Proceedings of the ICM2010 Satellite Conference
International Workshop on
Harmonic and Quasiconformal Mappings (HQM2010)
Editors: D. Minda, S. Ponnusamy, and N. Shanmugalingam
J. Analysis
Volume 18 (2010), 297–316

Harmonic Quasiconformal Mappings in Domains in $\mathbf{R}^n$

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Abstract. This a survey article on certain aspects of harmonic quasiconformal mappings in domains in $\mathbb{R}^n$. Emphasis is given on moduli of continuity results.

Keywords. Harmonic mappings, quasiconformal mappings, Hölder continuity.

2010 MSC. 30C65, 31B05.

1. Introduction

Harmonic quasiconformal (briefly, hqc) mappings in the plane were studied first by O. Martio in [23], today they are investigated both in the planar and the multidimensional setting from several different points of view. Among topics considered are: boundary behaviour, including Hölder and Lipschitz continuity and more general moduli of continuity, behavior with respect to natural metrics, especially quasihyperbolic metric, distortion estimates, bi-Lipschitz properties with respect to different metrics, characterization of boundary maps. Different tools are used: conformal moduli of curve families, Poisson kernels, estimates from the theory of second order elliptic operators, notions of capacity, subharmonic functions, Hilbert's transformation. Both theories of harmonic mappings and quasiconformal mappings are well developed, it is of interest to consider how these results can be strengthened in presence of both harmonicity and quasiconformality. Some of the results are unexpected and elegant, e.g. preservation of boundary modulus of continuity in the case of $\mathbb{B}^n$ ([3]), bi-Lipschitz property with respect to quasihyperbolic metric ($n = 2$) ([21]), Hölder continuity is preserved for uniformly perfect domains ([7]). For example, bi-Lipschitz properties of such mappings were studied extensively, see [15] and [26] for further references. Also, characterizations of the boundary mappings admitting hqc extension were given in [29] and [18].

ISSN 0971-3611

2. Quasiconformal mappings

Conformal invariance has played a predominant role in the study of geometric function theory during the past century. Some of the landmarks are the pioneering contributions of Grötzsch and Teichmüller prior to the Second World War, and the paper of Ahlfors and Beurling [2] in 1950. These results lead to farreaching applications and have stimulated many later studies [13]. For instance, Gehring and Väisälä [10], [33] have built the theory of quasiconformal mappings in $\mathbb{R}^n$ based on the notion of the modulus of a curve family introduced by Ahlfors and Beurling [2], which is an essential tool in investigation of quasiconformal mappings.

Let us consider a family Γ of curves in $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$. We say that a non-negative Borel measurable $\rho : \mathbb{R}^n \to \mathbb{R}$ is an admissible metric for Γ if

$$l_\rho(\gamma) = \int_\gamma \rho\, ds \geqslant 1 \text{ for each locally rectifiable } \gamma \in \Gamma.$$

Let $F(\Gamma)$ be the set of all admissible metrics for Γ. Finally, for each $p \geqslant 1$ we define p-modulus of Γ by

$$M_p(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_{\mathbb{R}^n} \rho^p dm.$$

The case $p = n$ is most important, in that case we write simply $M(\Gamma)$. $\frac{1}{M(\Gamma)}$ is the extremal length of Γ.

Definition 2.1. A domain $A \subset \overline{\mathbb{R}^n}$ is a ring if $C(A)$ has exactly two components.

If the components of $C(A)$ are C_0 and C_1, we denote $A = R(C_0, C_1)$, $B_0 = C_0 \cap \overline{A}$ and $B_1 = C_1 \cap \overline{A}$. To each ring $A = R(C_0, C_1)$, we associate the curve family $\Gamma_A = \Delta(B_0, B_1, A)$ and the capacity of A is defined by $\operatorname{cap} A = M(\Gamma_A)$. Next, the modulus of A is defined by $\operatorname{cap} A = \omega_{n-1}(\operatorname{mod} \mathrm{A})^{1-n}$.

Let $f : D \to D'$ be a homeomorphism. If Γ is a family of curves in D, then Γ' denotes the family $\{f \circ \gamma \,|\, \gamma \in \Gamma\}$ of curves in D'. We set

$$K_I(f) = \sup \frac{M(\Gamma')}{M(\Gamma)}, \quad K_O(f) = \sup \frac{M(\Gamma)}{M(\Gamma')},$$

where the suprema are taken over all families of curves $\Gamma \subset D$ such that $M(\Gamma)$ and $M(\Gamma')$ are not simultaneously 0 or ∞.

Note that both quantities are equal to one if f is conformal mapping.

Definition 2.2. If $f : D \to D'$ is a homeomorphism, $K_I(f)$ is the inner dilatation and $K_O(f)$ is the outer dilatation of f. The maximal dilatation of f is $K(f) = \max\{K_I(f), K_O(f)\}$. If $K(f) \leqslant K < \infty$, f is K-quasiconformal. f is quasiconformal (qc) if $K(f) < \infty$.

Equivalently, f is K-quasiconformal iff

$$\frac{M(\Gamma)}{K} \leqslant M(\Gamma') \leqslant K\, M(\Gamma),$$

for every family of curves Γ in D. This is the geometric definition of quasiconformal mappings.

Definition 2.3. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear bijection. The numbers

$$H_I(A) = \frac{|det(A)|}{l^n(A)}, \quad H_O(A) = \frac{|A|^n}{|det(A)|}, \quad H(A) = \frac{|A|}{l(A)},$$

where $l(A) = \inf_{||x||=1} ||Ax||$, are called the *inner*, *outer* and *linear dilatation* of A, respectively.

They have the following geometric interpretation: The image of the unit ball B^n under A is an ellipsoid $E(A)$. Let $B_I(A)$ and $B_O(A)$ be the inscribed and circumscribed balls of $E(A)$, respectively. Then

$$H_I(A) = \frac{m(E(A))}{m(B_I(A))} = \frac{a_1 \cdots a_{n-1}}{a_n^{n-1}},$$

$$H_O(A) = \frac{m(B_O(A))}{m(E(A))} = \frac{a_1^{n-1}}{a2 \cdots a_n}, \quad H(A) = \frac{a_1}{a_n},$$

where $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n$ are the semi-axes of $E(A)$.

