DOMAINS OF HOLOMORPHY FOR IRREDUCIBLE ADMISSIBLE UNIFORMLY BOUNDED BANACH REPRESENTATIONS OF SIMPLE LIE GROUPS

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Abstract. In this note, we address a question raised by Krötz on the classification of G-invariant domains of holomorphy for irreducible admissible Banach representations of connected non-compact simple real linear Lie groups G. When G is not of Hermitian type, we give a complete description of such G-invariant domains for irreducible admissible uniformly bounded representations on reflexive Banach spaces and, in particular, for all irreducible uniformly bounded Hilbert representations. When the group G is Hermitian, we determine such G-invariant domains only when the representations considered are highest or lowest weight representations.

1. Introduction

Let G be a connected non-compact simple real linear Lie group with Lie algebra $\mathfrak{g}$. Let K be a maximal compact subgroup of G with Lie algebra $\mathfrak{k}$. Further, let $G_{\mathbb{C}}$ be the universal complexification of G and $K_{\mathbb{C}}$, that of K, and let $\mathfrak{g}_{\mathbb{C}}$ denote the complexification of $\mathfrak{g}$. Let (π, V) be a Banach representation of G, i.e. assume there is a continuous action

$$G \times V \longrightarrow V, \qquad (g, v) \mapsto g \cdot v, \qquad g \in G, v \in V$$

of G on a complex separable Banach space V which gives rise to a group homomorphism $g \mapsto \pi(g)$ with $\pi(g)v = g \cdot v$. We remark here that a Banach space carrying an irreducible representation of a separable locally compact group is necessarily separable. For much of this introductory material [War72, Chapter 4] is a good reference. We call a vector $v \in V$ an *analytic vector* if the orbit map $\gamma_v : G \longrightarrow V$ of v, given by $g \longmapsto \pi(g)v$ and a priori continuous, extends to a holomorphic (V-valued) function on an open neighbourhood of G in $G_{\mathbb{C}}$ or equivalently, if γ_v is a (V-valued) real analytic map. Note that the space V^{ω} of analytic vectors for the representation (π, V) is a G-invariant subspace which is dense in V. Recall that a Banach representation (π, V) is called *admissible* if $\dim \mathrm{Hom_K(W, V_K)} < \infty$ for any finite-dimensional K-module W, where V_K is the space of K-finite vectors of (π, V). By an irreducible Banach representation (π, V) of G, we mean a topologically irreducible representation i.e. V has no non-trivial closed G-invariant subspace. Irreducible admissible Banach representations are the class of representations for which Harish-Chandra built his theory of harmonic analysis on real reductive groups. If π is irreducible and admissible, then we know that K-finite vectors are necessarily analytic vectors i.e., $V_K \subseteq V^{\omega}$.

So, given a non-zero vector $v \in V_K$, one might ask: to which $G \times K_{\mathbb{C}}$-invariant domain in $G_{\mathbb{C}}$ does its orbit map γ_v extend holomorphically. Towards describing such G-invariant domains, let Ξ denote the *crown domain* introduced in [AG90]. As we will see in some detail in Section 2, it is a G-invariant domain in $\mathbb{X}_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$ on which the G-action is proper, and which contains the Riemannian symmetric space $\mathbb{X} = G/K$. Further, let Ξ^+, Ξ^- be the related G-invariant domains in $\mathbb{X}_{\mathbb{C}}$, described in Section 2. We also write $\widetilde{\Xi} = q^{-1}(\Xi)$, $\widetilde{\Xi}^{\pm} = q^{-1}(\Xi^{\pm})$, where $q : G_{\mathbb{C}} \longrightarrow G_{\mathbb{C}}/K_{\mathbb{C}}$ is the canonical projection.

A first remark is that it is reasonable to expect that such a domain would be independent of the vector $v \in V_K$ because $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})\cdot v = V_K$ as $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$-modules. Here $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ is the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$. In fact, Krötz [Krö08, Theorem 5.1] proved a classification of such domains when π is an irreducible unitary representation and, further, proposed the following generalisation.

Conjecture: Let (π, V) be an irreducible admissible Banach representation of G. There exists a unique maximal $G \times K_{\mathbb{C}}$-invariant domain $D_\pi \subset G_{\mathbb{C}}$, such that for every non-zero vector $v \in V_K$, the orbit map $\gamma_v : g \mapsto \pi(g)v$ extends to a holomorphic map $\widetilde{\gamma}_v : D_\pi \longrightarrow V$. In more detail, we have

i) $D_\pi = G_{\mathbb{C}}$ if π is finite-dimensional.
ii) $D_\pi = \widetilde{\Xi}^+$ if G is Hermitian, and π is a highest weight representation.
iii) $D_\pi = \widetilde{\Xi}^-$ if G is Hermitian, and π is a lowest weight representation.
iv) $D_\pi = \widetilde{\Xi}$ in all other cases.

