

Now suppose that condition (2.6) fails for every sub-cube Q_j in $\mathcal{Q}_{\text{good}}$. Summing over all j, we therefore apply inequalities (2.3), (2.5), and (2.7) to obtain

$$\begin{aligned}
\frac{c_0\theta_0}{2-\theta_0}R^{n-p}N^p &\leq \big(\#\mathcal{Q}_{\text{good}}\big)\cdot c_0\Big(\frac{R}{N}\Big)^{n-p}\\
&\leq \sum_{Q_j\in\mathcal{Q}_{\text{good}}}\int_{Q_j}|\nabla u|^p\,dx \leq \|u\|_{W^{1,p}(Q)}^p \leq \gamma R^{n-p}.
\end{aligned}$$

So by choosing $N > [(c_0\theta_0)^{-1}\gamma(2-\theta_0)]^{\frac{1}{p}}$, we arrive at a contradiction. We therefore obtain the desired conclusion for the previous choice of $\epsilon = \frac{1}{N}$ and with the constant $c_0 := (16C)^{-p}[\theta_0(1-\lambda)(1-\theta)]^p$. ∎

2.2. Local boundedness and Caccioppoli inequalities. We now turn to the first part of Harnack's inequality (Theorem 1.7). We first motivate the ideas.

A function $u : \Omega \to \mathbb{R}$, *p*-harmonic or not, will have an essential supremum $h \in \mathbb{R}$ on a fixed ball $B(x_0, R) \subset \Omega$ if and only if the super-level set

$$\{x \in B(x_0, R) : u(x) > h\}$$

has zero measure, or equivalently, $(u - h)_+ := \max\{u - h, 0\}$ satisfies

$$\int_{B(x_0,R)} (u - h)_+^p \, dx \ = \ 0.$$

So to find an upper bound $h \in \mathbb{R}$ for u, we seek increasing sequences $\{h_k\}_{k=1}^\infty \subset \mathbb{R}$ converging to h and so that the truncations $(u - h_k)_+$ satisfy

$$\lim_{k \to \infty} \int_{B(x_0,R)} (u - h_k)_+^p \, dx \ = \ 0.$$

To this end, we turn to a lemma from elementary real analysis [21].

Lemma 2.7. *Let $b > 1$, $C_0 \geq 1$, and $\sigma > 0$ be given. If $\{Y_k\}_{k=0}^\infty$ is a sequence in $[0, \infty)$ whose terms satisfy the inequalities*

(2.8) $$Y_{k+1} \ \leq \ C_0 b^n Y_k^{1+\sigma}$$

(2.9) $$Y_0 \ \leq \ b^{-1/\sigma^2} C_0^{-1/\sigma}$$

then $Y_k \to 0$ as $k \to \infty$.

So to prove Theorem 1.7, it suffices to verify the hypotheses of Lemma 2.7 for the sequence of integral averages

$$Y_k \ := \ \fint_{B(x_0,R)} (u - h_k)_+^p \, dx.$$

In particular, inequality (2.8) requires that we compare the integral averages of truncations of varying levels, with the levels $\{k_n\}$ yet to be determined.

Once again we split the desired inequality into two parts. Already we know from Theorem 2.1 that the L^p-norms of Sobolev functions are controlled by the norms of their gradients. As suggested earlier, energy minimizers obey a partial converse, called the *Caccioppoli inequality*, from which the desired recurrence (2.8) follows. Indeed, the L^p-norms of truncations $(u - h)_+$ control the L^p-norms of their gradients. The proof uses Widman's "hole-filling" trick [46].

Lemma 2.8 (Caccioppoli inequality). *There exists $C = C(n,p) \geq 0$ so that every p-energy minimizer $u \in W^{1,p}(\Omega)$ satisfies*

$$\int_{B(x_0,r)} |\nabla(u - h)_+|^p \, dx \;\leq\; \frac{C}{(R-r)^p} \int_{B(x_0,R)} (u - h)_+^p \, dx$$

$$\int_{B(x_0,r)} |\nabla(u - h)_-|^p \, dx \;\leq\; \frac{C}{(R-r)^p} \int_{B(x_0,R)} (u - h)_-^p \, dx$$

for all $B(x,r) \subsetneq B(x,R) \subset \Omega$ and all $h \in \mathbb{R}$.

Proof of Lemma 2.8. If u is an energy minimizer, then so is $-u$, and from

$$(u - h)_- \;=\; -\min(u - h, 0) \;=\; \max(-u + h, 0) \;=\; (-u + h)_+.$$

It follows that the cases for $(u-h)_+$ and $(u-h)_-$ are symmetric, so without loss of generality, we show the first case. To this end, we show an initial inequality for intermediate radii, and then proceed by an iteration argument.

Fix $\rho', \rho \in (r, R)$ with $\rho' < \rho$ and fix a smooth function $\eta : \mathbb{R}^n \to [0, \infty)$ with support $\bar{B}(x_0, \rho)$ and satisfies $\eta|_{B(x_0,\rho')} \equiv 1$ and $\|\nabla\eta\|_\infty \leq c(\rho - \rho')^{-1}$ for some $c \geq 1$. Note that whenever $u(x) > h$, the function $v = u - (u - h)_+\eta$ satisfies

$$v(x) \;=\; u(x) - \big(u(x) - h\big)_+\eta(x) \;=\; \big(1 - \eta(x)\big)\big(u(x) - h\big) + h$$

so the product rule gives, at the same points x, the estimate

$$|\nabla v(x)| \;\leq\; \big(1 - \eta(x)\big)|\nabla u(x)| + |\nabla\eta(x)|\big(u(x) - h\big).$$

As a shorthand, put $B = B(x_0, \rho) \cap \{u > h\}$ and $B' = B(x_0, \rho') \cap \{u > h\}$. Since $u - v \in W_0^{1,p}\big(B(x_0, R)\big)$, the minimizing property of u implies that

$$(2.10) \quad \int_{B'} |\nabla u|^p \, dx \;\leq\; \fint_B |\nabla v|^p \, dx$$

$$\leq\; \int_B (1 - \eta)^p |\nabla u|^p \, dx \;+\; \int_{B'} |\nabla\eta|^p (u - h)_+^p \, dx$$

$$(2.11) \quad\qquad \leq\; \int_{B\setminus B'} |\nabla u|^p \, dx \;+\; \int_{B'} \frac{c^p (u - h)_+^p}{(\rho' - \rho)^p} \, dx$$

and adding $\int_{B'} |\nabla(u - h)_+|^p dx$ to both sides, the inequality becomes

$$(2.12) \quad \int_{B'} |\nabla u|^p \, dx \;\leq\; \frac{1}{2}\left\{ \int_B |\nabla u|^p \, dx \;+\; \int_{B'} \frac{c^p (u - h)_+^p}{(\rho' - \rho)^p} \, dx \right\}.$$

Fix $\lambda > 0$ so that $\lambda^p \in (\frac{1}{2}, 1)$. We now define a sequence of radii recursively by

$$\rho_0 \;=\; r$$

$$\rho_{n+1} - \rho_n \;=\; (1 - \lambda)\lambda^n(R - r)$$

and write $B_n := B(x_0, \rho_n)$. By iterating inequality (2.12), we obtain

$$
\begin{aligned}
\int_{B_0} |\nabla u|^p \, dx &\leq \frac{1}{2} \left\{ \int_{B_1} |\nabla u|^p \, dx + \int_{B_1} \frac{c^p (u-h)_+^p}{(\rho_2 - \rho_1)^p} \, dx \right\} \\
&= \frac{1}{2} \int_{B_1} |\nabla u|^p \, dx + \frac{1}{2\lambda^p} \frac{c^p}{(1-\lambda)^p} \int_{B_1} \frac{(u-h)_+^p}{(R-r)^p} \, dx \\
&\leq \frac{1}{2^n} \int_{B_n} |\nabla u|^p \, dx + \sum_{i=1}^{n} \frac{1}{(2\lambda^p)^n} \frac{c^p}{(1-\lambda)^p} \int_{B_n} \frac{(u-h)_+^p}{(R-r)^p} \, dx.
\end{aligned}
$$

Taking limits as $n \to \infty$, we obtain $\rho_n \to R$. The lemma follows. ∎

Remark 2.9. Note that the quasi-minimizing condition from Definition 1.2 can be used instead in estimate (2.10). For $K \geq 1$, inequality (2.12) then becomes

$$
\int_{B'} |\nabla u|^p \, dx \leq \frac{K}{K+1} \left\{ \int_{B} |\nabla u|^p \, dx + \int_{B'} \frac{c^p (u-h)_+^p}{(\rho' - \rho)^p} \, dx \right\}
$$

and the iteration proceeds, with different constants in the final estimate. We therefore conclude that Lemma 2.8 also holds true for quasi-minimizers.