Next we turn to a more interesting task: Finding conditions on quasiconformality in the case of a C^1 mapping in terms of its derivative. This is an analytic approach to quasiconformal mappings.

Module estimates will play crucial role here. In order to do that we define

$$H_O(f'(x)) = \frac{|(f'(x))|^n}{|J_f(x)|}, \qquad H_I(f'(x)) = \frac{|J_f(x)|}{l(f'(x))^n},$$

where $J_f(x) \neq 0$ is Jacobian of f.

Theorem 2.4. [33, Theorem 15.1] Suppose that $f : D \to D'$ is a diffeomorphism. Then

$$K_I(f) = \sup_{x \in D} H_I(f'(x)), \qquad K_O(f) = \sup_{x \in D} H_O(f'(x)).$$

Theorem 2.5. [33, Theorem 15.2] Let $f : D \to D'$ be a homeomorphism. If f is differentiable at a point $a \in D$ and if $K_O(f) < \infty$, then

$$|f'(a)|^n \leqslant K_O(f) \cdot |J_f(a)|.$$

Corollary 2.6. A diffeomorphism $f : D \to D'$ is K-qc iff the double inequality

$$\frac{|f'(x)|^n}{K} \leqslant |J_f(x)| \leqslant K \cdot l(f'(x))^n$$

holds for every $x \in D$.

Example 2.7. (Linear mapping) Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear bijection. Then $A'(x) = A$ for all $x \in \mathbb{R}^n$. From Theorem 2.4 we obtain

$$K_I(A) = H_I(A) \qquad K_O(A) = H_O(A).$$

Thus A is qc.

Example 2.8. (Stretching) Let $a \neq 0$ be a real number, and set $f(x) = |x|^{a-1}x$. Then from theorem 2.4 it follows that

$$K_I(f) = |a|, \qquad K_O(f) = |a|^{n-1} \qquad \text{if } |a| \geqslant 1,$$
$$K_I(f) = |a|^{1-n}, \qquad K_O(f) = |a|^{-1} \qquad \text{if } |a| \leqslant 1.$$

Thus f is qc. In Example 2.10 we prove the sharp Hölder estimate for f.

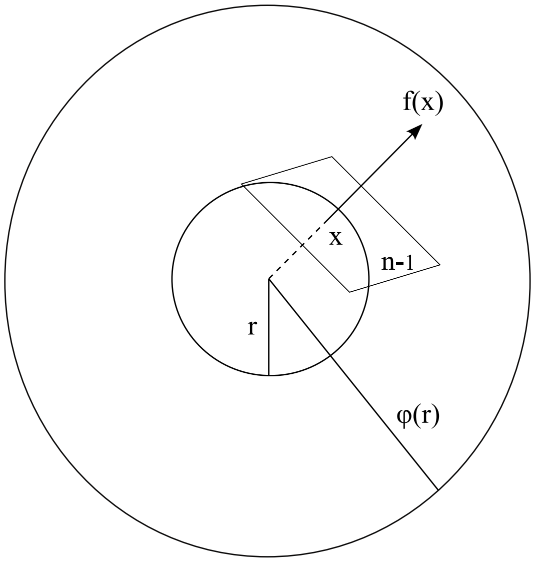

Figure 1. Radial mappings.

Example 2.9 (Radial mappings)**.** Now we consider radial mappings of a more general type:

$$f(x) = \varphi(|x|) \cdot \frac{x}{|x|},$$

where φ is continuously differentiable on $[0, +\infty)$, $\varphi'(r) > 0$ and $\varphi(0) = 0$ ([19]). Now we want to calculate the Jacobi matrix of f at a given point x. In fact, we are interested in $l(f'(x))$ and $|f'(x)|$. It is easier to work in a new rectangular coordinate system, one coordinate is along vector x, the other coordinates are in the tangent plane ($n-1$-dimensional hyperplane) to the sphere through x (see picture below).

Then we have

$$\left[\frac{\partial f}{\partial x}\right]_{i,j=1}^{n} = \begin{bmatrix} \varphi'(r) & & & 0 \\ & \frac{\varphi(r)}{r} & & \\ & & \ddots & \\ 0 & & & \frac{\varphi(r)}{r} \end{bmatrix}.$$

Indeed, the rate of stretching along the first coordinate axis is $\varphi'(r)$ by definition of f. In the tangent hyperplane we have a similarity transformation by a coefficient $\frac{\varphi(r)}{r}$ (see picture below), where Δr is "infinitesimal" displacement.

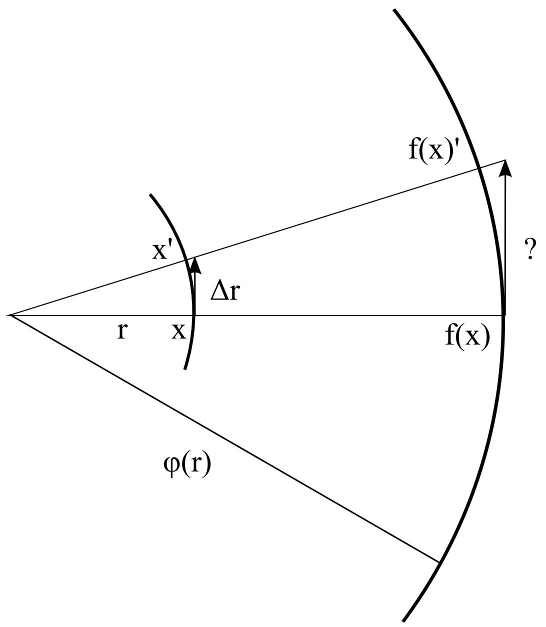

FIGURE 2. The stretching rate.

