Two-dimensional Picture Arrays and Parikh $q-$Matrices

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Abstract. Based on the idea of "$q-$count" of certain subwords of a word and generalizing the notion of Parikh matrix of a word, the notion of Parikh $q-$matrix of a word over an ordered alphabet was introduced. On the other hand, with a two-dimensional picture array of symbols arranged in rows and columns, two kinds of upper triangular matrices, known as row and column Parikh matrices have also been introduced and investigated. Here combining these two kinds of matrices of a picture array, we introduce row/column Parikh $q-$matrix of an array, leading to the concept of $q-$ambiguity of a picture array. Results relating to $q-$ambiguity of picture arrays are derived in the context of these Parikh $q-$matrices of arrays.

1. Introduction

Extending the notion of the classical Parikh vector [12], which gives a count of each distinct symbol in a word, Mateescu et al. [11] introduced the concept of Parikh matrix mapping, which associates with a word w over an ordered alphabet, a matrix, called Parikh matrix of the word w. This matrix gives more numerical information in terms of scattered subwords or simply called, subwords of the word w and includes the Parikh vector in the second diagonal above the main diagonal. It is known that the Parikh matrix mapping is not injective. Yet there are words whose Parikh vectors are the same but these words have distinct Parikh matrices.

Generalizing the concept of Parikh matrix mapping, Parikh $q-$matrix mapping was introduced in [7]. This mapping which is a morphism, associates with a word $w \in \Sigma_k^*$, a $k \times k$ upper-triangular matrix, where Σ_k is an ordered alphabet with k symbols. The entries in this matrix are non-negative polynomials with integer coefficients in a variable q and the polynomials represent a "$q-$count" of certain subwords of the word w. It is shown in [7] that the Parikh $q-$matrix mapping reduces to the Parikh matrix mapping when $q = 1$, by appropriately embedding the $k-$letter alphabet into a $(k + 1)-$letter alphabet. Injectivity is not possessed by the Parikh $q-$matrix mapping also but there are instances when two words have the same Parikh matrix but their Parikh $q-$matrices are different.

On the other hand a two-dimensional rectangular digitized picture array or simply an array, is an arrangement in rows and columns of symbols from a finite alphabet. Several combinatorial properties of arrays (or also called two-dimensional words) have been investigated in the literature (see, for example, [4, 5, 6, 9]). In [14], two kinds of matrices, known as row Parikh matrix and column Parikh matrix of a picture array, are defined and these matrices extend the notion of Parikh matrix from words to arrays. The notion of $M-$ambiguity of a picture array is thus introduced in [14] by considering two picture arrays to be $M-$ equivalent if their row Parikh matrices are the same and their column Parikh matrices are the same and in particular, conditions that ensure $M-$ambiguity in binary and ternary words are obtained.

Here, based on the notion of $q-$Parikh matrix [7] and row/column Parikh matrices [14] of a picture array, we associate with a picture array, two types of matrices. We call these as row Parikh $q-$matrix and column Parikh $q-$matrix, leading to the notions of $q-$row and $q-$column equivalences of two picture arrays. The notion of $q-$ambiguity of a picture array A is then introduced by defining array A to be $q-$ambiguous if there is another picture array B which is $q-$row and $q-$column equivalent to A. Several properties relating to $q-$ambiguity including conditions for $q-$ambiguity of row or column products for binary picture arrays are derived.