We now give a rigorous proof of the local boundedness property (Theorem 1.7) using iteration (Lemma 2.7).

Proof of Theorem 1.7. We begin by comparing integral averages of different levels. Fix $\rho', \rho \in (r, R)$ with $\rho' \in (\frac{\rho}{2}, \rho)$ and put $B = B(x_0, \rho)$ and $B' = B(x_0, \rho')$. For $l < h$, we have $(u-h)_+ \leq (u-l)_+$, so Hölder's inequality implies that

$$
\begin{aligned}
\fint_{B'} (u-h)_+^p \, dx &\leq \left\{ \fint_{B'} (u-h)_+^{p^*} \, dx \right\}^{\frac{p}{p^*}} \left(\frac{|B' \cap \{u > h\}|}{|B'|} \right)^{1 - \frac{p}{p^*}} \\
&\leq \left\{ \fint_{B'} (u-h)_+^{p^*} \, dx \right\}^{\frac{p}{p^*}} \left(\frac{1}{|B'|} \int_{B' \cap \{u > h\}} \frac{(h-l)^p}{(h-l)^p} \, dx \right)^{1 - \frac{p}{p^*}} \\
&\leq \left\{ \fint_{B'} (u-h)_+^{p^*} \, dx \right\}^{\frac{p}{p^*}} \left(\fint_{B'} \frac{(u-l)_+^p}{(h-l)^p} \, dx \right)^{\frac{p}{n}}
\end{aligned}
$$

From our choice of radii, we obtain $|B| \leq 2^n |B'|$, from which it follows that

$$
(2.13) \qquad \fint_{B'} (u-h)_+^p \, dx \leq \left\{ \fint_{B'} (u-h)_+^{p^*} \, dx \right\}^{\frac{p}{p^*}} \left(2^n \fint_{B} \frac{(u-l)_+^p}{(h-l)^p} \, dx \right)^{\frac{p}{n}}.
$$

Now let $\eta : \mathbb{R}^n \to [0, \infty)$ be a smooth cut-off function as in the proof of Lemma 2.8. For each $h \in \mathbb{R}$, the Sobolev function $w = (u-h)_+ \eta$ has compact support

in $B(x_0, R)$, so by the Sobolev embedding theorem (2.1), the product rule, and the Caccioppoli inequality (Lemma 2.8), we obtain

$$\left(\fint_{B'} (u-h)_+^{p^*}\, dx\right)^{\frac{p}{p^*}} \leq \left(\fint_{B} ((u-h)_+\eta)^{p^*}\, dx\right)^{\frac{p}{p^*}} \leq C\rho^p \fint_B |\nabla[(u-h)_+\eta]|^p\, dx$$

$$\leq C\rho^p\left\{\fint_B |\nabla(u-h)_+|^p\, dx + \fint_B \frac{(u-h)_+^p}{(\rho-\rho')^p}\, dx\right\}$$

$$\leq \frac{C\rho^p}{(\rho-\rho')^p}\fint_B (u-h)_+^p\, dx$$

We now combine (2.13) and the above inequality to obtain

$$\fint_{B'} (u-h)_+^p\, dx \leq \left\{\fint_{B'} (u-h)_+^{p^*}\, dx\right\}^{\frac{p}{p^*}} \left(2^n \fint_B \frac{(u-l)_+^p}{(h-l)^p}\, dx\right)^{\frac{p}{n}}$$

$$\leq \frac{C\rho^p}{(\rho-\rho')^p}\left(\fint_B (u-h)_+^p\, dx\right)\left(2^n \fint_B \frac{(u-l)_+^p}{(h-l)^p}\, dx\right)^{\frac{p}{n}}$$

from which we further obtain

$$(2.14) \qquad \left(\fint_{B'} (u-h)_+^p\, dx\right)^{\frac{1}{p}} \leq \frac{C\rho}{\rho-\rho'}\left(\fint_B (u-l)_+^p\, dx\right)^{\frac{1}{p}\left(1+\frac{p}{n}\right)}(h-l)^{-\frac{p}{n}}.$$

We are now ready to run the iteration. Choose radii and levels

$$\rho_k := \frac{R}{2}(1+2^{-k}) \quad\text{and}\quad h_k := d(1-2^{-k}),$$

where $d > 0$ is to be determined later. Now consider the sequence

$$Y_k := \left(\fint_{B_k} (u-h_k)_+^p\, dx\right)^{\frac{1}{p}}$$

where $B_k := B(x_0, \rho_k)$. Using balls B_k and B_{k+1} in place of B' and B, and levels h_{k+1} and h_k in place of l and h, respectively, we rewrite inequality (2.14) as

$$Y_k \leq 2^{k+1}Y_{k+1}^{1+\frac{p}{n}}\left(2^{-(k+1)}d\right)^{-\frac{p}{n}} \leq Cd^{-\frac{p}{n}}2^n Y_{n+1}^{1+\frac{p}{n}}.$$

Put $b = 2$ and $\sigma = \frac{p}{n}$, and choose d so that inequality (2.9) holds; it suffices that

$$d = C^{\frac{n}{p}}b^{\frac{1}{\sigma^2}}Y_0 = C\left(\fint_{B_0} (u-0)_+^p\, dx\right)^{\frac{1}{p}} = C\left(\fint_{B_0} u_+^p\, dx\right)^{\frac{1}{p}}.$$

So by Lemma 2.7, we conclude that the limit is

$$0 = \lim_{k\to\infty} Y_k = \left(\fint_{B(x_0,R)} (u-d)_+^p\, dx\right)^{\frac{1}{p}}$$

and as a result, for a.e. $x \in B(x_0, R)$, we have the inequality

$$u(x) \ \leq \ d \ = \ C\Big(\fint_B u_+^p \, dx\Big)^{\frac{1}{p}}. \qquad \blacksquare$$

The next result will be used to prove Corollary 1.10.

Corollary 2.10. *Let* $u : \Omega \to \mathbb{R}$ *be a p-energy minimizer, and fix* $B(x_0, R) \subset \Omega$ *and* $\tau > 0$. *There exists* $\lambda_0 = \lambda_0(p, n, \Omega) \in (0, 1)$ *so that if*

$$|\{u < \tau\} \cap B(x_0, R)| \ \leq \ \lambda_0 |B(x_0, R)|$$

holds, then we have

$$\inf_{B(x, \frac{1}{2}R)} u \ \geq \ \frac{\tau}{2}.$$

Proof. Put $B = B(x_0, R)$ and $B' = B(x_0, \frac{R}{2})$. By substituting $-u + \tau$ for u in the proof of Theorem 1.7, we obtain the inequality

$$\sup_{B'}(-u) \ \leq \ -\tau \ + \ C\Big(\fint_B (-u + \tau)_+^p \, dx\Big)^{\frac{1}{p}}$$

and from the elementary inequality $(\tau - u)_+ \leq \tau$, a further estimate gives

$$\inf_{B'} u \ \geq \ \tau - C\Big(\frac{1}{|B|}\int_B (\tau - u)_+^p \, dx\Big)^{\frac{1}{p}} \ \geq \ \tau - C\tau\Big(\frac{|\{u < \tau\} \cap B|}{|B|}\Big)^{\frac{1}{p}}.$$

So for $\lambda_0 = (2C)^{-p}$, the result follows from the hypothesis. $\qquad \blacksquare$

2.3. Expansion of Positivity and its Consequences. In the last section we used an iteration argument to show that *p*-energy minimizers are locally bounded. This means that their oscillations

$$\operatorname*{osc}_B u \ = \ \sup_{x,y \in B} |u(y) - u(x)| \ = \ \sup_B u - \inf_B u$$

are also a.e. locally finite. We now prove the weak Harnack inequality (Theorem 1.8) by using similar iteration techniques. The argument requires several steps:

1. To intermediate between the function values of an energy minimizer, we incorporate oscillations into the levels.

2. We show that energy minimizers satisfy an "expansion of positivity" property. Roughly speaking, this means that if the minimizer is positive on a large percentage of a ball, then it must also be positive (with a smaller lower bound) on a large percentage of successively larger balls.