Hence

$$J_f(x) = \varphi'(r) \cdot \left(\frac{\varphi(r)}{r}\right)^{n-1} = \det f'(x),$$
$$l(f'(x)) = l\left(\frac{\partial f}{\partial x}\right) = \min\left\{\varphi'(r), \frac{\varphi(r)}{r}\right\},$$

$$|f'(x)| = \max\left\{\varphi'(r), \frac{\varphi(r)}{r}\right\},$$

$$H_O(f'(x)) = \frac{|f'(x)|^n}{J_f(x)} = \frac{\max\left\{\varphi'(r), \frac{\varphi(r)}{r}\right\}^n}{\varphi'(r)\,\left(\frac{\varphi(r)}{r}\right)^{n-1}}.$$

This is bounded iff $\sup_{r>0} \frac{\varphi'(r)\cdot r}{\varphi(r)} < +\infty$ and $\sup_{r>0} \frac{\varphi(r)}{r\cdot\varphi'(r)} < +\infty$.

$$H_I(f'(x)) = \frac{J_f(x)}{l(f'(x)^n} = \frac{\varphi'(r)\,\left(\frac{\varphi(r)}{r}\right)^{n-1}}{\min\left\{\varphi'(r), \frac{\varphi(r)}{r}\right\}^n}$$
$$= \varphi'(r)\,\left(\frac{\varphi(r)}{r}\right)^{n-1}\cdot \max\left\{\frac{1}{\varphi'(r)}, \frac{r}{\varphi(r)}\right\}^n.$$

This is bounded iff $\sup_{r>0} \frac{\varphi(r)}{r\cdot\varphi'(r)} < +\infty$ and $\sup_{r>0} \frac{\varphi'(r)\cdot r}{\varphi(r)} < +\infty$, which is the same condition we obtained for H_O.

Conclusion: f is qc-mapping iff

$$\sup_{r>0} \frac{\varphi'(r)\cdot r}{\varphi(r)} < +\infty \qquad \text{and} \qquad \sup_{r>0} \frac{\varphi(r)}{r\cdot\varphi'(r)} < +\infty.$$

One can write this result in a different form:

Let $\alpha : (0,1) \to (u,v)$, where $0 < u < v < 1$. For $f(x) = |x|^{\alpha(|x|)-1}\,x$ we have $\varphi(r) = r^{\alpha(r)}$ and

$$\frac{r\,\varphi'(r)}{\varphi(r)} = r\cdot\alpha'(r)\cdot\ln(r) + \alpha(r)$$

and the condition for qc-ty of f is that

$$r\cdot\alpha'(r)\cdot\ln(r) + \alpha(r)$$

is bounded above and bounded away from zero.

This is, of course, true if $\alpha(r) = \alpha \in (0,1)$. In that case

$$J_f = \begin{bmatrix} \alpha\cdot r^{\alpha-1} & & & 0 \\ & r^{\alpha-1} & & \\ & & \ddots & \\ 0 & & & r^{\alpha-1} \end{bmatrix}.$$

Since $0 < \alpha < 1$, $l = \alpha\cdot r^{\alpha-1}$, $|f'(x)| = r^{\alpha-1}$, $J_f(x) = \alpha\cdot(r^{\alpha-1})^n$ and

$$H_O(f'(x)) = \frac{(r^{\alpha-1})^n}{\alpha\cdot(r^{\alpha-1})^n} = \frac{1}{\alpha}$$

and

$$H_I(f'(x)) = \frac{\alpha \cdot (r^{\alpha-1})^n}{(\alpha \cdot r^{\alpha-1})^n} = \frac{1}{\alpha^{n-1}}.$$

Note that $H_I \geqslant H_O$, and we see that the constant of qc-ty is $K = \frac{1}{\alpha^{n-1}}$. From here we have $\alpha = K^{1/(1-n)}$. The constant $\alpha = K^{1/(1-n)}$ is the best possible exponent.

Example 2.10 (Hölder continuity of radial mappings)**.** ([9, 19, Appendix]) Let $f(x) = |x|^{\alpha-1}\, x$, $x \in \mathbb{R}^n$, $\alpha \in (0,1)$. We prove that f is Hölder continuous with exponent α, i.e.

$$\forall x, y \in \mathbb{R}^n \qquad |f(x) - f(y)| \leqslant C\, |x-y|^\alpha$$

with $C = 2^{1-\alpha}$. Note that this Hölder estimate is sharp, since equality holds for $x = -y$. See also [9, 19].

We write the function f in the following form

$$f(x) = |x|^\alpha \, \frac{x}{|x|}. \tag{2.11}$$

Without loss of generality we can assume that $|x| \geqslant |y| > 0$. Put $k = \frac{|x|}{|y|}$ and $\tilde{x} = \frac{x}{|x|}$, $\tilde{y} = \frac{y}{|y|}$. Then $k \geqslant 1$ and $|\tilde{x}| = |\tilde{y}| = 1$. By the equality (2.11) we need to prove

$$\left| |x|^\alpha \, \frac{x}{|x|} - |y|^\alpha \, \frac{y}{|y|} \right| \leqslant C\, |x-y|^\alpha.$$

By dividing previous inequality by $|y|^\alpha$ it follows that

$$|k^\alpha \tilde{x} - \tilde{y}| \leqslant C\, |k\, \tilde{x} - \tilde{y}|^\alpha.$$

This inequality is equivalent with

$$|k^\alpha \tilde{x} - \tilde{y}|^2 \leqslant C^2\, |k\, \tilde{x} - \tilde{y}|^{2\alpha}.$$

By definition of the inner product it is equivalent with

$$C^2 \geqslant \frac{k^{2\alpha} - 2\, k^\alpha \, \langle \tilde{x}, \tilde{y} \rangle + 1}{(k^2 - 2\, k \langle \tilde{x}, \tilde{y} \rangle + 1)^\alpha}. \tag{2.12}$$

Inequality

$$C^2 \geqslant \max \frac{k^{2\alpha} - 2\, k^\alpha \, \langle \tilde{x}, \tilde{y} \rangle + 1}{(k^2 - 2\, k \langle \tilde{x}, \tilde{y} \rangle + 1)^\alpha}$$

ensures (2.12). Because $|\langle \tilde{x}, \tilde{y} \rangle| \leqslant 1$, it is sufficient to prove that

$$f(k,t) = \frac{k^{2\alpha} - 2\, k^\alpha t + 1}{(k^2 - 2k\, t + 1)^\alpha} \leqslant 2^{2-2\alpha}$$

for any $t \in [-1,1]$. The function $f(k,t)$ can be written in the form

$$f(k,t) = \frac{(k^\alpha - 1)^2 + 2\,k^\alpha\,(1-t)}{((k-1)^2 + 2\,k\,(1-t))^\alpha}.$$