In this note, we shall prove this for irreducible admissible uniformly bounded representations on reflexive Banach spaces. Notice that for a finite-dimensional representation π, the fact that $D_\pi = G_{\mathbb{C}}$ follows directly from the definitions. Henceforth, all the representations that we consider will be infinite-dimensional.

Now, we denote the continuous dual of a Banach space V by V'. We recall that a Banach space V is called *reflexive* if the canonical isometric embedding $v \mapsto (v, \cdot)$ of V into its continuous double dual V'' is surjective. Here $(\cdot,\cdot) : V \times V' \to \mathbb{C}$ is the duality bracket. Equivalently, a Banach space V is reflexive if the closed unit ball is compact in the weak-topology. Finite dimensional spaces, Hilbert spaces, L^p-spaces for $1 < p < \infty$, as well as uniformly convex Banach spaces are all reflexive. Spaces of continuous functions on infinite compact metric spaces, endowed with the supremum norm, L^1 spaces and L^∞ spaces are all not reflexive (see [Bre11, Sections 3.5, 3.7], for instance).

A Banach representation (π, V) of G is said to be *uniformly bounded* if there exists a constant $C > 0$ such that

$$\|\pi(g)\|_{op} \leq C \qquad \forall\, g \in G, \tag{1}$$

where $\|\cdot\|_{op}$ denotes the operator norm on the space of bounded linear operators on V. Note that $\|\pi(g)\|_{op} \leq C$ and $\|\pi(g)^{-1}\|_{op} \leq C$ for all $g \in G$ together force $C \geq 1$. Unitary representations on a Hilbert space are, of course, uniformly bounded.

Uniformly bounded Hilbert representations have been classically well-studied in the context of harmonic analysis on semisimple Lie groups - the Kunze-Stein phenomenon (see [Cow78] and the references therein), Langlands classification and $(\mathfrak{g}, K)$-module cohomology [BW80, Chapters IV, V]. Uniformly bounded representations on L^p-spaces, as well as on certain classes of uniformly convex Banach spaces, to our knowledge, have been studied in the context of Property (T) and rigidity (see [BFGM07], for instance). For a nice and concise overview of some of these aspects, we also refer to [Cow08].

We show in this note, that for an infinite-dimensional irreducible admissible uniformly bounded representation π of a connected non-compact simple real linear Lie group G, which is not of Hermitian type, on a reflexive Banach space, the associated $G \times K_{\mathbb{C}}$-invariant domain of holomorphy D_π is the domain $\widetilde{\Xi}$ mentioned earlier in the section. A non-trivial irreducible, uniformly bounded Hilbert representation is necessarily admissible (see [BW80, Theorem 5.2, Chapter IV]), and so our main result Theorem 3 gives, in particular, the G-invariant domains of holomorphy for all irreducible uniformly bounded Hilbert representations. We further emphasise that, already, the class of irreducible uniformly bounded Hilbert representations is a much larger class than the class of irreducible unitary representations. If G is Hermitian and π is a highest (respectively, lowest weight) uniformly bounded representation of G on a reflexive Banach space, then we show that $D_\pi = \widetilde{\Xi}^+$, (respectively, $D_\pi = \widetilde{\Xi}^-$). It is not yet clear to us how to handle the case when G is Hermitian but π is neither a highest nor a lowest weight representation. In the unitary setting of [Krö08], an $SL(2,\mathbb{R})$-reduction is used, together with the uniqueness of the direct integral decomposition for unitary representations. Since there is no analogous result on the uniqueness of direct integral decompositions in the Banach space setting, such a method does not extend. However, since the class of groups which are not Hermitian is a very large one, and since uniformly bounded representations are an important class, our results, while not complete, are, in our opinion, already of interest.

Finally, our methods are an extension of the circle of ideas developed in [KS04], [KO08], and [Krö08]. An important point in the proof is that the holomorphic extension of the orbit map γ_v of a non-zero K-finite vector v depends essentially on the smooth Fréchet structure of the Casselman-Wallach globalisation of the underlying $(\mathfrak{g}, K)$-module. Another key observation is that from the vanishing at infinity of the matrix coefficients for non-trivial irreducible, uniformly bounded representations, one can derive the appropriate properness of the G-action, which then leads to the determination of the desired G-invariant domain of holomorphy.

