Structural properties of word representable graphs

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Abstract. Given a word $w = w_1w_2...w_n$ of length n over an ordered alphabet Σ_k, we construct a graph $G(w) = (V(w), E(w))$ such that $V(w)$ has n vertices labeled $1, 2, \cdots, n$ and for $i, j \in V(w)$, $(i, j) \in E(w)$ if and only if w_iw_j is a scattered subword of w of the form a_ta_{t+1}, $a_t \in \Sigma_k$, for some $1 \leq t \leq k-1$ with the ordering $a_t < a_{t+1}$. A graph is said to be Parikh word representable if there exists a word w over Σ_k such that $G = G(w)$. In this paper we characterize all Parikh word representable graphs over the binary alphabet in terms of chordal bipartite graphs. It is well known that the graph isomorphism (GI) problem for chordal bipartite graph is GI complete. The GI problem for a subclass of (6, 2) chordal bipartite graphs has been addressed. The notion of graph powers is a well studied topic in graph theory and its applications. We also investigate a bipartite analogue of graph powers of Parikh word representable graphs. In fact we show that for $G(w)$, $G(w)^{[3]}$ is a complete bipartite graph, for any word w over binary alphabet.

1. Introduction

The Parikh vector mapping, an important tool in the theory of formal languages, introduced by R. J. Parikh in [7] - gives the number of occurrences of letters in the word as a numerical vector. After a long gap the Parikh matrix of a word has been introduced in [1] as an extension of Parikh vector mapping. The Parikh matrix mapping of a word gives more numerical properties of a word in terms of certain subwords of the given word. The Parikh vector appears in the Parikh matrix as the second upper diagonal. Hence two words having same the Parikh vector need not have the same Parikh matrix; in other words this mapping is not injective. Two words α and β are said to be amiable if and only if they have the same Parikh matrix [2].

In [2] the author had constructed a graph with vertex set being the set of all amiable words over binary alphabet and have shown that such a graph is connected.

Another type of word representable graphs has its roots in the study of Perkins semigroup [3, 4]. A graph $G = (V, E)$ is word representable [3, 4] if there exists a word w over the alphabet Σ such that letters x and y alternate in w if and only if $(x, y) \in E$ for each $x \neq y$.

In this paper we introduce another approach using Parikh matrices to represent graphs with words. Let $\Sigma_k = \{a_i | 1 \leq i \leq k\}$ be an alphabet with an ordering $a_i < a_{i+1}, 1 \leq i \leq k-1$. A graph $G = (V, E)$ is called Parikh word representable or simply word representable if there exists a word $w = w_1w_2...w_n \in \Sigma_k^*$ of length n such that V is the set of vertices $\{1, 2, \cdots, n\}$ and for $i, j \in V$, $(i, j) \in E$ iff w_iw_j is a scattered subword of w of the form a_ta_{t+1}, for some $1 \leq t \leq k-1$.

In this paper we characterize the class of all graphs that are word representable over an ordered binary alphabet. We show that these graphs are indeed (6, 2) chordal bipartite graphs with an additional property pertaining to the degree of the vertices. A graph G is called (6, 2) chordal bipartite if G is bipartite and for each cycle of length at least 6 there exists at least 2 chords. We also show that the class of word representable graphs is a proper subclass of bipartite permutation graphs.

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It was shown in [5] that the GI problem for chordal bipartite graphs is GI complete. We give necessary and sufficient condition for two word representable graphs $G_1(w_1)$ and $G_2(w_2)$ to be isomorphic to each other.

In general the problem of finding Hamiltonian cycle is NP complete [17, 18]. The only way to check whether a given graph has a Hamiltonian cycle or not is an exhaustive search. In this paper we characterize the class of Parikh word representable graphs that have a Hamiltonian cycle.

For the class of chordal graphs Duchet [13] proved that, for every positive integer m, if $G^{[m]}$ is chordal then so is $G^{[m+2]}$. Since power of a bipartite graph need not be bipartite, Chandran et al. [9] introduced the notion of bipartite power and proved that if G is k-chordal then so is $G^{[m]}$, for every positive integer $k \geq 4$ and odd positive m.

Since the closure property is not hereditary for graph classes, in [11] it was shown that interval bigraphs and bipartite permutation graphs are also closed under bipartite power defined in [9]. Following [9, 11], we show that the bipartite power of word representable graphs is also word representable. In fact we show that the 3^{rd} bipartite power of a word representable graph is a complete bipartite graph.

The paper is structured as follows: Section 2 provides some basic notations, definitions and some results on Parikh word representable graphs. We also characterize the class of Parikh word representable graphs in terms of $(6, 2)$ chordal bipartite graphs. We conclude this section by showing that the class of Parikh word representable graphs is a proper subclass of bipartite permutation graphs. In section 3 we discuss the graph isomorphism problem for the class of all connected Parikh word representable graphs. The Hamiltonian cycle problem for these class of Parikh word representable graphs has been studied in section 4. In section 5 we show that the bipartite power of Parikh word representable graph is also Parikh word representable and the 3^{rd} bipartite power of Parikh word representable graph is a complete bipartite graph. We end the paper with some concluding remarks.

2. Preliminaries

Let $\Sigma_k = \{a_1 < a_2 < ... < a_k\}$ be an ordered alphabet and $w = w_1w_2...w_n$ be a word of length n over Σ_k.

Definition 1. *For each word $w = w_1w_2...w_n$ of length n over Σ_k we define a simple graph $G = G(w)$ with n labeled vertices $1, 2, \cdots, n$, representing the positions of the letters w_i, $1 \leq i \leq n$ in w such that corresponding to each occurrence of the scattered subword a_ia_{i+1}, for some $1 \leq i \leq k-1$ in w, there is an edge between the corresponding positions. We say that the word w, represents the graph $G = G(w)$.*

Example 1. *Take $\Sigma_2 = \{a < b\}$ and the word $w = aabb$ of length 4. Then the corresponding graph having 4 vertices labeled $1, 2, 3$ and 4 is given in Figure 1.*

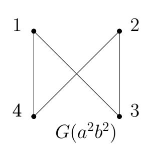

FIGURE 1

It is clear that the graph G has an edge between the vertices i and j (assuming $i < j$) if and only if $w_i = a$ and $w_j = b$. Also if there is an edge between the vertices i and j, then no vertex numbered $k(< i)$ is adjacent to the vertex i (For, if such a k exists, then we have an edge between k

and i where $k < i$, therefore we must have $w_k = a, w_i = b$ a contradiction to $w_i = a$). The graph G is unique.