3. By expanding finitely many times (where the number of expansions depends only on n, p, and Ω), we show that the minimizer must be controlled by a similar factor of its minimum.

The first step is given as a lemma. For a ball $B = B(x_0, R)$, we use the shorthand

$$M := \sup_B u \quad \text{and} \quad m := \inf_B u \quad \text{and} \quad \omega := M - m.$$

Lemma 2.11. *Let Ω be a domain in $\mathbb{R}^n$, let $p > 1$, and let $a \in (0,1)$. There exists a constant $\nu = \nu(n, p, a; \Omega) > 0$ such that for every ball $B(x_0, 2R)$ in Ω,*

1. *if a p-energy minimizer $u \in W^{1,p}(\Omega)$ satisfies the density condition*

$$\left|\{u > M - \xi\omega\} \cap B(x_0, R)\right| \leq \nu \left|B(x_0, R)\right|$$

   then for a.e. $x \in B(x_0, \frac{R}{2})$ we have the inequality

$$u(x) \leq M - a\xi\omega.$$

2. *if a p-energy minimizer $u \in W^{1,p}(\Omega)$ satisfies the density condition*

$$\left|\{u < m + \xi\omega\} \cap B(x_0, R)\right| \leq \nu \left|B(x_0, R)\right|$$

   then for a.e. $x \in B(x_0, \frac{R}{2})$ we have the inequality

$$u(x) \geq m + a\xi\omega.$$

Idea of proof. From the identities

$$\{u < m + \xi\omega\} = \{-u > -m - \xi\omega\} \quad \text{and} \quad \sup_B(-u) = -\inf_B u = -m,$$

we note that (2) follows from (1) by using $-u$ and $-m$ in place of u and M, respectively.

The proof of (1) uses iteration again, so for radii $\rho_k := \frac{R}{2}(1 - 2^k)$ and levels

$$h_k := (M - a\xi\omega) - 2^{-k}(1 - a)\xi\omega$$

we see that the following identities hold:

$$h_{k+1} - h_k = 2^{-(k+1)}(1 - a)\xi\omega$$
$$\rho_k - \rho_{k+1} = 2^{-(k+2)}R.$$

Hence, for the sequence of integrals

$$Y_k := h_0^{-p} \fint_{B_k} (u - h_k)_+^p \, dx$$

the result follows from a similar proof as in Theorem 1.7. For details, see [15, Chapter 10]. ∎

The next step is a standard but technical "telescoping" argument. For expositional reasons we divide it into two steps.

Theorem 2.12 (Expansion of Positivity). *Let $p > 1$ and let $B(x_0, 8R) \subset \Omega$. There exists $\sigma \in (0,1)$ so that if $u \in W^{1,p}(\Omega)$ is a positive p-energy minimizer that satisfies the density condition*

$$|\{u > h\} \cap B(x_0, R))| \geq \frac{1}{2}|B(x_0, R)|$$

for some $h > 0$, then for a.e. $x \in B(x_0, 2R)$, we have

$$u(x) \geq \sigma h.$$

Proof. *Step 1 of 2: Preliminary estimates.* Write $B_k = B(x_0, kR)$, for $k \in \mathbb{N}$. We first adjust the density condition into one for sub-level sets:

$$|B_4| = 4^n |B_1| \leq 2 \cdot 4^n |\{u > h\} \cap B_1| \leq 2 \cdot 4^n |\{u > h\} \cap B_4|$$

(2.15)
$$|\{u < h\} \cap B_4| \leq \left(1 - \frac{1}{2 \cdot 4^n}\right)|B_4|$$

For levels $0 < l < L < h$, we first show the estimate

(2.16) $$(L - l)\big|\{u < l\} \cap B_4\big| \leq CL\big|\{l < u \leq L\} \cap B_4\big|^{\frac{1}{q} - \frac{1}{p}}|B_4|^{1 - \frac{1}{q} + \frac{1}{p}}.$$

Indeed, consider the non-negative auxiliary function

$$\begin{aligned}
v := (u - L)_- - (u - l)_- &= -\min(u - L, 0) + \min(u - l, 0) \\
&= -\min(u, L) + \min(u, l) + (L - l)
\end{aligned}$$

which lies in $W^{1,p}(\Omega)$ and satisfies the identities

$$\begin{aligned}
\{|v| > 0\} &= \{u < L\} \subset \{u < h\}, \\
v|_{\{u > l\}} &= (u - L)_-|_{\{u > l\}}.
\end{aligned}$$

So the density condition (2.15) implies a density condition for v:

$$|\{|v| > 0\} \cap B_4| \leq |\{u < h\} \cap B_4| \leq \left(1 - \frac{1}{2 \cdot 4^n}\right)|B_4|.$$

For a fixed $q \in (1, p)$ and $t = q$, Hölder's inequality and Lemma 2.4 implies that

$$\begin{aligned}
(L - l)|\{u < l\} \cap B_4| &= \int_{\{u < l\} \cap B_4} (L - l)\,dx = \int_{\{u < l\} \cap B_4} v\,dx \\
&\leq \int_{B_4} |v|\,dx \leq |B_4|^{\frac{q-1}{q}}\left(\int_{B_4} |v|^q\,dx\right)^{\frac{1}{q}} \\
&\leq 4C'R|B_4|^{\frac{q-1}{q}}\left(\int_{B_4} |\nabla v|^q\,dx\right)^{\frac{1}{q}} \\
&= 4C'R|B_4|^{1 - \frac{1}{q}}\left(\int_{\{l < u \leq L\} \cap B_4} |\nabla v|^q\,dx\right)^{\frac{1}{q}}
\end{aligned}$$

where the last identity follows from the fact that v is constant on $\{u < l\}$ and $\{u > L\}$. Applying Hölder's inequality for $|\nabla v|^q \in L^{\frac{p}{q}}(B_4)$ once again, we obtain

$$(L-l)\frac{|\{u < l\} \cap B_4|}{|B_4|^{1-\frac{1}{q}}} \leq 4C'R\big|\{l < u \leq L\} \cap B_4\big|^{\frac{1}{q}-\frac{1}{p}}\left(\int_{\{l<u\leq L\}\cap B_4}|\nabla v|^p\,dx\right)^{\frac{1}{p}}.$$

Using the Caccioppoli inequality (Lemma 2.8) with B_4 and B_8, we now estimate

$$\begin{aligned}
\int_{\{l<u\leq L\}\cap B_4}|\nabla v|^p\,dx &\leq \int_{\{l<u\}\cap B_4}|\nabla v|^p\,dx\\
&= \int_{\{l<u\}\cap B_4}|\nabla(u-L)_-|^p\,dx \leq \int_{B_4}|\nabla(u-L)_-|^p\,dx\\
&\leq C(4R)^{-p}\int_{B_8}(u-L)_-^p\,dx \leq C(4R)^{-p}\int_{B_8}L^p\,dx\\
&\leq C(4R)^{-p}L^p|B_8| = C(4R)^{-p}2^nL^p|B_4|
\end{aligned}$$

and combining the above inequalities, we obtain (2.16):

$$\begin{aligned}
(L-l)\frac{|\{u<l\}\cap B_4|}{|B_4|^{1-\frac{1}{q}}} &\leq 4C'R\big|\{l<u\leq L\}\cap B_4\big|^{\frac{1}{q}-\frac{1}{p}}\left(C(4R)^{-p}2^nL^p|B_4|\right)^{\frac{1}{p}}\\
&= CL\big|\{l<u\leq L\}\cap B_4\big|^{\frac{1}{q}-\frac{1}{p}}|B_4|^{\frac{1}{p}}
\end{aligned}$$

where in the last line, $C'C^{\frac{1}{p}}$ is replaced by C.