2. Preliminaries

A word $w = w_1 w_2 \dots w_n \quad (n \geq 1)$ over a finite alphabet Σ, is a sequence of symbols $w_i \in \Sigma$ for $1 \leq i \leq n$. The set Σ^* is the collection of all words that can be formed over Σ and contains the empty word λ with no symbols. The notation $|w|$ denotes the length of the word w. The number of occurrences of a symbol a in a word w is denoted by $|w|_a$. For any word $w = w_1 w_2 \dots w_n$, we denote by w^t the word w written vertically. For example, if $w = aab$ over $\{a, b\}$, then w^t is $\begin{matrix} a \\ a \\ b \end{matrix}$. A word $u = x_1 x_2 \cdots x_n$ over an alphabet Σ with $x_i \in \Sigma$ for $1 \leq i \leq n$, is called a subword of w, if $w = v_1 x_1 v_2 x_2 \cdots v_{n-1} x_{n-1} v_n$ where $v_i \in \Sigma^*$. Also, the count of distinct occurrences of a word u as a subword in the word w is denoted by $|w|_u$. For other notions related to subwords of a word and Parikh matrix of a word, we refer to [11]. The notion of Parikh q-matrix mapping introduced in [7], is now recalled. Let $\mathscr{N}[q]$ be the collection of polynomials in the variable q with coefficients from the set $\mathscr{N}$ of all non-negative integers. Let $\mathscr{M}_k[q]$ be the set of $k \times k$ upper triangular matrices having entries in $\mathscr{N}[q]$. Throughout this paper, for $k \geq 1$, the ordered alphabet $\Sigma_k = \{a_1 < a_2 < \cdots < a_k\}$. When we use the alphabet $\{a < b\}$ or $\{a < b < c\}$, it is understood that $a = a_1, b = a_2$ and $c = a_3$.

Definition 1 *[7] The Parikh q-matrix mapping ψ_q is the morphism $\psi_q : \Sigma_k^* \longrightarrow \mathscr{M}_k[q]$ defined as follows: $\psi_q(\lambda) = I_k$, with I_k the $k \times k$ identity matrix and $\psi_q(a_j) = (m_{rs})_{1 \leq r,s \leq k}$ where*
(i) $m_{jj} = q$
(ii) $m_{ii} = 1$, for $1 \leq i \leq k$ and $i \neq j$
(iii) $m_{j(j+1)} = 1$, if $j < k$
and all other entries of $\psi_q(a_j)$ are zero.

Extending the mapping ψ_q from Σ_k to Σ_k^* we have $\psi_q(w_1 w_2 \cdots w_n) = \psi_q(w_1)\psi_q(w_2) \cdots \psi_q(w_n)$, for $w_i \in \Sigma_k$ where the operation on the right side is the usual matrix multiplication.

Example 1 *Consider the word $w = abaac$ over $\Sigma_3 = \{a < b < c\}$, setting $a_1 = a, a_2 = b, a_3 = c$. The Parikh q-matrix of w is given by*

$$\psi_q(w) = \begin{pmatrix} q & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & q & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} q & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} q & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q \end{pmatrix} = \begin{pmatrix} q^3 & 2q+q^2 & q \\ 0 & q & q \\ 0 & 0 & q \end{pmatrix}.$$

The Parikh q-matrix of a word w over Σ_k coincides with the usual Parikh matrix, when the q-matrix is evaluated at $q = 1$, treating the word w as a word over Σ_{k+1}.

Theorem 1 *([7]) Suppose $w \in \Sigma_k^*$. Consider w as a word over Σ_{k+1} and let $\psi_q(w)$ be the Parikh q-matrix in $\mathscr{M}_{k+1}[q]$. Then $\psi_q(w)$ evaluated at $q = 1$ is the Parikh matrix $\psi_{\mathscr{M}_k}(w)$.*

A notion called partial sum of two Parikh q-matrices was introduced in [1], motivated by a corresponding notion in [10] considered for words. This notion is now recalled.

Definition 2 *([1]) For two words x and y over Σ_k, let M_1 and M_2 be the corresponding Parikh $q-$matrices. Define $M_1 \oplus M_2 = M_3 = (c_{ij})_{k\times k}$ where c_{ij} is the usual sum of the corresponding entries of the matrices M_1 and M_2 except for the elements in the main diagonal of M_3 which are defined by $c_{ii} = q^{|x|_{a_i}+|y|_{a_i}}$, for $1 \le i \le k$.*