But every labeled graph need not be word representable. For example, the following graph is not word representable for any word over Σ_2.

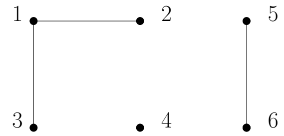

FIGURE 2. non representable graph

We call a graph G, Parikh word representable if there exists a word w that represents G. Throughout the rest of the paper we consider graphs which are representable using words over binary alphabet unless specified. We present some properties of Parikh word representable graphs.

Lemma 1. *A disconnected graph having more than one non trivial component is not word representable for any word over binary alphabet.*

Proof. It is enough to consider the case when the number of components is 2.

Let G be a disconnected graph with n vertices having two non trivial components G_1 and G_2. Therefore each of G_1 and G_2 must have at least one edge. If possible let G be word representable for a word $w = w_1 w_2 ... w_n$, where $w_i \in \Sigma_2$, for $1 \leq i \leq n$. Consider an edge (i, j) in G_1 and another edge (k, l) in G_2, where $1 \leq i, j, k, l \leq n$. Without loss of generality let $i < j$ and $k < l$. Then we have $w_i = a, w_j = b, w_k = a, w_l = b$. Clearly $j \neq k$, since j and k are in different components.

1. If $j < k$, then $i < l$ and this implies that there is a scattered subword ab corresponding to these positions i, l. Therefore there must be an edge between the vertex position i and l, a contradiction to the fact that they are in different components.
2. If $j > k$, this means that there is a scattered subword ab corresponding to these positions k, j. Therefore there must be an edge between the vertex position k and j, a contradiction to the fact that they are in different components.

Hence our assumption is wrong and the lemma follows. □

Therefore our main focus will be on the connected component of the given graph as the isolated vertices will contribute to a's as suffix or b's as prefix. Hence for a connected graph with n vertices if it is word representable for a word (say w, $|w| = n$), then the word w has to start with a and end with b. Now the vertex labeled 1 corresponding to this a is adjacent to each vertex which corresponds to b and therefore the degree of the vertex labeled 1 is $|w|_b$. Similarly the degree of the vertex labeled n is $|w|_a$. We also see that the vertices labeled 1 and n are adjacent and sum of degrees of these two vertices is $|w|_b + |w|_a = n$. Therefore we have the following.

Lemma 2. *A connected graph with n vertices representable by a word must have two adjacent vertices whose degree sum is n.*

We can generalize Lemma 1 for any arbitrary ordered alphabet.

Theorem 1. *Over an ordered alphabet Σ_k, $k > 1$ a disconnected graph having more than $(k-1)$ non trivial components can not be represented by a word.*

Proof. Let G be a Parikh word representable graph such that $G = G(w)$ for a word $w \in \Sigma_k^*$, having k non trivial components. G must have at least k edges and each edge corresponds to a scattered subword of the form $a_l a_{l+1}$, for $1 \leq l \leq k-1$. Since $|\Sigma| = k$, w can have $(k-1)$ such type of distinct scattered subwords. Thus by Pigeon-hole principle, there exists a scattered subword $a_i a_{i+1}$

that corresponds to 2 edges in G in different components. Now using the same argument in the proof of Lemma 1 we will end up with a contradiction.

$\square$

Another important property of Parikh word representable graphs concerning cycles is given by the following.

Proposition 1. *Any graph having an odd cycle is not word representable for any word over an arbitrary ordered alphabet.*

Proof. Let G be a graph which is word representable and let $C = \{c_1, c_2, ..., c_{2m+1}\}$ be an odd cycle of G where c_i, $1 \leq i \leq 2m+1$ are the labeled vertices. With out loss of generality, suppose the vertex c_1 corresponds to the letter $a_i \in \Sigma_k$. Then the vertex c_2 corresponds to the letter either a_{i-1} or $a_{i+1} \in \Sigma_k$. If c_2 corresponds to the letter a_{i-1}, c_3 corresponds to the letter either a_{i-2} or $a_i \in \Sigma_k$ and if c_2 corresponds to the letter a_{i+1}, c_3 corresponds to the letter either a_i or $a_{i+2} \in \Sigma_k$. Therefore the vertex c_3 corresponds to one of the letters a_{i-2}, a_i, $a_{i+2} \in \Sigma_k$. Proceeding in this way the vertex c_{2m+1} corresponds to the letter $a_{i+2j} \in \Sigma_k$, for some $j = 0, \pm 1, \pm 2, ..., \pm m$ and in each case the letters a_i and a_{i+2j} are not consecutive but there is an edge between the vertices c_1 and c_{2m+1}, a contradiction.

$\square$

Cor 1. *If $G = G(w)$ for $w \in \Sigma_k^*$ then G is bipartite.*

Let $G = (V, E)$ be a graph. Then its complement graph, denoted by $\overline{G}$ is the graph $\overline{G} = (V, \overline{E})$, where any two vertices are adjacent in $\overline{G}$ iff they are not adjacent in G. If a graph is bipartite, its complement need not be a bipartite graph. In [10] the authors have classified all graphs G such that both G and $\overline{G}$ are bipartite.

Theorem 2. ([10]) *All the graphs G such that both G and its complement denoted by $\overline{G}$ are bipartite are the graphs shown in Figure* 3.