Step 2 of 2: Telescoping sums. We now take levels $l_k := 2^{-k}h$ and l_{k+1} in place of L and l, respectively. For each $k \in \mathbb{N}$, we clearly have $l_k < h$ and

$$l_k - l_{k+1} = \frac{h}{2^{k+1}} = \frac{1}{2}l_k,$$

so inequality (2.16) can be rewritten as

$$\frac{h}{2^{k+1}}\Big|\Big\{u < \frac{h}{2^{k+1}}\Big\} \cap B_4\Big| \leq \frac{2Ch}{2^k}\Big|\Big\{\frac{h}{2^{k+1}} < u \leq \frac{h}{2^k}\Big\} \cap B_4\Big|^{\frac{1}{q}-\frac{1}{p}}|B_4|^{1-\frac{1}{q}+\frac{1}{p}},$$

hence $\quad \dfrac{|\{u < \frac{h}{2^{k+1}}\} \cap B_4|}{|B_4|} \leq C\dfrac{|\{\frac{h}{2^{k+1}} < u \leq \frac{h}{2^k}\} \cap B_4|^{\frac{1}{q}-\frac{1}{p}}}{|B_4|^{\frac{1}{q}-\frac{1}{p}}}.$

As a result, for each $N \in \mathbb{N}$ and each integer $0 \leq k \leq N$, we obtain

$$\begin{aligned}
\left(\frac{|\{u < \frac{h}{2^{N+1}}\} \cap B_4|}{|B_4|}\right)^{\frac{pq}{p-q}} &\leq \left(\frac{|\{u < \frac{h}{2^{k+1}}\} \cap B_4|}{|B_4|}\right)^{\frac{pq}{p-q}}\\
&\leq \frac{C}{|B_4|}\Big|\Big\{\frac{h}{2^{k+1}} < u \leq \frac{h}{2^k}\Big\} \cap B_4\Big|.
\end{aligned}$$

Summing the right hand side of the above inequality for $k = 0, 1, 2, \ldots N$, we observe that the sum is telescoping, so

$$\left(\frac{|\{u < \frac{h}{2^{N+1}}\} \cap B_4|}{|B_4|} \right)^{\frac{pq}{p-q}} \leq \frac{C}{N} \sum_{k=0}^{N} \frac{|\{\frac{h}{2^{k+1}} < u \leq \frac{h}{2^k}\} \cap B_4|}{|B_4|}$$

$$= \frac{C}{N} \frac{|\{\frac{h}{2^{N+1}} < u \leq h\} \cap B_4|}{|B_4|} \leq \frac{C}{N},$$

that is,

$$(2.17) \qquad \left| \left\{ u < \frac{h}{2^{N+1}} \right\} \cap B_4 \right| \leq CN^{\frac{1}{p}-\frac{1}{q}} |B_4|.$$

We note that the proof of Lemma 2.11 uses M, m, and ω only as numerical parameters, so letting ν be as in Lemma 2.11, choose $N \in \mathbb{N}$ sufficiently large so that $CN^{\frac{1}{p}-\frac{1}{q}} \leq \nu$. With $m = 0$ and $\omega = 2h$, Lemma 2.11 implies that

$$u(x) \geq \sigma h$$

holds for a.e. $x \in B(x_0, 2R)$, where $\sigma = 2^{-(N+1)}$. ∎

We are now ready to prove the weak Harnack inequality. The basic idea is to fix a ball and show an initial density estimate in a smaller ball. Using the previous theorem, we expand the validity of the estimate to the original ball.

Moreover, the number of expansions will depend only on n, p, and Ω. This motivates the use of an auxiliary function $N(r)$ to replace "radial supremum functions" $M(r)$.

Proof of Theorem 1.8. *Step 1 of 2: Initial density estimate.* Define functions

$$M(r) := \sup_{B(x_0,r)} u \quad \text{and} \quad N(r) := u(x_0)\left(1 - \frac{r}{R}\right)^{-\beta},$$

where $\beta \in (0, 1)$ is to be chosen later. Clearly we have

$$N(0) = u(x_0) \leq M(0).$$

By Theorem 1.7, $M(r)$ is bounded on $[0, R]$, whereas $N(r)$ is unbounded, so there must be a largest root $R_0 \in [0, R)$ for which $M(R_0) = N(R_0)$. Putting

$$M_* := 2^{\beta} N(R_0) \quad \text{and} \quad r_* := \frac{R - R_0}{4} \quad \text{and} \quad R_* := \frac{R + R_0}{2},$$

the Triangle inequality implies each $x \in B(x_0, R_0)$ satisfies $B(x, 2r_*) \subset B(x_0, R_*)$.

We claim that for $\xi := \sqrt{1 - 2^{-(\beta+1)}}$, there is a point $x \in B(x_0, R_0)$ so that

$$(2.18) \qquad |\{u > M_* - \xi M_*\} \cap B(x, r_*)| \geq \nu |B(x, r_*)|,$$

where $a = \xi$ and where $\nu \in (0,1)$ is chosen as in the proof of Lemma 2.11. Supposing otherwise, for each $x \in B(x_0, R_0)$ we apply Lemma 2.11, with $\omega = M_*$, and conclude that for a.e. $y \in B(x, \frac{r_*}{2})$ we have

$$u(y) \leq M_* - a\xi M_* = (1 - \xi^2)M_* = \left(1 - (1 - 2^{-(\beta+1)})\right)2^\beta N(R_0) = \frac{N(R_0)}{2}.$$

This is a contradiction, since $M(R_0) = N(R_0)$ is a supremum. The claim follows.

By hypothesis, u is an energy minimizer with $u = u_+$; the same applies to the function $v = (M_* - \xi M_*)^{-1}u$. Note also that $v \leq (2^\beta(1-\xi))^{-1}$ and that

$$|\{v > 1\} \cap B(x, r_*)| \geq \nu|B(x, r_*)|.$$

Applying Lemma 2.8 with $h = 0$ and balls $B(x, r_*)$ and $B(x, 2r_*)$, we have

$$\int_{B(x,r_*)} |\nabla v|^p \, dx \leq \frac{C}{r_*^p} \int_{B(x,2r_*)} v^p \, dx \leq \frac{C|B(x, 2r_*)|}{2^\beta r_*^p(1-\xi)} = \frac{C2^{n-\beta}}{1-\xi} r_*^{n-p}.$$

Applying Remark 2.6 with $\theta = \frac{1}{2}$ and $\lambda = [2^\beta(1-\xi)]^{-1}$, it follows that

$$\lambda(M_* - \xi M_*) = N(R_0)$$

and therefore the following density condition holds:

$$\left|\{u > N(R_0)\} \cap B(x, \epsilon r_*)\right| = |\{v > \lambda\} \cap B(x, \epsilon r_*)| \geq \frac{1}{2}|B(x, \epsilon r_*)|.$$

Step 2 of 2: Expanding to all of $B(x_0, R)$. Applying Theorem 2.12 to the previous inequality, it follows that for a.e. $y \in B(x, 2\epsilon r_*)$, we have

$$u(y) \geq \sigma N(R_0)$$

so trivially, u satisfies the density estimate

$$\left|\{u > \sigma N(R_0)\} \cap B(x, 2\epsilon r_*)\right| \geq \frac{1}{2}|B(x, 2\epsilon r_*)|.$$

We now choose the number of iterations $K \in \mathbb{N}$ so that

$$2^K \epsilon r_* = 2^K \epsilon \frac{R - R_0}{4} \geq 2R$$

and therefore K iterations of Theorem 2.12 implies that for a.e. $y \in B(x, 2^N \epsilon r_*)$, and hence a.e. $y \in B(x_0, R)$, the following estimate holds:

$$u(y) \geq \sigma^K N(R_0) = \sigma^K u(x)\left(1 - \frac{R_0}{R}\right)^\beta =: Cu(x).$$

Taking infima in y, the weak Harnack inequality follows. $\blacksquare$

Remark 2.13. It is possible to select β, σ, and R_0 to depend only on the parameters n, p, and Ω. We refer the interested reader to [15, Chapter 10, §9.3].