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2. Complex geometric setting

We begin by briefly describing the complex geometric setting, and refer to [KS05], [KO08] for comprehensive accounts. With G and K as before, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition such that K is the analytic subgroup corresponding to $\mathfrak{k}$, and let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ denote the set of restricted roots of the pair $(\mathfrak{g}, \mathfrak{a})$. Set $\Omega = \{Y \in \mathfrak{a} \,|\, |\alpha(Y)| < \frac{\pi}{2} \quad \forall \alpha \in \Sigma\}$. Ω is open, convex, and invariant under the Weyl group $W = W(\mathfrak{g}, \mathfrak{a})$ associated to Σ . Setting $\widetilde{\Xi} = G \exp(i\Omega) K_{\mathbb{C}} \subset G_{\mathbb{C}}$, we define the crown domain as $\Xi = \widetilde{\Xi}/K_{\mathbb{C}}$. This description is known as the *elliptic model* of the crown domain.

It was shown in [AG90] that Ξ is a Stein domain admitting a proper G-action. Notice that $\mathbb{X} \subset \Xi \subset \mathbb{X}_{\mathbb{C}}$. Now, if we define the set of elliptic elements in $\mathbb{X}_{\mathbb{C}}$ by $\mathbb{X}_{\mathbb{C},\,\mathrm{ell}} := G\exp(i\mathfrak{a})K_{\mathbb{C}}/K_{\mathbb{C}}$, then it is known that the crown domain Ξ is the maximal domain contained in $\mathbb{X}_{\mathbb{C},\,\mathrm{ell}}$ which admits a proper G-action but it is not a maximal domain in all of $\mathbb{X}_{\mathbb{C}}$ admitting a proper G-action. However, the polyhedron Ω in the elliptic model of the crown domain is maximal with respect to proper G-action. So to understand larger domains containing Ξ in $\mathbb{X}_{\mathbb{C}}$ on which G acts properly, we need an alternative description of the crown domain - the *unipotent model* of the crown domain. Fixing an order on the restricted root system $\Sigma(\mathfrak{g}, \mathfrak{a})$, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be the Iwasawa decomposition, where $\mathfrak{n}$ is the sum of the root-spaces corresponding to the positive restricted roots. Set Λ to be the connected component of $\{Y \in \mathfrak{n} \mid \exp(iY)K_{\mathbb{C}}/K_{\mathbb{C}} \in \Xi\}$ containing 0. Then, as described in [KO08, Section 8], we have the unipotent model $\Xi = G\exp(i\Lambda)K_{\mathbb{C}}/K_{\mathbb{C}}$ of the crown domain. Denote by $\partial\Xi$ the topological boundary of the crown domain Ξ. It is a union of G-orbits. The *elliptic* part $\partial_{\mathrm{ell}}\Xi$ of the boundary is then given by $\partial_{\mathrm{ell}}\Xi = G\exp(i\partial\,\Omega)K_{\mathbb{C}}$, and we define the *unipotent* part of the boundary to be $\partial_u\Xi := \partial\,\Xi\setminus\partial_{\mathrm{ell}}\Xi$. Indeed, it can be seen that $\partial_u\Xi = G\exp(i\partial\Lambda)K_{\mathbb{C}}/K_{\mathbb{C}}$. It was shown in [AG90] that the G-stabiliser of any point in $\partial_{\mathrm{ell}}\Xi$ is a non-compact subgroup of G. It therefore follows that for any G-invariant domain D with $\mathbb{X} \subseteq D \subseteq \mathbb{X}_{\mathbb{C}}$ on which G acts properly, we have that $D \cap \partial_{\mathrm{ell}}\Xi = \emptyset$. Further, $\partial_u\Xi \not\subset D$, and so if $D \not\subset \Xi$, then $D \cap \partial_u\Xi \neq \emptyset$