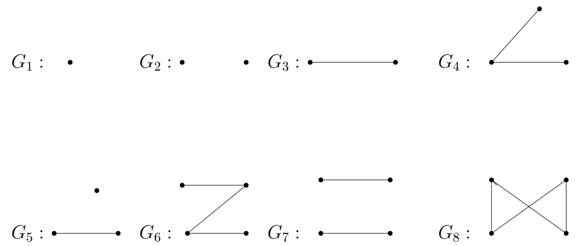

FIGURE 3

Thus we can deduce the following result pertaining to the complement of word representable graphs.

Lemma 3. *If a graph G is word representable, then its complement $\overline{G}$ is word representable iff G is one of the following:*

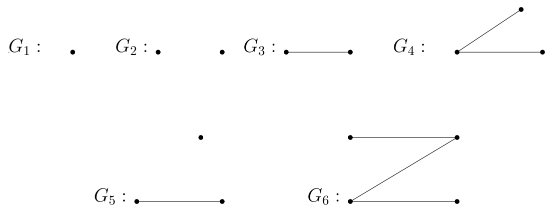

FIGURE 4

Proof. If part: Complements of the graphs given in Figure 4 are given below in Figure 5 which are word representable.

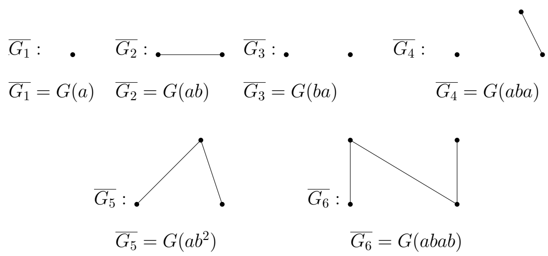

FIGURE 5

Only if part: Suppose $\overline{G}$ is word representable. Since G is word representable, it follows from Theorem 2, that G must be one of those 8 graphs. Now we see that the last two graphs in Theorem 2 are complement of each other and among these two G_7 is not word representable, since it has two non trivial components.

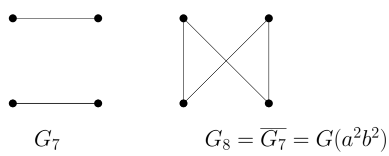

FIGURE 6

$\square$

Lemma 4. *A graph G is word representable for a word w if and only if every induced subgraph H of G is word representable for a word u (say), where u is a scattered subword of w over any arbitrary alphabet.*

Proof. If part: Let G be a graph and let every induced subgraph H of G represent a word u over any arbitrary alphabet Σ_k. Since the graph G is an induced subgraph of itself, $G = G(w)$ for some $w \in \Sigma_k^*$.

Only if part: Let the graph $G = G(w)$ for $w = w_1 w_2 ... w_n \in \Sigma_k^*$ of length n be word representable. Then the vertices of G can be labeled with $1, 2, ..., n$. Let H be an induced subgraph of G having vertices $i_1, i_2, ..., i_k$, $1 \leq k \leq n$ such that $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Then H represents the word $u = w_{i_1} w_{i_2} ... w_{i_k}$. And clearly u is a scattered subword of w. $\square$

For example, a Parikh word representable graph G and its proper induced subgraphs (up to isomorphism) are given below:

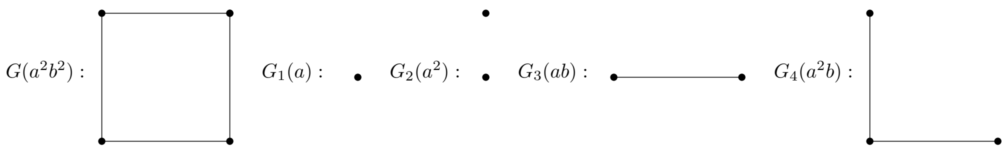

FIGURE 7

Cor 2. *If an induced subgraph H of a graph G is not word representable, then the main graph G is not word representable.*

Given a simple graph G with n vertices, can we label the vertices in such a way that the graph is word representable for a word over Σ_2? Before answering this we first observe the following.

1. A path P_n with $n \leq 4$ is word representable.

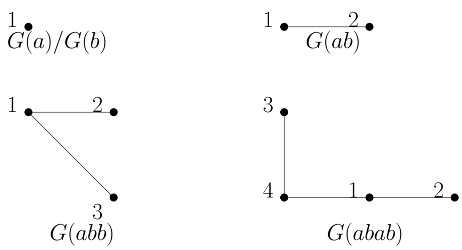

FIGURE 8

2. A path P_n with $n > 4$ is not word representable.
   For in a path, the degree of each vertex is 2 except for starting and ending vertex, therefore the degree sum of any two adjacent vertices is at most $4 < n$, a contradiction to Lemma 2. Thus $P_n, n > 4$ is not word representable over Σ_2.

3. A cycle C_n with $n > 4$ is not word representable.
   For in a cycle, the degree of each vertex is 2, therefore the degree sum of any two adjacent vertices is exactly $4 < n$, a contradiction to Lemma 2. Thus $C_n, n > 4$ is not word representable over Σ_2.

4. A star graph $K_{1,n}$ is word representable for all $n \geq 1$.

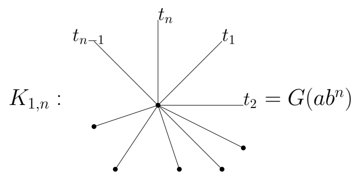

FIGURE 9

5. A tree with diameter[1] < 4 is word representable.

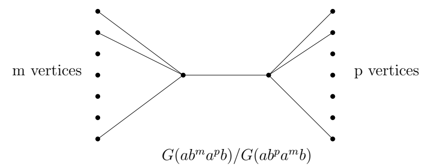

FIGURE 10. Tree with diameter < 4

6. A tree T_n, with diameter ≥ 4 is not word representable.
Since the diameter is ≥ 4, there exists a path P_m with $m > 4$ and $m \leq n$. Consider the induced subgraph P_m where $m > 4$, this is not word representable, hence by Cor 2, we can conclude that the tree T_n, with diameter ≥ 4 is not word representable.