We now prove the Strong Maximum Principle (Corollary 1.10).

Proof of Corollary 1.10. Suppose on the contrary that u is non-constant, and fix a ball $B = B(x, r) \subset \Omega$. By adding a constant, we may assume that $u \geq 0$. For $\tau := \max_B u$, there exists $\lambda \in (0, 1)$ so that

$$|\{u < \tau\} \cap B| \leq \lambda |B|.$$

We now run a similar argument to the proof of Theorem 2.12 with the above density condition in place of (2.15), with B in place of B_4, and with λ in place of the density value $1/2$. An analogue of (2.17) follows: for $1 < q < p$ and for sufficiently large $N \in \mathbb{N}$, we have

$$|\{u < 2^{-(N+1)}\tau\} \cap B| \leq \lambda_0 |B|$$

where $\lambda_0 \in (0, 1)$ is as in Corollary 2.10. The same Corollary 2.10 therefore gives

$$u(x) \geq 2^{-(N+2)}\tau > 0$$

for all $x \in B$, which is a contradiction.   ∎

2.4. Summary and Generalizations. As observed in this section, the De Giorgi method is technical in nature, but its basis rests on a few tools: the Caccioppoli inequality (Lemma 2.8) and various Sobolev inequalities.

It is interesting to note that the regularity theory for the p-energy integral (1.4) is *stable* in the sense that the fine properties enjoyed by minimizers also extend to similar functions. As observed in Remark 2.9, quasi-minimizers also satisfy the Caccioppoli inequality, with constants depending on the comparison constant K. As a result, Harnack's inequality (and local Hölder continuity) follow for quasi-minimizers, again with appropriate constants.

In another direction, one could consider a larger class of functionals and the regularity theory of their corresponding minimizers. A general discussion is treated in [21]; here we consider the simple case of

$$u \mapsto \mathcal{F}[u] := \int_\Omega \left(\frac{|\nabla u|^p}{p} + uF \right) dx,$$

where the *non-homogeneous* data is $F \in L^s(\Omega)$ with $s \geq \frac{p}{p-1}$. We define minimizers similarly, with $\mathcal{F}[u]$ replacing the p-energy integrals in Definition 1.2.

In this setting, one estimates the non-homogeneous term using the measure of a super-level set. More specifically, for $s > \frac{n}{p}$ one chooses $q \in (1, p)$ so that the Sobolev conjugate $q^* = \frac{nq}{n-q}$ satisfies $\frac{1}{s} + \frac{1}{q^*} = 1$. With the test function

$v = u - (u - h)_+ \eta$ as before, the Hölder and Sobolev inequalities give

$$
\begin{aligned}
\int_B (u - v) F \, dx &= \int_B (u - h)_+ \eta F \, dx \\
&\leq \|F\|_{L^s(B)} \big\|(u - h)_+ \eta\big\|_{L^{q^*}(B)} \\
&\leq \|F\|_{L^s(B)} \big\|\nabla[(u - h)_+ \eta]\big\|_{L^q(B)} \\
&\leq \|F\|_{L^s(B)} \big\|\nabla[(u - h)_+ \eta]\big\|_{L^p(B)} \big|\{u > h\} \cap B\big|^{\frac{1}{q} - \frac{1}{p}}
\end{aligned}
$$

and the remaining steps — the product rule, Young's inequality, etc — is similar in form. The Caccioppoli inequality (for minimizers) therefore reads as follows:

$$
\int_{B(x,r)} |\nabla(u - h)_+|^p \, dx \;\leq\; \int_{B(x,R)} \frac{(u - h)_+^p}{(R - r)^p} \, dx \;+\; \gamma \big|\{u > h\} \cap B(x,R)\big|^{1+\delta},
$$

for additional parameters $\delta = \delta(n, p, s) \in (0, 1)$ and $\gamma = \gamma(\|F\|_{L^s(B)}) \geq 0$.

One can further show that the minimizers associated to $\mathcal{F}$ also satisfy an analogous Harnack inequality. The main technical point lies in treating the *rate* of iteration: this is determined by the exponents δ and $\frac{n}{p}$, the latter induced by the p-energy integral (as in the proof of Theorem 1.7). For further details, see [21], [15], or [22].

3. Energy minimizers on metric measure spaces

3.1. Motivations: Sobolev inequalities and doubling measures.
In §2.4 we revisited the tools of proof for Harnack's inequality. Among the functional inequalities we used were:

1. various Sobolev embedding theorems (Theorem 2.1, Lemma 2.4) and as a consequence, the Caccioppoli inequality (Lemma 2.8);
2. the characterization of $W^{1,p}(\Omega)$ via modulus of continuity (Lemma 2.3) and its consequence, the local clustering property (Lemma 2.5).

More tacitly, additional properties of the "norm gradient" map $f \mapsto |\nabla f|$ include

(3) the product rule, as an inequality: $|\nabla(fg)| \leq |f| \, |\nabla g| + |g| \, |\nabla f|$;
(4) locality, in the sense that $|\nabla f| = 0$ holds on sets where f is constant.

We now discuss a framework that provides analogues of Sobolev functions in the setting of metric measure spaces. Moreover, we narrow our focus to spaces satisfying two key hypotheses — the doubling condition for measures and a Poincaré inequality, to be specific — and analyze how the corresponding *geometric* and *functional analytic* structures allow for similar tools of proof.

What follows is a survey of this subject and many details are omitted. In the area of analysis on metric spaces, we invite the reader to consult the articles [26], [29], [27], [25], and [28], and for topics specific to regularity theory, see [32], [2], and [3].

We begin with the geometry induced by the measure. A Radon measure μ on a metric space (X, d) — that is, a Borel regular measure that assigns positive, finite measure to compact sets — suffices for the usual integration theory. In the case of Lebesgue measure on $\mathbb{R}^n$, however, two properties were crucial:

1. scaling: in iteration arguments we often used, for $0 < r < R$, the property

$$(3.1) \qquad |B(x, R)| = \left(\frac{R}{r}\right)^n |B(x, r)|.$$

2. dyadic decompositions: for the local clustering property (Lemma 2.5) it was crucial to take smaller sub-cubes that possess comparable (same) measures and that partition a given cube.

On a metric space, we therefore consider measures with a fixed rate of growth. As first examined by Coifman and Weiss [9], [10], we require that the measure of a ball is controlled by the measure of smaller balls contained in it.

Definition 3.1. Let $c_\mu \geq 1$. A Borel measure μ on X is said to be *(c_μ)-doubling* if every ball $B(x, r)$ in X with $r > 0$ has positive, finite μ-measure and satisfies

$$(3.2) \qquad \mu\big(B(x, 2r)\big) \leq c_\mu \, \mu\big(B(x, r)\big)$$

and we define the *doubling exponent* of X to be $Q := \log_2(c_\mu)$.

By inspecting the proofs in §2, one sees that *exact* scaling, as in (3.1), is not essential. It suffices instead that balls and their sub-balls are comparable in measure, as in (3.2).

The doubling exponent plays the analogous role of dimension on metric measure spaces, and for $p \in (1, Q)$ we define (Sobolev) conjugate exponents as

$$p^* := \frac{Qp}{Q - p}.$$

For *connected* metric spaces, the doubling property (3.2) implies that locally, the μ-measures of balls are controlled by powers of their radii. As in Euclidean spaces, this provides useful comparisons between length and volume. We omit the proofs here; for details, see [27].