Now recall that a simple real Lie group is called *Hermitian* if the corresponding symmetric space G/K admits a G-invariant complex structure. In this case, as a $\mathfrak{k}_{\mathbb{C}}$-module, $\mathfrak{p}_{\mathbb{C}}$ splits into irreducible components $\mathfrak{p}_{\mathbb{C}}^{+}$, and $\mathfrak{p}_{\mathbb{C}}^{-}$, and we let $\mathsf{P}^{\pm}$ denote the corresponding analytic subgroups of $G_{\mathbb{C}}$. Then we set $\widetilde{\Xi}^{\pm} = GK_{\mathbb{C}}\mathsf{P}^{\pm}$. When G is Hermitian, the unipotent model allows for a description of the directions in which the crown domain Ξ can be extended to obtain the larger domains $\widetilde{\Xi}^{\pm}$ in such a way that it still admits a proper G-action. In fact, in this case we have a nice explicit description of the unipotent part $\partial_u\Xi$ of the boundary of the crown domain. We elaborate a little on this now. If G is Hermitian, then the restricted root system $\Sigma(\mathfrak{g}, \mathfrak{a})$ is either of type C_n or BC_n. We can then consider the set $\{\gamma_1, \dots, \gamma_n\}$ of long strongly orthogonal restricted roots. Fixing $E_j \in \mathfrak{g}_{\gamma_j}$ so as to form $\mathfrak{sl}_2$-triples with $\{E_j, \theta(E_j), [E_j, \theta(E_j)]\}$, θ being the Cartan involution, we have that $\Lambda = \bigoplus_{j=1}^{n}(-1,1)E_j$. So we have a description of the unipotent model $\Xi = G\exp(i\Lambda)K_{\mathbb{C}}/K_{\mathbb{C}}$ of the crown domain, and the unipotent part of the boundary is given by $\partial_u\Xi = G\exp(i\partial\Lambda)K_{\mathbb{C}}/K_{\mathbb{C}}$. The idea is that we can enlarge Ξ by enlarging Λ in such a way that G acts properly on the enlarged domain. Explicitly, setting

$$\Lambda^{+} = \bigoplus_{j=1}^{n}(-1,\infty)E_j, \quad \Lambda^{-} = \bigoplus_{j=1}^{n}(-\infty,1)E_j$$

we have the larger domains $\Xi^{\pm} = G\exp(i\Lambda^{\pm})K_{\mathbb{C}}/K_{\mathbb{C}}$ which still carry a proper G-action. In fact, Ξ^{+} and Ξ^{-} are Stein domains which are maximal domains in $\mathbb{X}_{\mathbb{C}}$ admitting a proper G-action. The relation to the elliptic model $\Xi = G\exp(i\Omega)K_{\mathbb{C}}/K_{\mathbb{C}}$ is also easy to see now, since, setting $T_j = \frac{1}{2}[E_j, \theta(E_j)]$, we have that $\Omega = \bigoplus_{j=1}^{n}(-\frac{\pi}{2}, \frac{\pi}{2})T_j$.

The symmetric space G/K depends only on the Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$, and the crown domain is a fibre bundle $G \times_K \mathrm{Ad}\,(K)\Omega$ over G/K with the convex balls $\mathrm{Ad}\,(K)\Omega$ as fibres. Hence the crown domain, as a topological space, is contractible to $\mathfrak{p} \times \mathrm{Ad}\,(K)\Omega$. Thus, as remarked in [Krö08, Remark 5.2], the domains Ξ, $\Xi^{\pm}$ are independent of the choice of the connected group G.

3. Holomorphic extensions of irreducible admissible representations

For use later in the section, we explain what we mean by going to infinity on the group G. Fix a positive system $\Sigma^+ = \Sigma^+(\mathfrak{g}, \mathfrak{a})$ of restricted roots, and let $\mathfrak{a}^+$ be the corresponding positive Weyl chamber. Set $A^+ = \exp(\mathfrak{a}^+)$. We have then the KAK-decomposition $G = K\overline{A^+}K$, with $\overline{A^+}$ denoting the closure of A^+. Let $B(\cdot,\cdot)$ denote the Killing form on $\mathfrak{g}$. Then the modified Killing form $\langle X, Y\rangle_\theta = -B(X, \theta Y)$, $X, Y \in \mathfrak{g}$, is an inner-product on $\mathfrak{g}$. One way is to consider the operator norm of the adjoint $\|Ad(g)\|$, for $g \in G$, on the Lie algebra $\mathfrak{g}$, considered as a real Hilbert space with respect to $\langle\cdot,\cdot\rangle_\theta$, and we write $g \to \infty$ to denote that $\|Ad(g)\|$ goes to infinity. Because of [vdBS87, Lemma 2.1, (iii)] and the KAK-decomposition of G, this is the same as saying $e^{\alpha(\log a)} \to \infty$ for all $\alpha \in \Sigma^+$, where $g = k_1 a k_2$, $k_1, k_2 \in K$, $a \in A^+$, $\log : A \to \mathfrak{a}$ is the inverse of the Lie algebra exponential map $\exp : \mathfrak{a} \to A$.