Example 2 *If we consider the words $x = ab^2$ and $y = bab$ over Σ_3, then we have*

$$M_1 = \psi_q(x) = \begin{pmatrix} q & q^2 & (1+q) \\ 0 & q^2 & (1+q) \\ 0 & 0 & 1 \end{pmatrix},\ M_2 = \psi_q(y) = \begin{pmatrix} q & q & 1 \\ 0 & q^2 & (1+q) \\ 0 & 0 & 1 \end{pmatrix},$$

$$\text{and } M_3 = M_1 \oplus M_2 = \begin{pmatrix} q^2 & (q+q^2) & (2+q) \\ 0 & q^4 & 2(1+q) \\ 0 & 0 & 1 \end{pmatrix}.$$

where M_3 is the partial sum of the two Parikh q-matrices M_1 and M_2, but one can verify that it is not a Parikh q-matrix. In fact, if M_3 is a Parikh q-matrix, then the entry in the second row and third column of the matrix M_3 corresponds to the q-counting polynomial $S_{w,b}$ [7] and this polynomial can have only number one as constant term as shown in page 486 in [1] while it is number 2 in M_3.

3. Row and column Parikh q-matrices of a picture array

We first recall the definition of a picture array or simply called an array when it is clear from the context.

Let m, n be two natural numbers. A picture array A over Σ_k of size $m \times n$ is a rectangular array of symbols from Σ_k in m number of rows and n number of columns and is of the form $A = \begin{matrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{matrix}$. We also write A as $A = (a_{ij})_{m\times n}$ or simply as $A = (a_{ij})$, if m, n are understood. A subarray X of A with two rows and two columns, referred to as a 2×2 subarray, which is said to occur in A, is an array in the form $\begin{matrix} a_{ij} & a_{ip} \\ a_{lj} & a_{lp} \end{matrix}$, for some i, j, p, l with $1 \le i < l \le m$ and $1 \le j < p \le n$.

Let $\mathscr{PA}_k$ denote the set of all picture arrays over Σ_k. We denote by $M_1 \circ M_2$, the column concatenation and by $M_1 \diamond M_2$, the row concatenation of picture arrays M_1, M_2 in $\mathscr{PA}_k$ respectively. We note that $M_1 \circ M_2$, is well defined if and only if the arrays have the same number of rows. Likewise $M_1 \diamond M_2$ is well defined if and only if the arrays have the same number of columns.

We now define row Parikh q-matrix and column Parikh q-matrix of a picture array.

Definition 3 *For $m, n \geq 1$, let $A \in \mathscr{PA}_k$ be an array over Σ_k. Let x_i be the horizontal words in the m rows of A and y_j be the vertical words in the n columns of A. Let $\psi_q(x_i)$ and $\psi_q(y_j^t)$ be the Parikh q-matrices of x_i and y_j^t respectively . Then the row Parikh q-matrix $M_r^q(A)$ is defined as $M_r^q(A) = \psi_q(x_1) \oplus \cdots \oplus \psi_q(x_m)$ and the column Parikh q-matrix $M_c^q(A)$ is defined as $M_c^q(A) = \psi_q(y_1^t) \oplus \cdots \oplus \psi_q(y_n^t)$.*

Example 3 *Consider the array A over $\Sigma_3 = \{a < b < c\}$ which is given by*

$$A = \begin{matrix} a & b & a & a & b \\ a & a & b & b & a \\ b & a & b & a & b \end{matrix}.$$

The Parikh q-matrices $\psi_q(x_i)$, $1 \leq i \leq 3$ of the words in the rows $x_1 = abaab, x_2 = aabba, x_3 = babab$ are respectively

$$\begin{pmatrix} q^3 & 2q^2+q^3 & 1+2q+q^2 \\ 0 & q^2 & 1+q \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} q^3 & 2q^2+q^3 & 1+2q+q^2 \\ 0 & q^2 & 1+q \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} q^2 & 2q^2 & 1+2q \\ 0 & q^3 & 1+q+q^2 \\ 0 & 0 & 1 \end{pmatrix}.$$

The row Parikh q-matrix of A is $\begin{pmatrix} q^8 & 6q^2+2q^3 & 3+6q+2q^2 \\ 0 & q^7 & 3+3q+q^2 \\ 0 & 0 & 1 \end{pmatrix}$.