Lemma 3.2. *Let X be a metric space and let μ a doubling measure on X with doubling exponent Q. For each ball $B_0 = B(x_0, r_0)$ in X, with $0 < r_0 < \infty$,*

1. *there exists $c = c(c_\mu, B_0) > 0$ so that, for all $x \in B_0$ and all $r \in (0, r_0)$,*

$$c\left(\frac{r}{r_0}\right)^Q \ \leq \ \frac{\mu(B(x,r))}{\mu(B_0)}$$

2. *If X is connected, there exist $A' = A'(c_\mu, B_0) > 0$ and $Q' = Q'(c_\mu, B_0) > 0$ so that, for all $x \in B_0$ and all $r \in (0, r_0)$,*

$$\frac{\mu(B(x,r))}{\mu(B_0)} \ \leq \ A'\left(\frac{r}{r_0}\right)^{Q'}$$

Metric spaces admitting a doubling measure are also *spaces of homogeneous type*, as introduced by Coifman and Weiss [9], [10] in their work in harmonic analysis. In terms of measure, the condition ensures that the space has good scaling properties, from which we obtain a rich theory of 'zeroth order' calculus.

In addition, Christ [8] proved that spaces X supporting a doubling measure allow a generalized dyadic decompositions $\{Q_a^k \subset X : k \in \mathbb{Z}, a \in I_k\}$, of scale δ^k, ratio λ_0, and constants $C_1, C_2, \eta \in (0, \infty)$ with the following properties:

- measurable partition: $\mu(X \setminus \bigcup_a Q_a^k) = 0$;
- partial ordering: if $l > k$ then either $Q_b^l \subset Q_a^k$ or $Q_b^l \cap Q_a^k = \emptyset$;
- nesting: for each (k, a) and $l < k$ there is a unique b so that $Q_a^k \subset Q_b^l$;
- scaling: the diameter of Q_a^k is at most $C_1 \delta^k$;
- roundness: each Q_a^k contains some ball $B(z_a^k, \lambda_0 \delta^k)$;
- annuli have comparable measure as their interior balls, in the sense that

$$\mu\big(\{x \in Q_a^k : d(x, X \setminus Q_a^k) < t\delta^k\}\big) \ < \ C_2 t^\eta \mu(Q_a^k).$$

So from the viewpoint of regularity theory, we therefore recover all necessary geometric properties for this class of measures.

3.2. Upper gradients, Poincaré inequalities and Newtonian spaces. We now consider analogues of the Sobolev spaces $W^{1,p}(\Omega)$. As indicated before, the distributional derivatives of a function may not be well-defined in this setting. To overcome this, one uses instead upper gradients, which are defined using line integrals and a Fundamental Theorem of Calculus, treated as an inequality.

Specifically, if $u : X \to \mathbb{R}$ is a Borel function, then we say that a Borel function $g : X \to [0, \infty]$ is an *upper gradient* of u if the inequality

$$(3.3) \qquad |u(\gamma(b)) - u(\gamma(a))| \ \leq \ \int_\gamma g\, ds \ := \ \int_a^b g(\gamma(t))\, dt$$

holds for all rectifiable curves $\gamma : [a, b] \to X$ parametrized by arc-length.

For technical reasons, we require a less restrictive notion of upper gradient, which in turn requires a means of measuring families of rectifiable curves. To

this end, let Γ be a collection of non-constant rectifiable curves on X. For $p \geq 1$, the *p-modulus* of Γ is defined as

$$\mathrm{mod}_p(\Gamma) \; := \; \inf_\rho \int_X \rho^p \, d\mu,$$

where the infimum is taken over all Borel functions $\rho : X \to [0, \infty]$ that satisfy

$$\int_\gamma \rho \, dx \; \geq \; 1.$$

It is well-known that p-modulus is an outer measure on $\mathcal{M}$, the family of all rectifiable curves on X; for details, see [27, Chapter 7].

Definition 3.3. We say that a Borel function $g : X \to [0, \infty]$ is a *(p-)weak upper gradient* of $u : X \to \mathbb{R}$ if Equation (3.3) holds for p-modulus a.e. curve $\gamma \in \mathcal{M}$. More precisely, this means that for the subcollection of curves Γ in $\mathcal{M}$ for which Equation (3.3) fails, we have $\mathrm{mod}_p(\Gamma) = 0$.

Example. If $u : X \to \mathbb{R}$ is *Lipschitz* — that is, if u satisfies the condition

$$\mathrm{Lip}(u) \; := \; \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} \; : \; x, y \in X, \; x \neq y \right\} \; < \; \infty$$

then $\mathrm{Lip}(u)$, called the *Lipschitz constant* of u, is an upper gradient of u.

In either definition, the notion of an upper gradient is meaningful only when the space X is *rectifiably connected*, that is: every pair of points can be joined by a rectifiable curve.

We now define an analogue of the Sobolev space $W^{1,p}(\mathbb{R}^n)$ on metric spaces, which is similar to the approach in Equation (1.3). For other formulations, see [6], [27], and [25].

Definition 3.4. Let $p \geq 1$. We say that a function $u : X \to \mathbb{R}$ lies in $\tilde{N}^{1,p}(X)$ if and only if $u \in L^p(X)$ and the quantity

$$\|u\|_{1,p} \; := \; \|u\|_p \; + \; \inf_g \|g\|_p$$

is finite, where the infimum is taken over all weak upper gradients g of u.

The *Newtonian space* $N^{1,p}(X)$ consists of equivalence classes of functions in $\tilde{N}^{1,p}(X)$. Here, two functions $u, v \in \tilde{N}^{1,p}(X)$ are *equivalent* if $u = v$ μ-a.e.

As defined in Definition 3.4, $\| \cdot \|_{1,p}$ is a norm and $N^{1,p}(X)$ is a Banach space with respect to this norm [44, Thm 3.7].

Remark 3.5. (A) Since there is no well-defined notion of "smooth function" on a metric space, the definition of the test function class $N_0^{1,p}(U)$ requires care. Roughly speaking, $N_0^{1,p}(U)$ is a similar space of equivalence classes, where the original functions are defined to have appropriate extensions of $N^{1,p}(X)$. For details, see [44].

(B) For each $u \in N^{1,p}(X)$, there exists a weak upper gradient g_u, called the *minimal upper gradient* of u, so that the infimum in $\|u\|_{1,p}$ is attained [25, Thm 7.16]. Indeed, it is a fact that $g_u = |\nabla u|$ on $X = \mathbb{R}^n$. We refer the details to [44].

Moreover, we have the following Leibniz product rule [45, Lemma 2.14]:

Lemma 3.6. *If $u \in N^{1,p}(X)$ and if $f : X \to \mathbb{R}$ is a bounded Lipschitz function, then $uf \in N^{1,p}(X)$ and its minimal upper gradient satisfy*

$$g_{uf} \leq g_u |f| + |u| \operatorname{Lip}(f).$$

We now formulate our second main hypothesis — the validity of a Poincaré inequality — in terms of weak upper gradients. Together with the doubling property (3.2), such inequalities determine a rich theory of first-order calculus on the underlying spaces.

Definition 3.7. We say that a metric measure space (X, d, μ) supports a *(weak)*[1] $(1, p)$-*Poincaré inequality* if μ is doubling and there exist $C \geq 0$, $\Lambda \geq 1$ so that

$$(3.4) \qquad \fint_B |u - u_B| \, d\mu \ \leq \ C \, r \left(\fint_{\Lambda B} g_u^p \, d\mu \right)^{\frac{1}{p}}$$

holds for all $u \in N_{\mathrm{loc}}^{1,p}(X)$ and for all balls B centered in X.

We now list several consequences of the Poincaré inequality on a metric measure space (X, d, μ):

1. It is easy to see from Definition 3.3 that g_u satisfies the locality property, as suggested before in §3.1: for a set $A \subset X$ of positive μ-measure, if $g_u = 0$ holds on every ball in A, then u is constant on A.

2. Such spaces X are *c-quasiconvex*; that is, every pair of points $x, y \in X$ can be joined by a curve in X whose length is at most $c \cdot d(x, y)$. (Here $c > 0$ depends only on the parameters of the hypotheses; see [12] or [6, Sect 17].) In particular, such spaces are connected, so the estimates of Lemma 3.2 apply to balls in X.