In this section, we relate the complex geometric setting discussed above to irreducible admissible G-representations. The first important observation in this direction is the following result on the holomorphic extension of the orbit map of a non-zero K-finite vector to the $G \times K_{\mathbb{C}}$-invariant domain $\widetilde{\Xi}$. We present the essential ideas of the proof, along the lines of the proof of [KS04, Theorem 3.1]. Note that while admissibility is a crucial assumption, we do not need the Banach spaces to be reflexive for the following theorem.

Theorem 1. *Let (π, V) be an irreducible admissible Banach representation of a connected, non-compact, simple Lie group G. If $0 \neq v \in V_K$ is a K-finite vector, then the orbit map $\gamma_v : G \longrightarrow V$ extends to a G-equivariant holomorphic map on $\widetilde{\Xi} = G\exp(i\Omega)K_{\mathbb{C}}$.*

Proof. Let V^∞ be the subspace of smooth vectors in V. Then $V_K \subset V^\infty$ and it is clear that $\gamma_v(G) \subset V^\infty$. Now, by the admissibility of π, V^∞ equipped with the topology induced by the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ becomes the (smooth) Casselman-Wallach globalisation of the Harish-Chandra module V_K (See [Cas89], [Wal92, Chapter 11] or see [BK14] for a different approach). On the other hand, since the topology on V^∞ as a Casselman-Wallach globalisation is finer than the topology on it induced from V, we only need to prove that the orbit map $\gamma_v : G \longrightarrow V^\infty$ extends to a G-equivariant map on $\widetilde{\Xi} = G\exp(i\Omega)K_{\mathbb{C}}$ which is holomorphic with respect to the topology of the Casselman-Wallach globalisation (π^∞, V^∞).

Now, by Casselman's submodule theorem, (π^∞, V^∞) is embedded (as a closed G-submodule) into a smooth principal series representation $(\pi^\infty_{\tau,\lambda}, \mathcal{H}^\infty_{\tau,\lambda})$ arising from a minimal parabolic subgroup P of G. In this way, we can assume that $v \in V^\infty \subseteq H^\infty_{\tau,\lambda}$. Then we can follow the argument in [KS04, Theorem 3.1] in order to conclude that $\gamma_v : G \longrightarrow V^\infty$ extends to a G-equivariant holomorphic map on $\widetilde{\Xi}$. □

This theorem tells us that the G-invariant domains of holomorphy that we seek necessarily contain the domain $\widetilde{\Xi}$. The question then is whether such domains can be larger than $\widetilde{\Xi}$, and if so, to understand the connection between the geometry of the domains and representation theory. This crucial link is established by using the vanishing property of matrix coefficients at infinity of the irreducible admissible representations under consideration, and relating this to properness of the G-action. We can then use the finer group theoretic structure according to whether G is Hermitian or non-Hermitian to establish precisely what the sought-after G-invariant domains of holomorphy are. Here is the result on vanishing at infinity of matrix coefficients that we need.

Proposition 1. *Let (π, V) be an infinite-dimensional uniformly bounded irreducible representation of G on a reflexive Banach space. Then all the matrix coefficients of π vanish at infinity.*

Proof. For the case of irreducible uniformly bounded Hilbert representations, see [BW80, Theorem 5.4]). For irreducible uniformly bounded representations on a reflexive Banach space, we refer to the recent article [BG16, Corollary 9.5]. We remark that since our group G is a connected simple linear Lie group, any proper normal subgroup is necessarily discrete, and which by [Kna02, Proposition 1.93. (c),(d)] is closed and lies in the centre of G. Further, by [Kna02, Theorem 6.31 (e)], the centre of G is contained in the maximal compact subgroup K. Thus, any proper normal subgroup of G is finite. We can then use the irreducibility of π to apply [BG16, Corollary 9.5] and conclude that the matrix coefficients for irreducible uniformly bounded representations on reflexive Banach spaces vanish at infinity. □

We now use this proposition to prove the following theorem on the properness of the G-action.

Theorem 2. *Let G be a non-compact simple real linear Lie group and let (π, V) be an infinite-dimensional uniformly bounded irreducible admissible representation of G on a reflexive Banach space. Then G acts properly on $D = \widetilde{D}/K_{\mathbb{C}}$, for any maximal $G \times K_{\mathbb{C}}$-invariant domain $\widetilde{D} \subset G_{\mathbb{C}}$ to which the orbit map γ_v of a non-zero K-finite vector v extends holomorphically.*

The first crucial step is the following lemma, which allows us to prove this theorem, not just for irreducible uniformly bounded Hilbert representations but also for irreducible admissible uniformly bounded representations on reflexive Banach spaces. This extends [Krö08, Lemma 4.2] proved there in the unitary case.