By differentiating $f(k,t)$ with respect to t we obtain

$$\frac{\partial f}{\partial t} = \frac{-2\,k\,((k^{\alpha+1}-1)(1-k^{\alpha-1}) + (1-\alpha)\,((k^\alpha-1)^2 + 2\,k^\alpha(1-t)))}{(k^2 - 2\,k\,t + 1)^{\alpha+1}} < 0.$$

Hence, $f(k,t)$ is decreasing as a function of t when $-1 \leqslant t \leqslant 1$. By this reason,

$$f(k,t) \leqslant f(k,-1) = \frac{k^{2\alpha} + 2\,k^\alpha + 1}{(k^2 + 2\,k + 1)^\alpha} = \left(\frac{k^\alpha + 1}{(k+1)^\alpha}\right)^2.$$

By concavity of function u^α follows

$$\frac{k^\alpha + 1}{(k+1)^\alpha} = \frac{\frac{k^\alpha+1}{2}}{\frac{(k+1)^\alpha}{2}} \leqslant \frac{\left(\frac{k+1}{2}\right)^\alpha}{\frac{(k+1)^\alpha}{2}} = 2^{1-\alpha}.$$

Consider a homeomorphism $f : D \to D'$. Suppose that $x \in D$, $x \neq \infty$ and $f(x) \neq \infty$. For each $r > 0$ such that $S^{n-1}(x,r) \subset D$ we set

(2.13) $$L(x,f,r) = \max_{|y-x|=r} |f(y) - f(x)|, \qquad l(x,f,r) = \min_{|y-x|=r} |f(y) - f(x)|.$$

Definition 2.14. The linear dilatation of f at x is the number

$$H(x,f) = \limsup_{r\to 0} \frac{L(x,f,r)}{l(x,f,r)}.$$

If $x = \infty$, $f(x) \neq \infty$, we define $H(x,f) = H(0, f\circ u)$ where u is the inversion $u(x) = \frac{x}{|x|^2}$. If $f(x) = \infty$, we define $H(x,f) = H(x, u\circ f)$.

Example 2.15. Mapping $f : B^n \to B^n$, $f(x) = |x|^{\alpha-1}x$ has $H(0,f) = 1$.

Theorem 2.16. [33, Theorem 22.3] *Suppose that $f : D \to D'$ is a homeomorphism such that one of the following conditions is satisfied for some finite K:*

1. $M(\Gamma_A) \leqslant K \cdot M(\Gamma'_A)$ *for all rings A such that* $\overline{A} \subset D$,
2. $K_O(f) \leqslant K$,
3. $K_I(f) \leqslant K$.

Then $H(x,f)$ is bounded by a constant which depends only on n and K.

3. Planar hqc mappings

In [23], Martio gave sufficient conditions for K-conformality of a harmonic mapping from the unit disk onto itself. He also posed the following question:

Question 1. If u is harmonic in the unit disk $\mathbb{D}$, and f is the boundary function on $\mathbb{T} = \partial\mathbb{D}$, find necessary and sufficient conditions that $\lim_{z\to\zeta} u_r(z)$ and $\lim_{z\to\zeta} u_\theta(z)$ exists at each $\zeta \in \mathbb{T}$.

In [29], Pavlović gave a characterization of the boundary function ensuring the quasiconformality of the boundary function.

Theorem 3.1. *Let u be a harmonic homeomorphism of $\mathbb{D}$. Then the following conditions are equivalent:*

(a) *u is qc;*
(b) *u is bi-Lipschitz in the euclidean metric;*
(c) *f is bi-Lipschitz and the Hilbert transformation of its derivative is in L^∞.*

In [18], the analogous result was proved for the half-plane. In [28], Partyka and Sakan gave explicit estimations of the bi-Lipschitz constants for u expressed by means of the maximal dilatation K of u and $|u^{-1}(0)|$. Additionally if $u(0) = 0$, they used Mori's theorem as in [29], to get asymptotically sharp estimates as $K \to 1$.

Definition 3.2. A mapping $f : (X, d_X) \to (Y, d_Y)$ is bi-Lipschitz if it is bijective and both f and f^{-1} are Lipschitz continuous.

Bi-Lipschitz property of harmonic quasiconformal mappings on the unit disc was investigated in [26]. In [21], the following theorem was proved.

Theorem 3.3. *Suppose D and D' are proper domains in $\mathbb{R}^2$. If $u : D \longrightarrow D'$ is K-qc and harmonic, then it is bilipschitz with respect to quasihyperbolic metrics on D and D'.*

The proof is based on the theorem of Astala and Gehring from [8].

In [26], Mateljević and Vuorinen extended Theorem 3.3 to domains in $\mathbb{R}^n$ under the hypothesis that $K < 2^{n-1}$ and u is of class $C^{1,1}$.

3.1. Moduli of continuity in Euclidean metric. It is well known that if f is a complex-valued harmonic function defined in a region G of the complex plane $\mathbb{C}$, then $|f|^p$ is subharmonic for $p \geq 1$, and that in the general case is not subharmonic for $p < 1$. However, if f is holomorphic, then $|f|^p$ is subharmonic for every $p > 0$. Here we consider k-quasiregular harmonic functions ($0 < k < 1$). We recall that a harmonic function is quasiregular if

$$|\bar{\partial} f(z)| \le k|\partial f(z)|, \qquad z \in G,$$

where

$$\bar{\partial} f(z) = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \quad \text{and} \quad \partial f(z) = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \qquad z = x + iy.$$

We prove that $|f|^p$ is subharmonic for $p \ge 4k/(1+k)^2 =: q$ as well as that the exponent q (< 1) is the best possible (see Theorem 3.4). The fact that $q < 1$ enables us to prove that if f is quasiregular in the unit disk $\mathbb{D}$ and continuous on $\overline{D}$, then $\tilde{\omega}(f, \delta) \le \text{const.}\omega(f, \delta)$, where $\tilde{\omega}(f, \delta)$ (respectively $\omega(f, \delta)$) denotes the modulus of continuity of f on $\mathbb{D}$ (respectively $\partial\mathbb{D}$); see Theorem 3.8.