7. A wheel graph $W_n, n > 2$ is not word representable over Σ_2 as the wheel graph is not bipartite.

We consolidate the above observations for some special classes of graphs in the following table.

Graph	existence of word over binary alphabet
Path P_n,$n \leq 4$	Yes
Path P_n,$n > 4$	No
Cycle C_3	No
Cycle C_4	Yes(a^2b^2)
Cycle C_n, $n > 4$	No
Complete graph K_2	Yes
Complete graph K_n, $n > 2$	No
Star graph S_n	Yes(ab^{n-1} or $a^{n-1}b$)
Trees T_n, diameter< 4	Yes
Trees T_n, diameter ≥ 4	No
Wheel W_n, $n > 2$	No
Complete bipartite graph $K_{m,n}$	Yes($a^m b^n$ or $a^n b^m$)

Thus we can conclude that not all graphs can be represented by a word.
The next theorem characterizes the class of all connected Parikh word representable graphs.

Theorem 3. *A connected bipartite graph is word representable if and only if it is (6, 2) chordal having two adjacent vertices whose degree sum is same as the number of vertices of the graph.*

Proof. **If part:** Let G be a connected (6, 2) chordal bipartite graph having two adjacent vertices whose degree sum is same as the number of vertices of the graph. Then we have the following two cases:

1. G does not have cycles of length more than 4.
2. G has cycles of length at least 6.

Case 1: If the graph does not have cycles of length more than 4, then we have the following sub cases:
 (a) If the graph does not contain a cycle.
 Then G is a tree. If diameter of G is ≥ 4, then G is trivially (6, 2) chordal but G does not have two adjacent vertices whose degree sum is same as the number of vertices ($n > 4$). If diameter of $G < 4$ then G is word representable (see Figure 10).
 (b) If the graph G contains only cycles of length 4.

[1]The diameter d of a graph is the length of the longest shortest path of the graph. i.e. $d = Max_{u,v\in G} d_G(u,v)$, where $d_G(u,v)$ is the distance between the vertices u and v in G.

(i) Let G be such a graph having more than one four cycle and two adjacent vertices whose degree sum is as same as the number of vertices in the graph G. Then the graph must be of the form given in Figure 11.

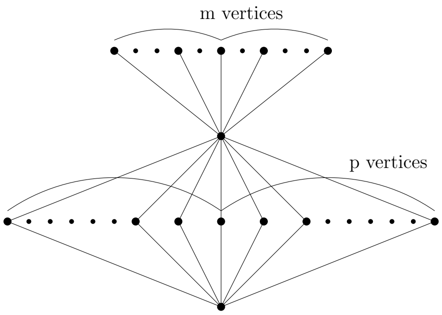

FIGURE 11

And this graph is word representable by either ab^mab^p or a^pba^mb.

(ii) If the graph G has only one four cycle having two adjacent vertices whose degree sum is as same as the number of vertices in the graph, then the graph will be one of the forms given in Figure 12.

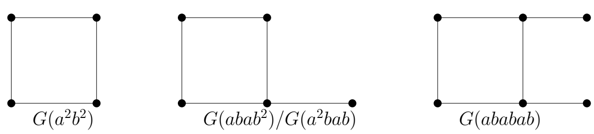

FIGURE 12

Case 2: Let G be a graph with cycles of length at least 6. Then we prove by induction on number of vertices.
Let G be a (6, 2) chordal bipartite graph with n vertices having 2 adjacent vertices x and y (say) such that $deg(x) + deg(y) = n$, where $deg(x)$ is the degree of the vertex x.
Base case: $n = 6$. The only (6, 2) chordal graphs having adjacent vertices whose degree sum is 6 are given in Figure 13.

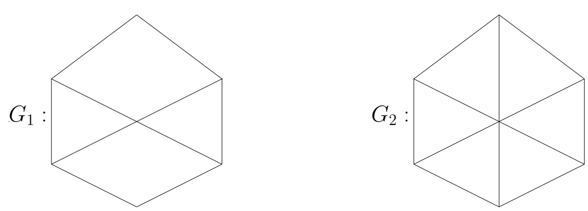

FIGURE 13

In Figure 13, one can easily verify that G_1 is word representable by the word a^2bab^2 and G_2 is word representable by the word a^3b^3.

Induction step: Suppose the statement is true for $n = k - 1$, i.e, all bipartite (6, 2) chordal graph with $k - 1$ vertices having 2 adjacent vertices x and y (say) such that $deg(x) + deg(y) = k - 1$ is word representable.

Let G be a (6, 2) chordal bipartite graph with k vertices, having 2 adjacent vertices x and y (say) such that $deg(x) + deg(y) = k$. Now delete a vertex v other than x and y from G. Let G' be the induced subgraph of $G - v$. We see that the vertex v must be adjacent to either x or y but not both (if v is adjacent to both x and y, then G contains a triangle, a contradiction to the fact that G is bipartite. And if v is adjacent to neither x nor y, then $deg(x) + deg(y) < k$, a contradiction). We see that

(a) G' is bipartite, since G is bipartite.
(b) G' is (6, 2) chordal, since (6, 2) chordal property is an hereditary property (i.e. any induced subgraph of a (6, 2) chordal graph is (6, 2) chordal).
(c) Also in G', $deg(x) + deg(y) = k - 1 = |V(G')|$.

Therefore by induction hypothesis G' is word representable and let G' is word representable by a word $u = u_1 u_2 ... u_{k-1}$.

Now since x and y are adjacent, without loss of generality, let the letter a corresponds to the position x and the letter b corresponds to the position y. Therefore we can take $u_1 = a$ and $u_{k-1} = b$. Also since either x or y is adjacent to v, let v be adjacent to x, therefore v has to be labeled with b.

Suppose $deg(v) = m$, $1 \leq m \leq |u|_a$. Then the word $w = au_2..u_p b u_{p+1}.u_{k-2}b$, where $|au_2..u_p|_a = m$, represents the graph G. Hence G is word representable.