In a similar manner, the column Parikh q-matrix of A is $\begin{pmatrix} q^8 & 1+5q+2q^2 & 4+2q \\ 0 & q^7 & 5+2q \\ 0 & 0 & 1 \end{pmatrix}$.

4. Picture arrays and q-ambiguity

We first extend the notion of M-ambiguity [14] of a picture array to q-ambiguity.

Definition 4 *For $m, n \geq 1$, two arrays $A, B \in \mathscr{PA}_k$ over Σ_k are said to be*

(i) q-row equivalent if $M_r^q(A) = M_r^q(B)$ and

(ii) q-column equivalent if $M_c^q(A) = M_c^q(B)$.

The arrays A and B are q-equivalent if A and B are both q-row equivalent and q-column equivalent and we say that A (as well as B) is q-ambiguous. An array $A \in \mathscr{PA}_k$ is said to be q-unambiguous if it is not q-ambiguous.

Example 4 *The array $A_1 \in \mathscr{PA}_3$ given by $A_1 = \begin{matrix} a & b & b & a \\ b & a & a & b \end{matrix}$ is q-ambiguous, since $M_r^q(A_1) = M_r^q(A_2) = \begin{pmatrix} q^4 & 2q+2q^2 & 2+2q \\ 0 & q^4 & 2+2q \\ 0 & 0 & 1 \end{pmatrix}$ and $M_c^q(A_1) = M_c^q(A_2) = \begin{pmatrix} q^4 & 2+2q & 2 \\ 0 & q^4 & 4 \\ 0 & 0 & 1 \end{pmatrix}$ where the array A_2 is given by $\begin{matrix} b & a & a & b \\ a & b & b & a \end{matrix}$.*

In [14], the authors have given a condition for M-ambiguity of $A \in \mathscr{PA}_2$, which is stated in the following theorem.

Theorem 2 *([14]) An array $A \in \mathscr{PA}_2$ over $\Sigma_2 = \{a < b\}$ is M-ambiguous if either the 2×2 subarray W_1 or the 2×2 subarray W_2, occurs in A, where $W_1 = \begin{matrix} a & b \\ b & a \end{matrix}$ and $W_2 = \begin{matrix} b & a \\ a & b \end{matrix}$.*

In the case of Parikh q-matrices for arrays, we observe the following interesting property in Lemma 1.

Lemma 1 *An array in $\mathscr{PA}_2$, having either* $W_1 = \begin{matrix} a & b \\ b & a \end{matrix}$ *or* $W_2 = \begin{matrix} b & a \\ a & b \end{matrix}$ *as its subarray, need not be q-ambiguous, treating the array as an array over* Σ_3.

Proof Consider an array $A = \begin{matrix} a & b & a \\ b & a & b \end{matrix} \in \mathscr{PA}_2$ over $\Sigma_3 = \{a < b < c\}$. Although the array A has a subarray $\begin{matrix} a & b \\ b & a \end{matrix}$ which is q-ambiguous, A is q-unambiguous which can be seen as follows; The arrays having the same q-column equivalent Parikh matrix as that of A, are the following arrays:

$$B_1 = \begin{matrix} b & a & a \\ a & b & b \end{matrix}, \; B_2 = \begin{matrix} a & a & b \\ b & b & a \end{matrix}.$$

But each of these arrays has a row Parikh q-matrix which is different from the row Parikh q-matrix of A and hence A is $q-$unambiguous. ■

We recall now two results from [2, 3].

Lemma 2 *[3] Let z be a word over $\Sigma_2 = \{a < b\}$ with the property $|z|_a = |z|_b$. Then $x = abzba$ and $y = bazab$ are $q-$equivalent, treating x, y as words over Σ_3.*

Lemma 3 *([2]) Let x, y be two words over Σ_2. Let $|x|_a = |x|_b$ and $|y|_a = |y|_b$. Then xy and yx are $q-$equivalent, treating the words as words over Σ_3.*

We now provide in Lemma 4, a special kind of q-ambiguous array in $\mathscr{PA}_2$, treating the array as an array over Σ_3. We make use of Lemmas 2 and 3.