Theorem 3.4. [20] *If f is a complex-valued k-quasiregular harmonic function defined on a region $G \subset \mathbb{C}$, and $q = 4k/(k+1)^2$, then $|f|^q$ is subharmonic. The exponent q is optimal.*

Recall that a continuous function u defined on a region $G \subset \mathbb{C}$ is subharmonic if for all $z_0 \in G$ there exists $\varepsilon > 0$ such that

$$u(z_0) \le \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{it})\, dt, \qquad 0 < r < \varepsilon, \tag{3.5}$$

If $u(z_0) = |f(z_0)|^2 = 0$, then (3.5) holds. If $u(z_0) > 0$, then there exists a neighborhood U of z_0 such that u is of class $C^2(U)$ (because the zeros of u are isolated), and then we may prove that $\Delta u \ge 0$ on U. Thus the proof reduces to proving that $\Delta u(z) \ge 0$ whenever $u(z) > 0$. In order to do this we will calculate Δu.

It is easy to prove that If $u > 0$ is a C^2 function defined on a region in $\mathbb{C}$, and $\alpha \in \mathbb{R}$, then next two statements holds

$$\Delta(u^\alpha) = \alpha u^{\alpha-1}\Delta u + \alpha(\alpha-1)u^{\alpha-2}|\nabla u|^2, \tag{3.6}$$

$$|\nabla u|^2 = 4|\partial u|^2 \quad \text{and} \quad \Delta u = 4\partial\bar{\partial} u. \tag{3.7}$$

For a continuous function $f : \overline{\mathbb{D}} \to \mathbb{C}$ harmonic in $\mathbb{D}$ we define two moduli of continuity:

$$\omega(f, \delta) = \sup\{|f(e^{i\theta}) - f(e^{it})| : |e^{i\theta} - e^{it}| \le \delta,\ t, \theta \in \mathbb{R}\}, \quad \delta \ge 0,$$

and

$$\tilde{\omega}(f, \delta) = \sup\{|f(z) - f(w)| : |z - w| \le \delta,\ z, w \in \overline{\mathbb{D}}\}, \quad \delta \ge 0.$$

Clearly $\omega(f,\delta) \le \tilde{\omega}(f,\delta)$, but the reverse inequality need not hold. To see this consider the function

$$f(re^{i\theta}) = \sum_{n=1}^{\infty} \frac{(-1)^n r^n \cos n\theta}{n^2}, \qquad re^{i\theta} \in \overline{\mathbb{D}}.$$

This function is harmonic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. The function $v(\theta) = f(e^{i\theta})$, $|\theta| < \pi$, is differentiable, and

$$\begin{aligned} \frac{dv}{d\theta} &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin n\theta}{n} \\ &= \frac{\theta}{2}, \qquad |\theta| < \pi. \end{aligned}$$

This formula is well known, and can be verified by calculating the Fourier coefficients of the function $\theta \to \theta/2,\ |\theta| < \pi$. It follows that

$$|f(e^{i\theta}) - f(e^{it})| \le (\pi/2)|\theta - t|, \qquad -\pi < \theta,\, t < \pi,$$

and hence $\omega(f,\delta) \le M\delta,\ \delta > 0$, where M is an absolute constant. On the other hand, the inequality $\tilde{\omega}(f,\delta) \le CM\delta$, $C = \text{const}$, does not hold because it implies that $|\partial f/\partial r| \le CM$, which is not true because

$$\frac{\partial}{\partial r} f(re^{i\theta}) = \sum_{n=1}^{\infty} \frac{r^{n-1}}{n}, \quad \text{for } \theta = \pi,\ 0 < r < 1.$$

However, as was proved by Rubel, Shields and Taylor [30], and Tamrazov [32], if f is a holomorphic function, then $\tilde{\omega}(f,\delta) \le C\omega(f,\delta)$, where C is independent of f and δ. Here is an extension that result to quasiregular harmonic functions.

Theorem 3.8. [20] *Let f be a k-quasiregular harmonic complex-valued function which has a continuous extension on $\overline{\mathbb{D}}$, then there is a constant C depending only on k such that $\tilde{\omega}(f,\delta) \le C\omega(f,\delta)$.*

This theorem was deduced from Theorem 3.4 by use of some simple properties of the modulus $\omega(f,\delta)$. Let

$$\omega_0(f,\delta) = \sup\{|f(e^{i\theta}) - f(e^{it})| : |\theta - t| \le \delta,\ t,\theta \in \mathbb{R}\}.$$

It is easy to check that

$$C^{-1}\omega_0(f,\delta) \le \omega(f,\delta) \le C\omega_0(f,\delta), \tag{3.9}$$

where C is an absolute constant, and that

$$\omega_0(f,\delta_1+\delta_2) \le \omega_0(f,\delta_1) + \omega_0(f,\delta_2), \quad \delta_1,\, \delta_2 \ge 0.$$

Hence, $\omega_0(f, 2^n\delta) \leq 2^n\omega_0(f,\delta)$, and hence $\omega_0(\lambda\delta) \leq 2\lambda\omega_0(\delta)$, for $\lambda \geq 1, \delta \geq 0$. From these inequalities and (3.9) it follows that

$$\omega(f,\lambda\delta) \leq 2C\lambda\omega(f,\delta), \quad \lambda \geq 1, \delta \geq 0, \tag{3.10}$$

and

$$\omega(f,\delta_1+\delta_2) \leq C\omega(f,\delta_1) + C\omega(f,\delta_2), \quad \delta_1,\, \delta_2 \geq 0, \tag{3.11}$$

where C is an absolute constant. As a consequence of (3.10) we have, for $0 < p < 1$,

$$\int_x^\infty \frac{\omega(f,t)^p}{t^2}\,dt \leq C\frac{\omega(f,x)^p}{x}, \quad x > 0, \tag{3.12}$$

where C depends only on p.