Converse part: Let G be a connected bipartite graph and let it is word representable by a word w. Then by Lemma 2, it follows that G must have two adjacent vertices whose degree sum is same as the number of vertices in G. To prove the second condition, if possible let G be not (6, 2) chordal, then there exists a cycle C of length at least 6 in G such that the cycle C has at most one chord. Now consider the induced subgraph of this cycle C. If the cycle C is of length exactly 6, then it is not word representable by base case. If the cycle is of length more than 6, then the degree sum of any two adjacent vertices in the cycle is at most 6. Therefore by Lemma 2 the cycle C is not word representable. Hence by Cor 2 the graph G is not word representable, a contradiction. □

One can easily check that none of the two conditions in Theorem 3 can be dropped. For example,

Graph-A Graph-B

FIGURE 14

Graph-A in Figure 14, is (6, 2) chordal bipartite but it is not word representable, and the Graph-B has two adjacent vertices whose degree sum is 6 (the number of vertices) still it is not word representable.

A graph $G = (V, E)$ with $V = \{1, 2, ..., n\}$ is a permutation graph if there exists a permutation π over V such that $\{i, j\} \in E$ if and only if $(i - j)(\pi(i) - \pi(j)) < 0$. A graph is a bipartite permutation graph if it is bipartite and a permutation graph. An interval graph is the intersection graph of a family of intervals on the real line. It has one vertex for each interval in the family, and an edge between every pair of vertices corresponding to intervals that intersect. A graph is a bipartite interval graph if it is bipartite and an interval graph. A bipartite graph is chordal if every induced cycle is of length four. Let the classes of bipartite permutation graphs, bipartite interval graphs and chordal bipartite graphs are denoted by BP, IB and CB respectively. Hell and Haung [15] have shown the following hierarchy among these classes.

$$BP \subset IB \subset CB.$$

We denote the class of all Parikh word representable graphs by PWG. We conclude this section by showing that the class of Parikh word representable graphs is a proper sub class of bipartite permutation graphs. We recall the following.

Theorem 4. ([15]) *The following statements are equivalent, for a bipartite graph G:*

1. *G is a permutation graph. i.e. both G and $\overline{G}$ are comparability graph.*
2. *$\overline{G}$ is a comparability graph[2].*

Theorem 5. *The class of Parikh word representable graphs(PWG) is a proper sub class of bipartite permutation graph(BP).*

Proof. Let G be a Parikh word representable graph by a word $w = w_1 w_2 ... w_n$ with n vertices $1, 2, ..., n$. To show that G is a bipartite permutation graph, by the previous theorem it is enough to show that $\overline{G}$ is a comparability graph.

If possible, suppose $\overline{G}$ is not a comparability graph. Then there exist three vertices say $1 \leq i < j < l \leq n$ such that $(i, j), (j, l) \in E(\overline{G})$ but $(i, l) \notin E(\overline{G})$. Now since $(i, j), (j, l) \in E(\overline{G})$, $(i, j), (j, l) \notin E(G)$. We also have $i < j < l$. There are two cases either $w_j = a$ or $w_j = b$.

1. If $w_j = a$, we must have $w_l = a$, since $(j, l) \notin E(G)$. On the other hand w_i can be either a or b. In both cases there is no edge between i and l. i.e. $(i, l) \notin E(G)$. This implies that $(i, l) \in E(\overline{G})$, a contradiction.
2. If $w_j = b$, we must have $w_i = b$, since $(i, j) \notin E(G)$. On the other hand w_l can be either a or b. In both cases there is no edge between i and l. i.e. $(i, l) \notin E(G)$. This implies that $(i, l) \in E(\overline{G})$, a contradiction.

Thus our assumption is wrong. Hence $\overline{G}$ is a comparability graph. This shows that G is a bipartite permutation graph.

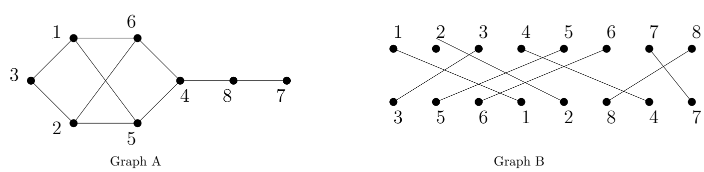

FIGURE 15

The Graph A in Figure 15 is an example of bipartite permutation graph (whose permutation representation is shown in Graph B in Figure 15) which is not Parikh word representable, as it does not have two adjacent vertices whose degree sum is 8.

Therefore, the class of Parikh word representable graphs is a proper sub class of bipartite permutation graph. □

Thus we have

$$PWG \subset BP \subset IB \subset CB.$$

[2]A comparability graph is an undirected graph that connects pairs of elements that are comparable to each other in a partial order. Comparability graphs have also been called transitively orientable graphs.

3. Graph Isomorphism

The graph isomorphism problem is to decide whether two finite graphs are isomorphic or not. Since it is neither known to be NP complete nor to be tractable, we will try to find some partial result for the class of Parikh word representable graphs. Two words w_1 and w_2 are said to be M-ambiguous if they have the same Parikh matrix (i.e) $|w_1|_a = |w_2|_a$, $|w_1|_b = |w_2|_b$ and number of ab's in w_1 and w_2 as a scattered subword are same.

Here it is natural to check whether graphs corresponding to M-ambiguous words w_1 and w_2 are isomorphic to each other or not. We observe the following.

1. For $abbba \sim_M babab$, (where $\sim_M$ means M-ambiguous, i.e. the Parikh matrices of the two words $abbba$ and $babab$ are same.) the corresponding graphs which are non isomorphic are given below.