Lemma 1. *Let (π, V) be an infinite-dimensional irreducible admissible uniformly bounded representation of G on a reflexive Banach space. Then G acts properly on $V \setminus \{0\}$.*

Proof. The G-action on $V \setminus \{0\}$ is proper if and only if for every compact subset W of $V \setminus \{0\}$, the set $W_G = \{g \in G \,|\, \pi(g)W \cap W \neq \emptyset\}$ is compact. Notice that since the G-action is continuous, if W is compact, then W_G is always closed. We claim that W_G is also compact. Suppose W_G is not compact for a compact W. Then there exist sequences $(g_n)_{n\in\mathbb{N}}$ in W_G and $(v_n)_{n\in\mathbb{N}}$ in W such that $\lim_{n\to\infty} g_n = \infty$ but $\pi(g_n)v_n \in W$ for all $n \in \mathbb{N}$. By the compactness of W we may assume, by going to subsequences if necessary, that there exist $v, w \in W$ such that $\lim_{n\to\infty} v_n = v$ and $\lim_{n\to\infty} \pi(g_n)v_n = w$. Now, since π is uniformly bounded, we have that $\|\pi(g_n)v_n - \pi(g_n)v\| \leq C \|v_n - v\|$, and so it follows that $\lim_{n\to\infty} \pi(g_n)v = w$. Since $w \neq 0$, there is an $f \in V'$ such that $(w, f) \neq 0$.

But then $\lim_{n\to\infty}(\pi(g_n)v, f) = (w, f) \neq 0$ - a contradiction to the vanishing of all the matrix coefficients at infinity, guaranteed by Proposition 1. This concludes the proof. □

Now we are in a position to prove Theorem 2. This proof crucially uses the fact that the representation is uniformly bounded, thus extending [KO08, Theorem 4.3].

Proof of Theorem 2: We know that the orbit map $\gamma_v : G \to V$ extends to a G-equivariant holomorphic map $\widetilde{D} \to V$. Then using the identity theorem for holomorphic maps we can deduce, as in the proof of [KO08, Theorem 4.3], that $\pi(\tilde{z})v \neq 0$ for all $\tilde{z} \in \widetilde{D}$.

Now suppose that G does not act properly on D. Then there exist sequences $(g_n)_{n\in\mathbb{N}}$ in G with $g_n \to \infty$, $(z_n)_{n\in\mathbb{N}}$ in D such that $z_n \to z \in D$ and $g_n z_n \to w \in D$. We choose pre-images $\tilde{z}_n, \tilde{z}, \tilde{w}$ in $\widetilde{D}$ of z_n, z, w, respectively, such that $\tilde{z}_n \to \tilde{z}$. Then a sequence $(k_n)_{n\in\mathbb{N}}$ in $K_{\mathbb{C}}$ can also be chosen so that $g_n\tilde{z}_n k_n \to \tilde{w}$, from which it follows that $\pi(g_n\tilde{z}_n k_n)v \to \pi(\tilde{w})v$. For a uniformly bounded representation, we note that sub-multiplicativity of the operator norm gives us also a bound from below:

$$\frac{1}{C} \leq \|\pi(g)\| \leq C \qquad \forall g \in G,$$

where C is as in (1). So it follows that $\frac{1}{C}\|\pi(\tilde{z}_n k_n)v\| \leq \|\pi(g_n\tilde{z}_n k_n)v\| \leq C\|\pi(\tilde{z}_n k_n)v\|$. Then since $\pi(\tilde{z})v \neq 0\ \forall \tilde{z} \in \widetilde{D}$, $\exists\ \alpha_1, \alpha_2 > 0$ such that $\alpha_1 < \|\pi(\tilde{z}_n k_n)v\| < \alpha_2$, for all $n \in \mathbb{N}$. Let W denote the linear span of the K-translates of v. From these considerations and the fact that $\tilde{z}_n \to \tilde{z}$, we obtain that $\pi(\tilde{z}_n)_{|W} - \pi(\tilde{z})_{|W} \to 0$, and hence that $\beta_1 < \|\pi(k_n)v\| < \beta_2$ for some constants $\beta_1, \beta_2 > 0$. As $v \in V_K$, W is finite-dimensional, and so the closure U of the sequences $(\pi(g_n\tilde{z}_n k_n)v)_{n\in\mathbb{N}}$ and $(\pi(\tilde{z}_n k_n)v)_{n\in\mathbb{N}}$ in V is a compact subset of $V \setminus \{0\}$. But then the set $U_G = \{g \in G \mid \pi(g)U \cap U \neq \emptyset\}$ contains the unbounded sequence $(g_n)_{n\in\mathbb{N}}$, and this is a contradiction to Lemma 1, thus proving the theorem. □