4. HQC mappings in domains in $\mathbb{R}^n$

Let $B^n = \{x \in \mathbb{R}^n : |x| < 1\}$ and $d\sigma$ is the normalized surface measure on the unit sphere S^{n-1}. Let us consider the following question:

Question 2. If u is a hqc mapping on the unit ball and $u|_{S^{n-1}}$ has some continuity property, does it follow that u has the same property on the ball?

If this property is Lipschitz continuity, the answer is affirmative [4, 6]. If this property is the modulus of continuity, the answer is also affirmative [3].

It is known, even for $n = 2$, that Lipschitz continuity of $\phi : T \to C$, where $T = \{z \in C : |z| = 1\}$, does not imply Lipschitz continuity of $u = P[\phi]$.

Here, for any $n \geq 2$,

$$P[\phi](x) = \int_{S^{n-1}} P(x,\xi)\phi(\xi)d\sigma(\xi), \;\; x \in B^n$$

where $P(x,\xi) = \frac{1-|x|^2}{|x-\xi|^n}$ is the Poisson kernel for the unit ball, and $\phi : S^{n-1} \to R^n$ is a continuous mapping.

It was shown in [4] that Lipschitz continuity is preserved by harmonic extension, if the extension is quasiregular. The analogous statement is true for Hölder continuity without assumption of quasiregularity.

Theorem 4.1. *Assume $\phi : S^{n-1} \to R^n$ satisfies a Lipschitz condition:*

$$|\phi(\xi) - \phi(\eta)| \leq L|\xi - \eta|, \;\; \xi, \eta \in S^{n-1}$$

and assume $u = P[\phi] : B^n \to R^n$ is K-quasiregular. Then

$$|u(x) - u(y)| \leq C'|x - y|, \;\; x, y \in B^n$$

where C' depends on L, K and n only.

In [6], the authors gave the estimate $L_u \le KL_\phi$, where L_u, L_ϕ denote the Lipschitz constants of u, ϕ, respectively. Also in [6], **Question 1** is solved and extended to the n-dimensional case.

4.1. Moduli of continuity of hqc mappings in higher dimensions. The following problem is considered: can one control the modulus of continuity ω_u of a harmonic quasiregular (briefly, hqr) mapping u in $\mathbb{B}^n$ by the modulus of continuity ω_f of its restriction to the boundary $\mathbb{S}^{n-1}$, i.e. is it true that $\omega_u \le C\omega_f$?

In fact this problem has been studied extensively for harmonic functions and mappings without assumption of quasiregularity. We recall some of known results. For the unit ball the answer is positive in the case $\omega(\delta) = \delta^\alpha$, $0 < \alpha < 1$ (Hölder continuity) and negative in the case $\omega(\delta) = L\delta$ (Lipschitz continuity). In fact, for bounded plane domains the answer is always negative for Lipschitz continuity (see [1]). However, it is proved in [11], that for general plane domains one has "logarithmic loss of control": $\omega_u(\delta) \le C\omega_f(\delta)\log(1/\delta)$. In [3], the following results were proved.

Theorem 4.2. *There is a constant $q = q(K, n) \in (0, 1)$ such that $|u|^q$ is subharmonic in $\Omega \subset \mathbb{R}^n$ whenever $u : \Omega \to \mathbb{R}^n$ is a K-quasiregular harmonic map.*

Theorem 4.3. *If $u : \overline{\mathbb{B}}^n \to \mathbb{R}^n$ is a continuous map which is K-quasiregular and harmonic in $\mathbb{B}^n$, then $\omega_u(\delta) \le C\omega_f(\delta)$ for $\delta > 0$, where $f = u|_{\mathbb{S}^{n-1}}$ and C is a constant depending only on K, ω_f and n.*

In the case $n = 2$, these theorems were proved by using properties of analytic functions [20]. In [3], Theorem 4.2 was proved by using a linear algebra extremal problem. In [17], the optimal constant q is find in simple way.

We note that every every continuous map $u : \overline{\mathbb{B}}^n \to \overline{\Omega}$ which is hqc in $\mathbb{B}^n$, where Ω is bounded and has C^2 boundary, is Lipschitz continuous, see [16]. Also, every holomorphic quasiregular mapping on a domain $\Omega \subset \mathbb{C}^n$ $(n > 1)$ with C^2 boundary is Lipschitz continuous, see [31]. The same paper contains an example of a holomorphic quasiregular map in a domain $\Omega \subset \mathbb{C}^2$ (with nonsmooth boundary) which is not Lipschitz.

In view of the above, one is tempted to make the following conjecture: every hqc map $u : \overline{\mathbb{B}^n} \to \Omega$ is Lipschitz continuous. However, this is false, as we show by an example for $n = 3$ given in [3] by V. Božin.

Example 4.4. We use the following notation: $X = (x, y, z)$, $\Pi^+ = \{(x, y, z) : z > 0\}$. We construct a mapping $f : \overline{\Pi^+} \to \mathbb{R}^3$ such that

1. f is continuous on $\overline{\Pi^+}$.
2. f is not Lipschitz on $L = \{(0, 0, z) : 0 \le z \le 1\}$.

3. f is hqc on Π^+.

Of course, then the same is true for the restriction of f to the closed unit ball centered at $(0,0,1)$.

Set $g(X) = X/|X|^3$, this mapping is, up to a constant, the gradient of a harmonic function $1/|X|$, and therefore harmonic for $X \neq 0$. We have

$$Dg(X) = \frac{1}{|X|^3}(I - 3U_X^T \cdot U_X),$$

where $U_X = X/|X|$. Note that $|g(X)| \leq 1/|X|^2$ and $\|Dg(X)\| \leq C_0/|X|^3$. Now set

$$f(X) = f_0(X) + \sum_{n=1}^{\infty} f_n(X),$$

where $f_0(X) = (x, y, -2z)$, $f_n(X) = c_n g(X - X_n)$ and $X_n = (0, 0, -r_n)$. We are going to show that one can choose sequences r_n and c_n such that the above three conditions are satisfied, for the moment we require that they are strictly positive and that $\lim_{n\to\infty} r_n = 0$ monotonically.