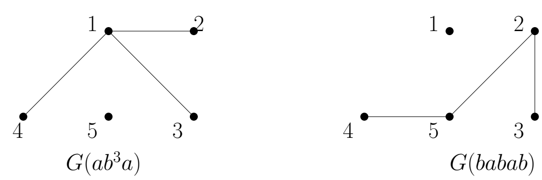

FIGURE 16

2. Two isomorphic graphs having different labeling need not be word representable by M-ambiguous words.

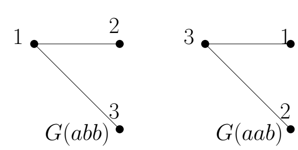

FIGURE 17

Definition 2. *An involution is a mapping $\theta : \Sigma \to \Sigma$ such that θ^2 equals the identity mapping, $\theta(\theta(a)) = a$, for all $a \in \Sigma$. A mapping $\theta : \Sigma^* \to \Sigma^*$ is a morphism if $\theta(uv) = \theta(u)\theta(v)$ and an anti morphism if $\theta(uv) = \theta(v)\theta(u)$ for all $u, v \in \Sigma^*$.*

Now we provide a sufficient condition for two graphs to be isomorphic each other.

Lemma 5. *Let x and y be two words over the binary alphabet $\{a < b\}$ such that $x = \theta(y)$, then $G(x) \simeq G(y)$ (i.e. $G(x)$ is isomorphic to $G(y)$), where $G(x)$ denotes the graph corresponding to the word x.*

Proof. Let $x = x_1x_2...x_{n-1}x_n$, where $x_i \in \{a < b\}$, $1 \le i \le n$ and $G(x)$ be the corresponding graph with the labeled vertices $1, 2, ..., n$.

Then $\theta(x) = \theta(x_n)\theta(x_{n-1})...\theta(x_2)\theta(x_1) = y$ and $G(y)$ be the corresponding graph with the labeled vertices $1, 2, ..., n$.

Now we define a map $\phi : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ by $\phi(i) = n - i + 1$.

To show that ϕ is an isomorphism,

1. Without loss of generality, suppose there is an edge between i and j and $1 \le i < j \le n$. Then $x_i = a$ and $x_j = b$. We also have $y_{n-j+1} = \theta(x_j) = a$ and $y_{n-i+1} = \theta(x_i) = b$, since $n - j + 1 < n - i + 1$, there is an edge between $n - j + 1 (= \phi(j))$ and $n - i + 1 (= \phi(i))$.

2. Without loss of generality, Suppose l and m, $1 \leq l < m \leq n$ are not adjacent, then there are three cases:
   (a) $x_l = a$ and $x_m = a$, then $y_{n-l+1} = \theta(x_l) = b$ and $y_{n-m+1} = \theta(x_m) = b$ and therefore $n-l+1(= \phi(l))$ and $n-m+1(= \phi(m))$ are not adjacent.
   (b) $x_l = b$ and $x_m = b$, then $y_{n-l+1} = \theta(x_l) = a$ and $y_{n-m+1} = \theta(x_m) = a$ and therefore $n-l+1(= \phi(l))$ and $n-m+1(= \phi(m))$ are not adjacent.
   (c) $l < m$ and $x_l = b$ and $x_m = a$, then $y_{n-l+1} = \theta(x_l) = a$ and $y_{n-m+1} = \theta(x_m) = b$ and since $n-m+1 < n-l+1$ therefore $n-l+1(= \phi(l))$ and $n-m+1(= \phi(m))$ are not adjacent.

□

Lemma 6. *Let w and w' be two words over the binary alphabet $\{a < b\}$ such that $G(w) \simeq G(w')$ and the graphs are connected, then $w = \theta(w')$.*

Proof. We prove by induction on the length of the word.

Base case: $G(ab) \simeq G(ab)$. Clearly $ab = \theta(ab)$.

Induction hypothesis: Suppose it is true for all words with $|w| < n$.

Let w and w' be of length n and $G(w) \simeq G(w')$ and the graphs are connected. Then we have $w = aw_1b$ and $w' = aw'_1b$. Therefore $G(aw_1b) \simeq G(aw'_1b)$. Let $G_1 = G(aw'_1b)$ and $V(G_1) = X_1 \cup Y_1$, $G_2 = G(aw'_1b)$, $V(G_2) = X_2 \cup Y_2$, then $G_1 \simeq G_2$. Also let $x_1 \in X_1$ corresponds to the starting letter a in w and $y_1 \in Y_1$ corresponds to the ending letter b in w, we see that x_1 is adjacent to all the vertices in Y_1 and y_1 is adjacent to all the vertices in X_1. Similarly in G_2, let $x_2 \in X_2$ corresponds to the starting letter a in w' and $y_2 \in Y_2$ corresponds to the ending letter b in w', we see that x_2 is adjacent to all the vertices in Y_2 and y_2 is adjacent to all the vertices in X_2. Let the isomorphic map be ϕ. Since isomorphism preserves the structure of a graph, either $\phi(X_1) = X_2, \phi(Y_1) = Y_2$ or $\phi(X_1) = Y_2, \phi(Y_1) = X_2$.

Case 1: If $\phi(X_1) = X_2$, $\phi(Y_1) = Y_2$, then without loss of generality we can assume that $\phi(x_1) = x_2$, $\phi(y_1) = y_2$. Now if we delete the vertices x_1, y_1 from G_1 and x_2, y_2 from G_2 we still have isomorphism between $G_1 \setminus \{x_1, y_1\}$ and $G_2 \setminus \{x_2, y_2\}$(the same map ϕ will work). This implies that $G(w_1) \simeq G(w'_1)$. By induction hypothesis, we have $w_1 = \theta(w'_1)$.

Therefore $\theta(w') = \theta(aw'_1b) = a\theta(w'_1)b = aw_1b = w$.

Case 2: If $\phi(X_1) = Y_2$, $\phi(Y_1) = X_2$, then $|X_1| = |Y_2|$ and $|Y_1| = |X_2|$. Since x_1 is adjacent to all the vertices in Y_1 and y_2 is adjacent to all the vertices in X_2, also we have $|Y_1| = |X_2|$. Without loss of generality we can assume $\phi(x_1) = y_2$ and similarly $\phi(y_1) = x_2$. Proceeding in a similar way as in case (i) we will get $G(w_1) \simeq G(w'_1)$. By induction hypothesis, we have $w_1 = \theta(w'_1)$.