---

[1]Here "weak" refers to the possibility that $\Lambda > 1$.

3. $N^{1,p}(X)$ also allows a characterization in terms of modulus of continuity, as in Lemma 2.3. In fact, one can show that an analogous inequality

$$|u(x) - u(y)| \leq Cd(x,y)\big(g(x) + g(y)\big)$$

arises precisely from taking *localized maximal functions* of g_u on balls with radius comparable to the distance $d(x,y)$. For more details, see [25].

As a last consequence, we also recover a version of the Sobolev embedding theorem [26, Thm 5.1]; see also [32, Eq (2.11)]. We recall here that Sobolev exponents are given as $q^* = Qq/(Q - q)$.

Lemma 3.8. *Let (X, d, μ) be a metric measure space that supports a $(1, p)$-Poincaré inequality. Fix $p \in (1, Q)$. Then there exists $\epsilon > 0$ so that for $p - \epsilon < q < p$, there exist $c > 0$ and $\Lambda' \geq 1$ so that the inequality*

$$\left(\fint_B |u|^t \, d\mu\right)^{\frac{1}{t}} \leq cr\left(\fint_{\Lambda B} g_u^q \, d\mu\right)^{\frac{1}{q}}$$

holds for all balls B with $3B \subset X$, all $t \in [1, q^*]$, and all $u \in N_0^{1,p}(3\Lambda B)$.

We note that the lemma relies on a deep theorem of Keith and Zhong [31, Thm 1.0.1], regarding the open-endedness of (weak) $(1, p)$-Poincaré inequality.

3.3. Energy minimizers revisited. With the notion of (weak) upper gradients in hand, we arrive at similar notions of p-energy minimizers from Euclidean spaces. The discussion here follows [32].

Here and in what follows, we assume that (X, d) is a metric space that supports a doubling measure μ and a (weak) $(1, p)$-Poincaré inequality, and that all constants $C > 0$ depend only on the parameters of the previous assumptions. Similarly to the Euclidean case, we also assume that $1 < p < Q$.

Definition 3.9. Let U be a domain in X. We call $u \in N^{1,p}(U)$ a *p-energy minimizer* if for all bounded subdomains $U' \subset U$ and all functions $v \in N^{1,p}(U)$ with $u - v \in N_0^{1,p}(U)$, the following inequality holds:

$$\int_{\Omega' \cap \{u \neq v\}} g_u^p \, d\mu \leq \int_{\Omega' \cap \{u \neq v\}} g_v^p \, d\mu.$$

Using now the tools of proof available in our setting, we may follow the previous approach in §2. The following facts are proven analogously.

- **Caccioppoli's inequality**: there exists a constant $C > 0$ so that for every p-energy minimizer $u \in N^{1,p}(U)$, every pair of balls $B(x, r) \subset B(x, R) \subset U$, and every level $h \in \mathbb{R}$, we have

$$\int_{B(x,r)} g_{(u-h)_+}^p \, d\mu \leq \frac{C}{(R - r)^p} \int_{B(x,R)} (u - h)_+^p \, d\mu.$$

- **Local boundedness**: there is a constant $C_1 > 0$ so that for every p-energy minimizer $u \in N^{1,p}(U)$ and every ball $B(x, 2r) \subset U$, we have

$$\sup_{B(x,r)} u \ \leq \ C_1 \Big\{ \fint_{B(x,2r)} u_+^p \, d\mu \Big\}^{\frac{1}{p}}.$$

- **(Weak) Harnack Inequality**: there exists $C_2 > 0$ so that for every positive p-energy minimizer $u \in N^{1,p}(U)$ and every $B(x, 2r) \subset U$, we have

$$C_2 \Big\{ \fint_{B(x,r)} u_+^p \, d\mu \Big\}^{\frac{1}{p}} \ \leq \ \inf_{B(x,2r)} u$$

  from which we conclude that

$$\sup_{B(x,\frac{r}{2})} u \ \leq \ \frac{C_1}{C_2} \Big\{ \inf_{B(x,2r)} u \Big\}.$$

- **Maximum Principle**: Let Ω be a domain in $\mathbb{R}^n$, If a p-energy minimizer $u : \Omega \to \mathbb{R}$ attains its maximum or minimum in the interior of Ω, then u is constant on Ω.

- **Hölder continuity**: There exist $C \geq 1$ and $\alpha > 0$ so that for all p-energy minimizers $u \in N^{1,p}(U)$ and all balls $B(x, r) \subset B(x, R) \subset U$, we have

$$\operatorname*{osc}_{B(x,r)} u \ \leq \ C \Big\{ \operatorname*{osc}_{B(x,R)} u \Big\} \Big(\frac{r}{R} \Big)^{\alpha}.$$

  In particular, every p-energy minimizer has an a.e. representative that is locally Hölder continuous, with exponent α.

Lastly, we note that similar results also hold for a larger class of functionals, with similar growth in g_u. For details, see [22].

3.4. Further regularity problems. As indicated before, there is no adequate notion of higher-order derivatives on metric measure spaces, even for the class of spaces that support Poincaré inequalities. Q questions of $C^{1,\alpha}$-regularity are therefore not well-posed in our setting. However, it does make sense to ask whether p-energy minimizers are locally *Lipschitz* continuous. (This corresponds to the case $\alpha = 1$ in the discussion above.)

The difficulty here is that on Euclidean spaces, standard proofs of Lipschitz continuity of p-harmonic functions (or other solutions of elliptic PDEs) require estimates for second-order derivatives. There are honest obstructions to this improved continuity. An example provided in [33, p. 150] shows a space that supports a 2-Poincaré inequality and a 2-harmonic function which fails to be locally Lipschitz continuous.

As a result, the positive results in the field are necessarily special cases. Only a few are known, and mostly in the "linear" case of $p = 2$.

- Petrunin [41] proved that 2-harmonic functions on Alexandrov spaces are also locally Lipschitz continuous. Roughly speaking, such spaces exhibit negative curvature in the sense of comparison with triangles, and Rajala [42] has previously shown that such spaces also support weak $(1, p)$-Poincaré inequalities.

- Koskela, Rajala, and Shanmugalingam [33] proved that if the space supports a 2-Poincaré inequality and a certain heat kernel estimate, then 2-harmonic functions (in the weak sense) are locally Lipschitz continuous. For the non-homogeneous case, this has recently been extended by Jiang [30].

As a clarification, we note that Cheeger [6] has proven a generalization of the classical Rademacher theorem — that Lipschitz functions on $\mathbb{R}^n$ are a.e. differentiable — in the setting of metric spaces that support Poincaré inequalities.

In particular, there is a well-defined *linear* differential map $f \mapsto Df$, with similar properties as the Euclidean gradient and that extends uniquely to functions in $N^{1,p}(X)$ as well. This leads to the formal definition of a weak solution.

Definition 3.10. Call $u \in N^{1,p}(U)$ a *weak solution* of the *p*-Laplace equation if

$$\int_U |Du|^{p-2} Du \cdot D\varphi \, d\mu$$

holds for for all Lipschitz functions $\varphi : U \to \mathbb{R}$ with compact support in U.

Lipschitz continuity is better understood for these functions than for *p*-energy minimizers. Indeed, Mäkäläinen [37] showed that for $p > 1$, the Lipschitz continuity of weak solutions to the *p*-Laplace equation is equivalent to certain growth properties of the non-homogeneous data, in terms of nonlinear potentials.

Though nontrivial, it remains a fact that weak solutions are *quasi-minimizers*, similarly defined as before, of the *p*-energy functional in Definition 3.9. Moreover, the comparison constant $K \geq 1$ depends on the parameters of our hypotheses; for details, see [6]. It remains unclear if the two notions are equal.

We close this survey with the case of *p*-energy minimizers, which remains open:

Question 3.11. *Let $p > 1$. On a metric space supporting a (weak) $(1, p)$-Poincaré inequality, what additional conditions are necessary or sufficient to guarantee local Lipschitz continuity of p-energy minimizers?*

References

[1] A. BELLAÏCHE, *The tangent space in sub-Riemannian geometry,* In Sub-Riemannian geometry, Progr. Math. 144, Birkhäuser, Basel, 1996, 1–76.