We claim that the condition

$$\sum_{n=1}^{\infty} \frac{c_n}{r_n^2} < +\infty \qquad (C)$$

is sufficient for continuity up to the boundary. Indeed, for every $X \in \overline{\Pi^+}$,

$$\sum_{n=1}^{\infty} |f_n(X)| = \sum_{n=1}^{\infty} \frac{c_n}{|X - X_n|^2} \leq \sum_{n=1}^{\infty} \frac{c_n}{r_n^2} < +\infty$$

and therefore the series $\sum_{n=1}^{\infty} f_n(X)$ converges absolutely and uniformly on $\overline{\Pi^+}$.

Next, the condition

$$\sum_{n=1}^{\infty} \frac{c_n}{r_n^3} = +\infty \qquad (NL)$$

is sufficient for property 2. In fact, for $X = (0,0,z) \in \Pi^+$ we have $\frac{\partial}{\partial z} f_n(X) = -\frac{2c_n}{(z+r_n)^3} e_3$, where $e_3 = (0,0,1)$ and

$$\frac{\partial}{\partial z} f(X) = -\left[2 + \sum_{n=1}^{\infty} \frac{c_n}{(z + r_n)^3}\right] \cdot e_3.$$

Hence, if (NL) holds, then $|\frac{\partial}{\partial z} f(0,0,z)| \to \infty$ as $z \to 0$ and this implies that f is not Lipschitz continuous on L.

Now we look for a sufficient condition for quasiconformality. First note that a matrix $A \in M_3(\mathbb{R})$, $A \neq 0$ is K-quasiconformal iff its normalization $\tilde{A} = A/\|A\|$

is K-quasiconformal. Also, for each compact $H \subset GL_3(\mathbb{R})$ there is a $K \geq 1$ such that every $A \in H$ is K-quasiconformal.

We omit an easy proof of the following lemma, which is true for vectors in any real inner product space, although for our purposes $M_3(\mathbb{R}) \cong \mathbb{R}^9$ is the only case of interest.

Lemma 4.5. *Let $A \in M_3(\mathbb{R})$, $A \neq 0$ and $0 < \epsilon < \sqrt{2}$. Then for any matrices $B_1, \dots, B_k$ satisfying $\|\tilde{A} - \tilde{B}_j\| < \epsilon$, $1 \leq j \leq k$ we have $\|\tilde{A} - \tilde{B}\| < \epsilon$, where $B = B_1 + \cdots + B_k$.*

Let $A_0 = \operatorname{diag}(1, 1, -2) = Df_0$. Choose $0 < \delta < \sqrt{2}$ such that $H = \{B : \|\tilde{A}_0 - B\| \leq \delta\}$ is a compact subset of $GL_3(\mathbb{R})$. Hence, there is a $K_0 > 1$ such that every $B \in H$ is a K_0-qc matrix.

Set $A_X = I - 3U_X^T \cdot U_X$, $X \in \overline{\Pi^+}$, $X \neq 0$. Of course, $A_X = A_{U_X}$. Clearly, A_X is continuous in X and $A_{(0,0,1)} = A_0$. Hence there is an $\epsilon > 0$ such that $\|\tilde{A}_X - \tilde{A}_0\| < \delta$ whenever $|X - e_3| < \epsilon$. Equivalently, there is an $\eta > 0$ such that $\tan \angle(e_3, X) < \eta$ implies $\|\tilde{A}_X - \tilde{A}_0\| < \delta$ or, in coordinates, $\|\tilde{A}_X - \tilde{A}_0\| < \delta$ whenever $X = (x, y, z) \in \overline{\Pi^+}$ satisfies $\frac{\sqrt{x^2+y^2}}{z} < \eta$.

Next choose $\beta > 0$ such that $\|A_0 - B\| \leq \beta$ implies $\|\tilde{A}_0 - \tilde{B}\| < \delta$.

Now we can show that the condition

$$2\sqrt{2}C_0 \sum_{r_n \leq \frac{\rho}{\eta}} \frac{c_n}{(r_n + \rho)^3} \leq \beta \quad \text{for all} \quad \rho > 0 \qquad (QC_1)$$

is sufficient for quasiconformality of f. First note that, for $z > 0$, $Df(0, 0, z)$ is a constant multiple of A_0. Now consider $X = (x, y, z) \in \Pi^+$ with $\rho = \sqrt{x^2 + y^2} > 0$. Then

$$\begin{aligned} Df(X) &= Df_0 + \sum_{n=1}^{\infty} Df_n(X) \\ &= A_0 + \sum_{n=1}^{\infty} c_n A_{X - X_n} \\ &= A_0 + \sum_{r_n \leq \frac{\rho}{\eta}} c_n A_{X - X_n} + \sum_{r_n > \frac{\rho}{\eta}} c_n A_{X - X_n} \\ &= A_0 + R + T \end{aligned}$$

The sum T is finite (possibly empty) and for each term in that sum we have

$$\frac{\sqrt{x^2 + y^2}}{z + r_n} \leq \frac{\rho}{r_n} \leq \eta.$$

Hence, $\|\tilde{A}_{X-X_n} - \tilde{A}_0\| < \delta$ for each term in that sum. Now we estimate norm of R:

$$\begin{aligned}
\|R\| &\leq \sum_{r_n \leq \frac{\rho}{\eta}} c_n \|A_{X-X_n}\| \\
&\leq C_0 \sum_{r_n \leq \frac{\rho}{\eta}} \frac{c_n}{|X - X_n|^3} \\
&= C_0 \sum_{r_n \leq \frac{\rho}{\eta}} \frac{c_n}{[\rho^2 + (z + r_n)^2]^{3/2}} \\
&\leq C_0 \sum_{r_n \leq \frac{\rho}{\eta}} \frac{c_n}{(\rho^2 + r_n^2)^{3/2}} \\
&\leq 2\sqrt{2} C_0 \sum_{r_n \leq \frac{\rho}{\eta}} \frac{c_n}{(r_n + \rho)^3} \\
&\leq \beta.
\end{aligned}$$

Hence $\|(A_0+R)-A_0\| \leq \beta$ and therefore $\|\tilde{A}_0 - \widetilde{A_0 + R}\| < \delta$. Now we see that $Df(X)$ can be represented as a sum of terms satisfying assumptions of Lemma, therefore $\|\widetilde{Df(X)} - \tilde{A}_0\| < \delta$, hence $\widetilde{Df(X)} \in H$. So, $\widetilde{Df(X)}$ is a K_0-qc matrix and so is $Df(X)$. Since it is easy to verify that f is one-to-one it follows that f is a K_0-quasiconformal map.