Therefore $\theta(w') = \theta(aw'_1b) = a\theta(w'_1)b = aw_1b = w$.

□

Lemma 7. *$G(b^kxa^l) \simeq G(b^mxa^n) \simeq G(b^m\theta(x)a^n) \simeq G(b^k\theta(x)a^l)$, where $x \in \{a < b\}^*$ and $k + l = m + n$, $k, l, m, n \geq 0$.*

Proof. $G(b^kxa^l) =$ union of $(k + l)$ isolated vertices and the graph of the word x.
$\simeq$ union of $m + n$ isolated vertices and the graph of the word x, as $k + l = m + n$.
$= G(b^mxa^n)$
$=$ union of $(m + n)$ isolated vertices and the graph of the word x.
$\simeq$ union of $(m + n)$ isolated vertices and the graph of the word $\theta(x)$, since $G(x) \simeq G(\theta(x))$.
$= G(b^m\theta(x)a^n)$
$\simeq G(b^k\theta(x)a^l)$. □

For example, the graphs of the words ba^2b, bab^2, a^2ba, ab^2a in Figure 18 are isomorphic to each other.

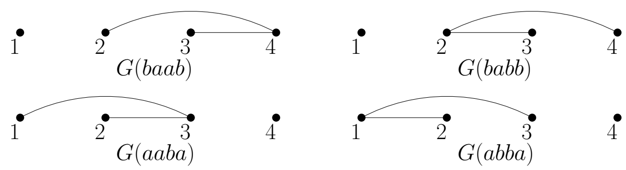

FIGURE 18

The next theorem will give us a necessary and sufficient condition for any two Parikh word representable graphs to be isomorphic.

Based on Theorem 3, Lemmas 5 and 6 we deduce the following.

Theorem 6. *Two connected (6, 2) chordal bipartite graphs each having two adjacent vertices whose degree sum is same as the number of vertices are isomorphic iff the corresponding words represented by those graphs are involution of each other.*

4. Hamiltonian cycle problem

A Hamiltonian cycle of a graph $G = (V, E)$ is a cycle which traverses each vertices of G exactly once. A graph is said to be Hamiltonian if it contains a Hamiltonian cycle. The Hamiltonian cycle problem is a problem to decide whether a given graph is Hamiltonian or not. In general the class of (6, 2) chordal bipartite graphs are not Hamiltonian. In [16] The authors have shown that the expanding condition is necessary for (6, 2) chordal bipartite graphs to have a Hamiltonian cycle, where the expanding condition is defined as follows.

Definition 3. ([16]) *Assume that $G = (V^+, V^-, E)$ is a bipartite graph with $2n$ vertices and $|V^+| = |V^-| = n$. We call the following condition an expanding condition for V^+ of G:*

$$\forall X \subset V^+,\ |X| < |N(X)|$$

This condition is equivalent to $\forall X \subset V^-,\ |X| < |N(X)|$.

Now we give a necessary and sufficient condition for a Parikh word representable graph to have a Hamiltonian cycle as follows.

Theorem 7. *A Parikh word representable graph by a word w over binary alphabet Σ_2 with $2n$ vertices, n vertices in each partite set, is Hamiltonian if and only if the followings hold:*

1. *$w = a^2w'b^2$, for some $w' \in \Sigma_2^*$.*
2. *all the prefixes of w has more number of a's than b's.*

Proof. If part: Let G be word representable by a word $w = w_1w_2w_3 \cdots w_{2n-2}w_{2n-1}w_{2n}$ satisfying the above two conditions. We see that all the prefixes starts with a^2 and $|w|_a \geq 2$, $|w|_b \geq 2$. Clearly $w_1 = w_2 = a$ and $w_{2n-1} = w_{2n} = b$. To show that G is Hamiltonian it is enough to show that there exists a Hamiltonian cycle. We have $|w|_a = |w|_b = n$. To find the Hamiltonian cycle of G, we will start with the vertex corresponding to the 1^{st} a in w. Now choose the edge between the vertices corresponding to the 1^{st} a and 1^{st} b. Suppose 1^{st} b occurs in the t_1^{th} position in w. i.e. $w_{t_1} = b$, $3 \leq t_1 \leq 2n$. Now since $u_1 = w_1w_2w_3 \cdots w_{t_1}$ is a prefix of w, $|u_1|_a > |u_1|_b$ and $w_2 = a$, we choose the edge between the vertices corresponding to the 1^{st} b and 2^{nd} a. Then choose the edge between the vertices corresponding to the 2^{nd} a and 2^{nd} b. Then we have the following two cases:

case 1: If the 2^{nd} b occurs in w_{2n} position, then there is no more a and b. Therefore now choose the the edge between the vertices corresponding to the 2^{nd} b and 1^{st} a. And this will complete the Hamiltonian cycle.

case 2: If the word w has more than 2 a's and b's. Then the 2^{nd} b will occurs in some w_{t_2} position where $t_1 < t_2 < 2n$. Now considering the prefix $u_2 = w_1w_2w_3...w_{t_2}$ and using the condition $|u_2|_a > |u_2|_b$ we assured that there is another $a(3^{rd})$ before the 2^{nd} b. Choose the edge between the vertices corresponding to the 2^{nd} b and 3^{rd} a. Since $|w|_a = |w|_b$ 3^{rd} b must be there.

If 3^{rd} b occurs in w_{2n} position, using case 1 we will get the cycle, otherwise iteratively proceeding as in case 2 we will end up with a Hamiltonian cycle.

Only if part: Suppose a Parikh word representable graph $G = G(w)$ is Hamiltonian. Since G is Hamiltonian, it has a Hamiltonian cycle that traverses all the vertices exactly once. Therefore each vertex has to be of degree atleast 2. Hence w must be of the form $w = a^2w'b^2$, otherwise the graph will have pendent vertices, a contradiction.