Lemma 4 *The arrays M_1 and M_2 given below, over the binary alphabet $\Sigma_2 = \{a < b\}$ are q-ambiguous, treating the arrays as arrays over $\Sigma_3 = \{a < b < c\}$.*

$$M_1 = \begin{matrix} a & b & x & b & a \\ z_1 & z_2 & w & z_2 & z_1 \\ b & a & y & a & b \end{matrix}, \qquad M_2 = \begin{matrix} b & a & y & a & b \\ z_2 & z_1 & w & z_1 & z_2 \\ a & b & x & b & a \end{matrix}$$

where $z_1, z_2 \in \Sigma_2$, $z_1 \neq z_2$, and $x, y, w \in \Sigma_2^*$ such that $|x| = |y| = |w|$, $|x|_a = |x|_b$, $|y|_a = |y|_b$ and $|w|_a = |w|_b$.

Proof We note that the first two and the last two columns of M_1 and M_2 are the same. But they are in a different order. Also using Lemma 3, we have $\psi_q(xwy) = \psi_q(xyw) = \psi_q(yxw) = \psi_q(ywx)$. Therefore, $M_c^q(M_1) = M_c^q(M_2)$.

To prove q-row equivalence, we use the Lemma 2. Thus
$\psi_q(z_1z_2wz_2z_1) = \psi_q(z_2z_1wz_1z_2)$. Also the first and third rows of the array M_1 are same as the third and first rows of the array M_2 respectively. Hence $M_r^q(M_1) = M_r^q(M_2)$. ■

Using the Lemma 3, conditions for two arrays are now obtained such that the array obtained by their column concatenation is q-ambiguous.

Theorem 3 *Let A, B be two arrays of sizes $m \times n$ and $m \times l$ respectively in $\mathscr{PA}_2$ over $\Sigma_2 = \{a < b\}$ such that the i^{th} row of each of the arrays A, B has the same number a's and b's. Then the arrays $A \circ B$ and $B \circ A$ are q-ambiguous, treating the arrays as arrays over $\Sigma_3 = \{a < b < c\}$.*

Proof Since the i^{th} row of each of the arrays A, B has the same number a's and b's, the i^{th} row of each of $A \circ B$ and $B \circ A$ has the same number a's and b's. Therefore $\psi_q(x_iy_i) = \psi_q(y_ix_i)$, where x_i and y_i are the words in the i^{th} rows of A and B respectively. Thus using the Lemma 3, we have

$$\begin{aligned} M_r^q(A \circ B) &= \psi_q(x_1y_1) \oplus \psi_q(x_2y_2) \oplus \cdots \oplus \psi_q(x_my_m) \\ &= \psi_q(y_1x_1) \oplus \psi_q(y_2x_2) \oplus \cdots \oplus \psi_q(y_mx_m) = M_r^q(B \circ A). \end{aligned}$$

Therefore $A \circ B$ and $B \circ A$ are q-row equivalent.
Since both the arrays $A \circ B$ and $B \circ A$ have the same columns but in a different order, we have $M_c^q(A \circ B) = M_c^q(B \circ A)$. Thus they are q-column equivalent. Hence the arrays $A \circ B$ and $B \circ A$ are q-ambiguous. ■

Remark 1 *Corresponding result similar to the Theorem 3 remains true for $A \diamond B$, where A, B be two arrays of sizes $m \times n$ and $l \times n$ respectively in $\mathscr{PA}_2$ over $\Sigma_2 = \{a < b\}$ such that the i^{th} column of each of A, B has the same number a's and b's.*

We recall now the q-weak ratio property [2] of words.

Definition 5 *Two words $x, y \in \Sigma_k^*$ satisfy a q-weak ratio property, if they have weak ratio property [13] and satisfy one of the following conditions: for each $1 \leq i \leq k-1$,*

(i) $|x|_{a_i} = |x|_{a_{i+1}}$ *and* $|y|_{a_i} = |y|_{a_{i+1}}$.