Now we are in a position to use the geometry of the situation according to if G is Hermitian or not Hermitian, along with Theorem 1 and Theorem 2 to determine the G-invariant domains of holomorphy. First, suppose G is non-Hermitian. Then as in [Krö08, Lemma 4.4], we have that for any root $\alpha \in \Sigma$ and $Y \in \mathfrak{g}_\alpha$, there exists an $m \in M = Z_K(\mathfrak{a})$ such that $\mathrm{Ad}\,(m)Y = -Y$. For the sake of the reader, we give a few more references and details here. If $\dim \mathfrak{g}_\alpha > 1$, then this follows from the fact [Kos04, Theorem 2.13] that the identity component of M acts transitively on the unit sphere in the root space $\mathfrak{g}_\alpha$. If not, we first observe that the M-group of $SL(3,\mathbb{R})$ is $(\mathbb{Z}/2\mathbb{Z})^2$ and see by an easy calculation that the result holds for $SL(3,\mathbb{R})$. Then since G is non-Hermitian, we use the classification of non-compact real simple Lie algebras [Kna02, Appendix C] to see that the only Lie algebras that we need to consider are the following: $\mathfrak{sl}(n,\mathbb{R})$ for $n \geq 3$ (split real form), $\mathfrak{so}(p,q)$ for $0, 2 \neq p, q$ and $p+q > 2$, $E\ I$ (split real form), $E\ II$ (quaternion type), $E\ V$ (split real form), $E\ VI$ (quaternion type), $E\ VIII$ (split real form), $E\ IX$ (quaternion type), $F\ I$ (split real form) and G (split real form). Except in the orthogonal cases in the above list, we can put $Y \in \mathfrak{g}_\alpha$ with $\dim \mathfrak{g}_\alpha = 1$ into a Lie sub-algebra isomorphic to $\mathfrak{sl}(3,\mathbb{R})$, as in the proof of [Krö08, Lemma 4.4]. Further, [Joh87, Theorem 3.5] describes the component group of the M-groups as being a certain number of copies of $\mathbb{Z}/2\mathbb{Z}$, with a precise description of how many copies occur. In the split cases, by [Joh87, Theorem 3.5 , Corollary 3.ii.] and reading off the rank of these groups from [Kna02, Appendix C], we see that the M-group of $SL(3,\mathbb{R})$ embeds into the M-group of the given group. In the three quaternionic exceptional cases in the above list, we see by [Joh87, Theorem 3.5, Corollary 3.iii], that the component group of M is $(\mathbb{Z}/2\mathbb{Z})^2$ and so the M-group of $SL(3,\mathbb{R})$ embeds into the M-group of these groups. The case of the orthogonal groups in the above list can be done by explicit calculation.

The above discussion, together with a certain precise description of the boundary of $\widetilde{\Xi}$ in the $SL(2,\mathbb{R})$ case, leads, as in [Krö08, Theorem 4.1], to the geometric result that $D \subset \Xi$ for any G-invariant domain D such that $\mathbb{X} \subset D \subset \mathbb{X}_{\mathbb{C}}$ on which G acts properly. Therefore, we have that $\widetilde{D}/K_{\mathbb{C}}$ is contained in the crown domain Ξ. Now suppose $\widetilde{D}$ is a maximal $G \times K_{\mathbb{C}}$-invariant domain to which γ_v extends holomorphically. Observe that by Theorem 1, $\widetilde{D}$ necessarily contains the domain $\widetilde{\Xi}$. Theorem 2 then tells us that G acts properly on $\widetilde{D}/K_{\mathbb{C}}$, and so it follows that it is equal to Ξ. This yields our main result in the non-Hermitian case.