To prove the condition 2, we use the fact that the expanding condition is necessary for a graph G to have a Hamiltonian cycle. For the contrary suppose the condition 2 does not hold. Then there exists a prefix u of w such that $|u|_a \leq |u|_b$ and $w = ux$, $x \in \Sigma_2^+$. Also u can be written as $u = a^{m_1}b^{n_1}a^{m_2}b^{n_2}\cdots a^{m_t}b^{n_t}$ such that $m_1+m_2+\cdots+m_t \leq n_1+n_2+\cdots+n_t$. Let $G = (X^+, X^-, E)$, where $X^+ =$ vertices corresponding to a's, $X^- =$ vertices corresponding to b's and E denotes the edges. Consider $X =$ vertices corresponding to these $n_1, n_2, ..., n_t$ b's. Clearly $N(X) =$ vertices corresponding to these $m_1, m_2, ..., m_t$ a's and $|N(X)| \leq |X|$ where $X \subset X^-$. Therefore the expanding condition fails. This implies that G has no Hamiltonian cycle, a contradiction. Thus the condition 2 is established. $\square$

5. Bipartite power

The notion of graph power and its algorithmic application (see [8] and the references therein) have been well studied in graph theory. Graph power for chordal graphs is used to construct graphs with higher boxicity [9]. Also recognition of power of a graph is NP complete. Several graph classes are closed under power operation that is, for some graph classes C, $G \in C$ implies $G^k \in C$, $k \in \mathbb{N}$. Now we recall the definition of power of a graph.

Definition 4. *Given a graph G and a positive integer m, G^m is a graph with $V(G^m) = V(G)$, $E(G^m) = \{(u,v)|u,v \in V(G), d_G(u,v) \leq m\}$, where $d_G(u,v)$ denotes the distance between the two vertices u and v in G. The graph G^m is called the m-th power of G. For example,*

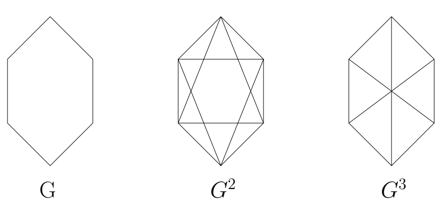

FIGURE 19

If a bigraph has at least three vertices, any k^{th} power of this graph must contain an odd cycle. This shows that the class of bigraphs is not closed under power operation. To preserve the bipartitedness, Sunil Chandran in [9] introduced the notion of bipartite powering to preserve the bipartitedness of a bipartite graph as follows.

Definition 5. ([9]) *Given a bipartite graph G and an odd positive integer m, $G^{[m]}$ is a bipartite graph with $V(G^{[m]}) = V(G)$, $E(G^{[m]}) = \{(u,v)|u,v \in V(G), d_G(u,v)$ is odd and $d_G(u,v) \leq m\}$. The graph $G^{[m]}$ is called the m-th bipartite power of G.*

It is known that several important graph classes such as the class of unit interval graphs, interval graphs, strongly chordal graphs are closed under the k^{th} power for any k, [12, 13, 14]. Also in case of bipartite graphs, the class of chordal bipartite graphs(CB), interval bipartite graphs(IB) and bipartite permutation graphs(BP) are closed under k^{th} power for any odd positive integer k [9, 11].

Since closure property for graph classes is not hereditary, we show that PWG is also closed under the odd k^{th} bipartite power. Indeed we show that $G^{[k]}$ is complete for any odd $k \geq 3$ which follows from the following theorem.

Theorem 8. *For any Parikh word representable connected graph G, $G^{[3]}$, the 3^{rd} bipartite power of G is Parikh word representable and it is the complete bipartite graph $K_{m,n}$, where $m = |w|_a$, $n = |w|_b$, w represents the graph G.*

Proof. Let G be a word representable connected graph by a word $w = w_1 w_2 ... w_{m+n}$ such that $|w|_a = m$, $|w|_b = n$. Since G is connected, we must have $w = aw'b$, for some $w' \in \Sigma_2^*$. Also G is bipartite. Let $G = X \cup Y$, where $X =$ the set of all vertices represented by a's, $Y =$ the set of all vertices represented by b's. Then we have $|X| = m$ and $|Y| = n$.

Now consider $G^{[3]}$, then we have

$$V(G^{[3]}) = V(G), E(G^{[3]}) = \{(u, v)|u, v \in V(G), d_G(u, v) \text{ is odd and } d_G(u, v) \leq 3\}$$

In $G^{[3]}$, there will be no edge between any two vertices from the same bipartite set X or Y, because $d_G(u, v)$ is always even for every pair $u, v \in X$ or Y. Therefore $G^{[3]}$ remains bipartite with the same bipartite sets X and Y.

To show that every vertex in X (represented by a) is adjacent to every vertex in Y(represented by b) in $G^{[3]}$, we have the following two cases:

Case 1: If a vertex in X is adjacent to a vertex in Y in G, then they are adjacent in $G^{[3]}$ also.

Case 2: Let there be a vertex v_a in X which is labeled 'i' (say) in the graph G that is not adjacent to a vertex v_b in Y labeled 'j' (say), then $1 \leq j < i \leq m + n$. Then $d_G(v_a, v_b) = 3$. This implies that there will be an edge between these two vertices in $G^{[3]}$.

Since these two vertices are arbitrary, we can say that every vertex in X is adjacent to every vertex in Y in $G^{[3]}$. i.e. $G^{[3]}$ is complete bipartite graph $K_{m,n}$.

And $G^{[3]}$ is word representable by the word $a^m b^n$. □

6. Concluding remarks

This paper introduces a new notion of Parikh word representable graphs and some properties of the class of these graphs. A major result concerning the characterization of Parikh word representable graphs over binary ordered alphabet is given. Also the GI problem for this special class of graphs over binary ordered alphabet has been studied. It is interesting to see if we can generalize the results we obtained for the class of Parikh word representable graphs over binary ordered alphabet to an arbitrary ordered alphabet.

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