[2] A. BJÖRN, J. BJÖRN, and N. SHANMUGALINGAM, The Dirichlet problem for p-harmonic functions on metric spaces, *J. Reine Angew. Math.*, **556**(2003), 173–203.

[3] A. BJÖRN and N. MAROLA, Moser iteration for (quasi)minimizers on metric spaces, *Manuscripta Math.*, **121**(2006), 339–366.

[4] L. CAPOGNA, Regularity of quasi-linear equations in the Heisenberg group, *Comm. Pure Appl. Math.*, **50**(1997), 867–889.

[5] L. CAPOGNA, Regularity for quasilinear equations and 1–quasiconformal maps in Carnot groups, *Math. Ann.* **313**(1999), 263–295.

[6] J. CHEEGER, Differentiability of Lipschitz functions on metric measure spaces, *Geom. Funct. Anal.* **9**(1999), 428–517.

[7] J. CHEEGER and B. KLEINER, Generalized differential and bi-Lipschitz nonembedding in L^1. *C. R. Math. Acad. Sci. Paris*, **343**(2006), 297–301.

[8] M. CHRIST, A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral, *Colloq. Math.*, **60-61**(1990), 601–628.

[9] R. COIFMAN and G. WEISS, *Analyse harmonique non-commutative sur certains espaces homogenes,* Lecture Notes in Mathematics 242, Springer-Verlag, Berlin, 1971.

[10] R. COIFMAN and G. WEISS, Extensions of Hardy spaces and their use in analysis, *Bull. Amer. Math. Soc.* **83**(1977), 569–645.

[11] T. COLDING and W. MINICOZZI, Harmonic functions on manifolds, *Ann.of Math.* **146**(2)(1997), 725–747.

[12] G. DAVID and S. SEMMES, Quantitative rectifiability and Lipschitz mappings, *Trans. Amer. Math. Soc.* **337**(1993), 855–889.

[13] *E. De Giorgi*, Sulla differenziabilità e lanaliticità delle estremali degli integrali multipliregolari, *Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat.* **3**(3)(1957), 25–43.

[14] E. DIBENEDETTO, $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations, *Nonlinear Anal.* **7**(1983), 827–850.

[15] E. DIBENEDETTO, *Partial differential equations*, 2nd Ed., Cornerstones. Birkhäuser, Boston, 2010.

[16] E. DIBENEDETTO, U. GIANAZZA, and V. VESPRI, Local clustering of the non-zero set of functions in $W^{1,1}(E)$, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.*, **17**(2006), 223–225.

[17] A. DOMOKOS and J. MANFREDI, $C^{1,\alpha}$–regularity for p–harmonic functions in the Heisenberg group for p near 2, In: *The p-harmonic equation and recent advances in analysis,* Contemp. Math. 370, Amer. Math. Soc., Providence, RI, 2005, 17–23.

[18] L. EVANS and R. GARIEPY, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.

[19] C. FEFFERMAN and E. STEIN, H^p spaces of several variables, *Acta Math.* **129**(1972), 137–193.

[20] M. GIAQUINTA and E. GIUSTI, Quasiminima, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **1**(1984), 79–107.

[21] E. GIUSTI, *Direct methods in the calculus of variations*, World Scientific Publishing Co. Inc., River Edge, NJ, 2003.

[22] J. GONG, J. MANFREDI, and M. PARVIAINEN, Nonhomogeneous variational problems and quasiminimizers on metric spaces, *Manuscripta Math.* to appear.

[23] A. GRIGOR'YAN, The heat equation on noncompact Riemannian manifolds, *Mat. Sb.*, **182**(1991), 55–87.

[24] M. GROMOV, Carnot-Carathéodory spaces seen from within, In: *Sub-Riemannian geometry,* Progr. Math. 144, Birkhäuser, Basel, 1996, 79–323.

[25] P. HAJŁASZ, Sobolev spaces on metric-measure spaces, In: *Heat kernels and analysis on manifolds, graphs, and metric spaces,* Contemp. Math. 338, Amer. Math. Soc., Providence, RI, 2003, 173–218.

[26] P. HAJŁASZ and P. KOSKELA, *Sobolev met Poincaré,* Mem. Amer. Math. Soc. **145**, no. 688, 2000.

[27] J. HEINONEN, *Lectures on analysis on metric spaces*, Universitext, Springer-Verlag, New York, 2001.

[28] J. HEINONEN, Nonsmooth calculus, *Bull. Amer. Math. Soc.*, **44**(2007), 163–232.

[29] J. HEINONEN and P. KOSKELA, Quasiconformal maps in metric spaces with controlled geometry, *Acta Math.*, **181**(1998), 1–61.

[30] R. JIANG, Lipschitz continuity of solutions of poisson equations in metric measure spaces, arXiv preprint http://arxiv.org/pdf/1004.1101v1.

[31] S. KEITH and X. ZHONG, The Poincaré inequality is an open ended condition, *Ann. of Math.* **167**(2)(2008), 575–599.

[32] J. KINNUNEN and N. SHANMUGALINGAM, Regularity of quasi-minimizers on metric spaces, *Manuscripta Math.* **105**(2001), 401–423.

[33] P. KOSKELA, K. RAJALA, and N. SHANMUGALINGAM, Lipschitz continuity of Cheeger-harmonic functions in metric measure spaces, *J. Funct. Anal.* **202**(2003), 147–173.

[34] O. LADYŽENSKAJA and N. URALTSEVA, On the Hölder continuity of the solutions and the derivatives of linear and quasi-linear equations of elliptic and parabolic types, *Trudy Mat. Inst. Steklov.* **73**(1964), 172–220.

[35] J. LEE and A. NAOR, l_p metrics on the Heisenberg group and the Hoemans-Linial conjecture, *FOCS*(2006), 99–108.

[36] J. LEWIS, Regularity of the derivatives of solutions to certain degenerate elliptic equations, *Indiana Univ. Math. J.* **32**(1983), 849–858.

[37] T. MÄKÄLÄINEN, Removable sets for Hölder continuous p-harmonic functions on metric measure spaces, *Ann. Acad. Sci. Fenn. Math.* **33**(2008), 605–624.

[38] N. MEYERS and J. SERRIN, $H = W$, *Proc. Nat. Acad. Sci. U.S.A.* **51**(1964), 1055–1056.

[39] G. MINGIONE, A. ZATORSKA-GOLDSTEIN, and X. ZHONG, Gradient regularity for elliptic equations in the Heisenberg group, *Adv. Math.* **222**(2009), 62–129.

[40] J. MOSER, A new proof of De Giorgis theorem concerning the regularity problem for elliptic differential equations, *Comm. Pure Appl. Math.* **13**(1960), 457–468.

[41] A. PETRUNIN, Harmonic functions on Alexandrov spaces and their applications, *Electron. Res. Announc. Amer. Math. Soc.*, **9**(2003), 135–141.

[42] K. RAJALA, *Alexandrov spaces, upper gradients, and the Poincaré inequality,* Masters thesis, University of Jyväskylä, 2000.

[43] L. SALOFF-COSTE, Uniformly elliptic operators on Riemannian manifolds, *J. Differential Geom.* **36**(1992), 417–450.

[44] N. SHANMUGALINGAM, Newtonian spaces: an extension of Sobolev spaces to metric measure spaces, *Rev. Mat. Iberoamericana*, **16**(2000), 243–279.

[45] N. SHANMUGALINGAM, Harmonic functions on metric spaces, *Illinois J. Math.* **45**(2001), 1021–1050.

[46] K.-O. WIDMAN, Hölder continuity of solutions of elliptic systems, *Manuscripta Math.* **5**(1971), 299–308.

[47] S.-T. YAU, *Nonlinear analysis in geometry*, Monographies de LEnseignement Mathématique 33, Geneva, 1986.

[48] W. ZIEMER, *Weakly differentiable functions*, Graduate Texts in Mathematics 120, Springer-Verlag, New York, 1989.

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