Theorem 3. *Let G be a connected non-compact real simple linear Lie group which is not Hermitian. If (π, V) is an infinite-dimensional irreducible admissible uniformly bounded representation of G on a reflexive Banach space, then the corresponding G-invariant domain of holomorphy is given by $D_\pi = \widetilde{\Xi}$. In particular, this is the case for all infinite-dimensional irreducible uniformly bounded Hilbert representations of G.* □

We conclude by considering the case when G is Hermitian. Here we use the notation set up in Section 2. Suppose (π, V) is a highest (respectively, lowest) weight infinite-dimensional irreducible admissible uniformly bounded G-representation on a reflexive Banach space V, i.e. $\mathsf{P}^{\pm}$ acts finitely on V_K. Using this, and Theorem 1 as well as the fact that V_K is also naturally a $K_{\mathbb{C}}$-module, we can conclude as in [Krö08] that, for $0 \neq v \in V_K$, the orbit map γ_v extends holomorphically to the $G \times K_{\mathbb{C}}$-invariant domain $GK_{\mathbb{C}}\mathsf{P}^{\pm} = \widetilde{\Xi}^{\pm}$. Now since G is Hermitian, it is in fact true that if a G-invariant domain D such that $\mathbb{X} \subset D \subset \mathbb{X}_{\mathbb{C}}$ admits a proper G-action, then either $D \subset \widetilde{\Xi}^{+}/K_{\mathbb{C}}$ or $D \subset \widetilde{\Xi}^{-}/K_{\mathbb{C}}$. Thus we can conclude, using Theorem 2, that

Proposition 2. *For a Hermitian group G, we have that the G-invariant domains of holomorphy $\widetilde{D}_\pi = \widetilde{\Xi}^{\pm}$ depending on whether (π, V) is of highest weight or lowest weight, respectively.* □

References

[AG90] D. Akhiezer and S. Gindikin, *On Stein extensions of real symmetric spaces*, Math. Ann. **286** (1990), 1–12.

[BFGM07] U. Bader, A. Furman, T. Gelander, and N. Monod, *Property (T) and rigidity for actions on Banach spaces*, Acta Math. **198** (2007), 57–105.

[BG16] U. Bader and T. Gelander, *Equicontinuous actions of semisimple groups*, To appear in Groups, Geometry and Dynamics, arXiv Preprint 1408.4217, 2016.

[BK14] J. Bernstein and B. Krötz, *Smooth Fréchet globalizations of Harish-Chandra modules*, Israel J. Math. **199** (2014), 45 – 111.

[Bre11] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Universitext, Springer, 2011.

[BW80] A. Borel and N. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, vol. 94, Ann. of Math. Stud., 1980.

[Cas89] W. Casselman, *Canonical extensions of Harish-Chandra modules to representations of G*, Can. J. Math. **XLI** (1989), no. 3, 385–438.

[Cow78] M. Cowling, *The Kunze-Stein phenomenon*, Ann. of Math. **107** (1978), no. 2, 209–234.
[Cow08] ———, *Applications of representation theory to harmonic analysis of Lie groups (and vice versa)*, Representation Theory and Complex Analysis, Lecture Notes in Mathematics, vol. 1931, Springer-Verlag Berlin Heidelberg, 2008.
[Joh87] K.D. Johnson, *A strong generalization of Helgason's theorem*, Trans. Am. Math. Soc. **304** (1987), no. 1, 171–192.
[Kna02] A. W. Knapp, *Lie groups beyond an introduction*, Progress in Mathematics, no. 140, 2nd edition Birkhäuser, 2002.
[KO08] B. Krötz and E.M. Opdam, *Analysis on the crown domain*, Geom. Funct. Anal. **18** (2008), 1326 – 1421.
[Kos04] B. Kostant, *A branching law for subgroups fixed by an involution and a noncompact analogue of the Borel-Weil theorem*, Noncommutative Harmonic Analysis, Progress in Mathematics, vol. 220, Birkhäser, 2004.
[Krö08] B. Krötz, *Domains of holomorphy for irreducible unitary representations of simple Lie groups*, Invent. Math. **172** (2008), 277 – 288.
[KS04] B. Krötz and R.J. Stanton, *Holomorphic extensions of representations: (I) automorphic functions*, Ann. of Math. **159** (2004), 641 – 724.
[KS05] ———, *Holomorphic extensions of representations: (II) geometry and harmonic analysis*, Geom. Funct. Anal. **15** (2005), 190 – 245.
[vdBS87] E.P. van den Ban and H. Schlichtkrull, *Asymptotic expansions and boundary values of eigenfunctions on Riemannian symmetric spaces*, J. Reine Angew. Math. **380** (1987), 108–165.
[Wal92] N. R. Wallach, *Real reductive groups*, vol. II, Academic Press, Inc., 1992.
[War72] G. Warner, *Harmonic analysis on semi-simple Lie groups*, vol. I, Springer–Verlag, Berlin, Heidelberg, New York, 1972.

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