It is easily seen that the condition (QC_1) is equivalent to the following one:

$$\sum_{k=n}^{\infty} c_k \leq M r_n^3 \quad \text{for all } n \geq 1 \qquad (QC)$$

where M is constant depending on η, C_0 and β. However, the exact value of M is of no importance. Namely, once we have sequences r_n and c_n satisfying (C), (NL) and (QC) for some M, it suffices to multiply c_n with suitably small constant to get M as small as desired, since conditions (C) and (NL) are invariant under such change of c_n.

Observe that the sequences $r_n = 2^{-2^n/3}$ and $c_n = 2^{-2^n}$ satisfy the conditions (C), (NL) and (QC), hence there is a non-Lipschitz qhc map on $\mathbb{B}^n$ continuous up to the boundary.

As concluding remark we note that an analogous construction can be carried in any dimension $k \geq 2$. Also, by multiplying the z-component of our function by a factor $-1/2$ and taking a tail of the series $\sum_{n=1}^{\infty} f_n(X)$, one can get the constant of quasiconformality as close to 1 as desired.

5. Moduli of continuity of quasiconformal mappings in higher dimensions

Clearly, for general quasiconformal mappings $u : \Omega_1 \to \Omega_2$ one can not expect that the modulus of continuity behaves as in the above theorem, even for $\Omega_1 = \mathbb{B}^n$. However, for bounded Ω_2, Hölder continuity of $u|_{\partial\Omega_1}$ implies Hölder continuity of u, but with possibly different Hölder exponent, see [27] and [24].

The following theorem is the main result in [24].

Theorem 5.1. *Let D be a bounded domain in $\mathbb{R}^n$ and let f be a continuous mapping of $\overline{D}$ into $\mathbb{R}^n$ which is quasiconformal in D. Suppose that, for some $M > 0$ and $0 < \alpha \leq 1$,*

$$|f(x) - f(y)| \leq M|x - y|^\alpha \tag{5.2}$$

whenever x and y lie on ∂D. Then

$$|f(x) - f(y)| \leq M'|x - y|^\beta \tag{5.3}$$

for all x and y on $\overline{D}$, where $\beta = \min(\alpha, K_I^{1/(1-n)})$ and M' depends only on M, α, n, $K(f)$ and $\operatorname{diam}(D)$.

The exponent β is the best possible, as an example of a radial quasiconformal map $f(x) = |x|^{\alpha-1}x$, $0 < \alpha < 1$, of $\overline{\mathbb{B}^n}$ onto itself shows (see [33], p. 49). Also, the assumption of boundedness is essential. Indeed, one can consider $g(x) = |x|^a x$, $|x| \geq 1$ where $a > 0$. Then g is quasiconformal in $D = \mathbb{R}^n \setminus \overline{\mathbb{B}^n}$ (see [33], p. 49), it is identity on ∂D and hence Lipschitz continuous on ∂D. However, $|g(te_1) - g(e_1)| \asymp t^{a+1}$, $t \to \infty$, and therefore g is not globally Lipschitz continuous on D.

P. Koskela posed the following question:

Question 3. Is it possible to replace β with α if we assume, in addition to quasiconformality, that f is harmonic?

In the special case $D = \mathbb{B}^n$ this was proved, for arbitrary moduli of continuity $\omega(\delta)$, in [3]. Our main result is that the answer is positive, if ∂D is a uniformly perfect set (cf. [12]). In fact, we prove a more general result, including domains having a thin, in the sense of capacity, portion of the boundary. However, this generality is in a sense illusory, because any hqc mapping extends harmonically and quasiconformally across such portion of the boundary.

In the case of smooth boundaries much better regularity up to the boundary can be deduced, see [14]; related results for harmonic functions were obtained by [1].

A compact set $K \subset \mathbb{R}^n$, consisting of at least two points, is α-uniformly perfect ($\alpha > 0$) if there is no ring R separating K (i.e. such that both components of $\mathbb{R}^n \setminus R$ intersect K) such that $\mathrm{mod}(R) > \alpha$. We say that a compact $K \subset \mathbb{R}^n$ is uniformly perfect if it is α-uniformly perfect for some $\alpha > 0$.

Here D denotes a bounded domain in $\mathbb{R}^n$. Let

$$\Gamma_0 = \{x \in \partial D : \mathrm{cap}\, \overline{B}(x, \epsilon) \cap \partial D = 0 \text{ for some } \epsilon > 0\},$$

and $\Gamma_1 = \partial D \setminus \Gamma_0$. The following result is proved in [7].

Theorem 5.4. *Assume $f : \overline{D} \to \mathbb{R}^n$ is continuous on $\overline{D}$, harmonic and quasiconformal in D. Assume f is Hölder continuous with exponent α, $0 < \alpha \leq 1$, on ∂D and Γ_1 is uniformly perfect. Then f is Hölder continuous with exponent α on $\overline{D}$.*

If Γ_0 is empty we obtain the following

Corollary 5.5. *If $f : \overline{D} \to \mathbb{R}^n$ is continuous on $\overline{D}$, Hölder continuous with exponent α, $0 < \alpha \leq 1$, on ∂D, harmonic and quasiconformal in D and if ∂D is uniformly perfect, then f is Hölder continuous with exponent α on $\overline{D}$.*

6. Open problems

1. Is Hölder continuity on the boundary preserved for hqc mappings in any bounded domain?
2. Generalize characterizations from [18] and [29] to $n > 2$.
3. Characterize moduli of continuity of boundary values of functions in $hqc(\mathbb{B}^n)$. Note that these include non-Lipschitz ones.
4. Is any hqc mapping bi-Lipschitz continuous with respect to quasihyperbolic metric in bounded domains in $\mathbb{R}^n$ for $n > 2$?

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