(ii) $\frac{S_{x,a_i}(q)}{S_{y,a_i}(q)} = \frac{q^{|x|_{a_{i+1}}} - q^{|x|_{a_i}}}{q^{|y|_{a_{i+1}}} - q^{|y|_{a_i}}}$, *provided* $q \neq 0$.

and it is denoted by $u \sim_{qwr} v$.

Using this q-weak ratio property, the authors characterized in [2], the words over a binary alphabet whose Parikh q-matrices commute.

Lemma 5 ([2]) *If two words u_1 and u_2 over binary alphabet Σ_2 satisfy q-weak ratio property, then their Parikh q-matrices commute, which means $\psi_q(u_1u_2) = \psi_q(u_2u_1)$.*

We extend Lemma 5 for a binary picture array as follows.

Theorem 4 *Let $A, B \in \mathscr{PA}_2$ be two arrays of sizes $m \times n$ and $m \times l$ respectively with $\Sigma_2 = \{a < b\}$ such that the i^{th} row of A and the i^{th} row of B satisfy q-weak ratio property. Therefore the arrays $A \circ B$ and $B \circ A$ are q-ambiguous, treating the arrays as arrays over $\Sigma_3 = \{a < b < c\}$.*

Proof For $1 \leq i \leq m$, let x_i and y_i be the words in the i^{th} rows of A and B respectively so that the words in the i^{th} rows of $A \circ B$ and $B \circ A$ are x_iy_i and y_ix_i respectively.
Let the i^{th} row of A and the i^{th} row of B satisfy q-weak ratio property. Then by Lemma 5 we have $\psi_q(x_iy_i) = \psi_q(y_ix_i)$. Now we see that

$$\begin{aligned} &M_r^q(A \circ B) = \psi_q(x_1y_1) \oplus \psi_q(x_2y_2) \oplus \cdots \oplus \psi_q(x_my_m) \\ = \ &\psi_q(y_1x_1) \oplus \psi_q(y_2x_2) \oplus \cdots \oplus \psi_q(y_mx_m) = M_r^q(B \circ A). \end{aligned}$$

Therefore $A \circ B$ and $B \circ A$ are q-row equivalent.

Since the columns of the arrays $A \circ B$ and $B \circ A$ are the same, arranged in different order, we have $M_c^q(A \circ B) = M_c^q(B \circ A)$. Thus they are q-column equivalent. ■

Remark 2 *The converse of Theorem 4 need not be true always, as seen from the following example.*

Example 5 *Consider* $A = \begin{matrix} a & b & b \\ b & b & a \end{matrix}$ *and* $B = \begin{matrix} b & b & a \\ a & b & b \end{matrix}$. *Then we have*

$$A \circ B = \begin{matrix} a & b & b & b & b & a \\ b & b & a & a & b & b \end{matrix}, B \circ A = \begin{matrix} b & b & a & a & b & b \\ a & b & b & b & b & a \end{matrix}.$$

Since the rows of $A \circ B$ *and* $B \circ A$ *are the same except that they are in different order, we have* $M_r^q(A \circ B) = M_r^q(B \circ A)$. *Also* $A \circ B$ *and* $B \circ A$ *have the same columns again in different order so that* $M_c^q(A \circ B) = M_c^q(B \circ A)$. *Hence the arrays* $A \circ B$ *and* $B \circ A$ *are q-ambiguous. But the first rows of* A *and* B *are abb and bba respectively. It can be shown that they do not satisfy the q-weak ratio property.*

Acknowledgement

The authors thank the reviewers for their very useful comments which helped to revise the paper and present it in a better form. This work was supported by National Natural Science Foundation of China (61320106005 and 91530320) and the Innovation Scientists and Technicians Troop Construction Projects of Henan Province (154200510012). The author K.G. Subramanian expresses his gratitude to University Grants Commission (India), for the award of Emeritus Fellowship (No.F.6-6/2016-17/EMERITUS-2015-17-GEN-5933 / (SA-II)) to him to carry out his research work in the Department of Mathematics, Madras Christian College.

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