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Smooth structures on a fake real projective space

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We show that the group of smooth homotopy 7-spheres acts freely on the set of smooth manifold structures on a topological manifold M which is homotopy equivalent to the real projective 7-space. We classify, up to diffeomorphism, all closed manifolds homeomorphic to the real projective 7-space. We also show that M has, up to diffeomorphism, exactly 28 distinct differentiable structures with the same underlying PL structure of M and 56 distinct differentiable structures with the same underlying topological structure of M.

1. Introduction

Throughout this paper M^m will be a closed oriented m-manifold and all homeomorphisms and diffeomorphisms are assumed to preserve orientation, unless otherwise stated. Let $\mathbb{R}\mathbf{P}^n$ be real projective n-space. López de Medrano [10] and C.T.C. Wall [17,18] classified, up to PL homeomorphism, all closed PL manifolds homotopy equivalent to $\mathbb{R}\mathbf{P}^n$ when $n > 4$. This was extended to the topological category by Kirby–Siebenmann [9, p. 331]. Four-dimensional surgery [4] extends the homeomorphism classification to dimension 4.

In this paper we study up to diffeomorphism all closed manifolds homeomorphic to $\mathbb{R}\mathbf{P}^7$. Let M be a closed smooth manifold homotopy equivalent to $\mathbb{R}\mathbf{P}^7$. In section 2, we show that if a closed smooth manifold N is PL-homeomorphic to M, then there is a unique homotopy 7-sphere $\Sigma^7 \in \Theta_7$ such that N is diffeomorphic to $M\#\Sigma^7$, where Θ_7 is the group of smooth homotopy spheres defined by M. Kervaire and J. Milnor in [7]. In particular, M has, up to diffeomorphism, exactly 28 distinct differentiable structures with the same underlying PL structure of M.

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In section 3, we show that if a closed smooth manifold N is homeomorphic to M, then there is a unique homotopy 7-sphere $\Sigma^7 \in \Theta_7$ such that N is diffeomorphic to either $M\#\Sigma^7$ or $\widetilde{M}\#\Sigma^7$, where $\widetilde{M}$ represents the non-zero concordance class of PL-structure on M. We also show that the group of smooth homotopy 7-spheres Θ_7 acts freely on the set of smooth manifold structures on a manifold M.

2. Smooth structures with the same underlying PL structure of a fake real projective space

We recall some terminology from [7]:

Definition 2.1. ([7])

(a) A homotopy m-sphere Σ^m is a smooth closed manifold homotopy equivalent to the standard unit sphere $\mathbb{S}^m$ in $\mathbb{R}^{m+1}$.
(b) A homotopy m-sphere Σ^m is said to be exotic if it is not diffeomorphic to $\mathbb{S}^m$.
(c) Two homotopy m-spheres Σ^m_1 and Σ^m_2 are said to be equivalent if there exists a diffeomorphism $f : \Sigma^m_1 \to \Sigma^m_2$.

The set of equivalence classes of homotopy m-spheres is denoted by Θ_m. The equivalence class of Σ^m is denoted by $[\Sigma^m]$. M. Kervaire and J. Milnor [7] showed that Θ_m forms a finite abelian group with group operation given by connected sum $\#$ except possibly when $m = 4$ and the zero element represented by the equivalence class of $\mathbb{S}^m$.

Definition 2.2. Let M be a closed PL-manifold. Let (N, f) be a pair consisting of a closed PL-manifold N together with a homotopy equivalence $f : N \to M$. Two such pairs (N_1, f_1) and (N_2, f_2) are equivalent provided there exists a PL homeomorphism $g : N_1 \to N_2$ such that $f_2 \circ g$ is homotopic to f_1. The set of all such equivalence classes is denoted by $\mathcal{S}^{PL}(M)$.

Definition 2.3 *(Cat = Diff* or *PL-structure sets).* Let M be a closed *Cat*-manifold. Let (N, f) be a pair consisting of a closed *Cat*-manifold N together with a homeomorphism $f : N \to M$. Two such pairs (N_1, f_1) and (N_2, f_2) are concordant provided there exists a *Cat*-isomorphism $g : N_1 \to N_2$ such that the composition $f_2 \circ g$ is topologically concordant to f_1, i.e., there exists a homeomorphism $F : N_1 \times [0, 1] \to M \times [0, 1]$ such that $F_{|N_1\times 0} = f_1$ and $F_{|N_1\times 1} = f_2 \circ g$. The set of all such concordance classes is denoted by $\mathcal{C}^{Cat}(M)$.

We will denote the class in $\mathcal{C}^{Cat}(M)$ of (N, f) by $[N, f]$. The base point of $\mathcal{C}^{Cat}(M)$ is the equivalence class $[M, Id]$ of $Id : M \to M$.

We will also denote the class in $\mathcal{C}^{Diff}(M)$ of $(M^n\#\Sigma^n, \mathrm{Id})$ by $[M^n\#\Sigma^n]$. (Note that $[M^n\#\mathbb{S}^n]$ is the class of (M^n, Id).)

Definition 2.4. Let M be a closed PL-manifold. Let (N, f) be a pair consisting of a closed smooth manifold N together with a PL-homeomorphism $f : N \to M$. Two such pairs (N_1, f_1) and (N_2, f_2) are PL-concordant provided there exists a diffeomorphism $g : N_1 \to N_2$ such that the composition $f_2 \circ g$ is PL-concordant to f_1, i.e., there exists a PL-homeomorphism $F : N_1 \times [0, 1] \to M \times [0, 1]$ such that $F_{|N_1\times 0} = f_1$ and $F_{|N_1\times 1} = f_2 \circ g$. The set of all such concordance classes is denoted by $\mathcal{C}^{PDiff}(M)$.

Definition 2.5. Let M^m be a closed smooth m-dimensional manifold. The inertia group $I(M) \subset \Theta_m$ is defined as the set of $\Sigma \in \Theta_m$ for which there exists a diffeomorphism $\phi : M \to M\#\Sigma$.

The concordance inertia group $I_c(M)$ is defined as the set of all $\Sigma \in I(M)$ such that $M\#\Sigma$ is concordant to M.

The key to analyzing $\mathcal{C}^{Diff}(M)$ and $\mathcal{C}^{PDiff}(M)$ are the following results.

Theorem 2.6. *(Kirby and Siebenmann, [9, p. 194]) There exists a connected H-space Top/O such that there is a bijection between $\mathcal{C}^{Diff}(M)$ and $[M, Top/O]$ for any smooth manifold M with $\dim M \geq 5$. Furthermore, the concordance class of given smooth structure of M corresponds to the homotopy class of the constant map under this bijection.*

Theorem 2.7. *(Cairns–Hirsch–Mazur, [6]) Let M^m be a closed smooth manifold of dimension $m \geq 1$. Then there exists a connected H-space PL/O such that there is a bijection between $\mathcal{C}^{PDiff}(M)$ and $[M, PL/O]$. Furthermore, the concordance class of the given smooth structure of M corresponds to the homotopy class of the constant map under this bijection.*

Theorem 2.8. *([7])* $\Theta_7 \cong \mathbb{Z}_{28}$.

We now use the Eells–Kuiper μ invariant [3,15] to study the inertia group of smooth manifolds homotopy equivalent to $\mathbb{R}\mathbf{P}^7$. We recall the definition of the Eells–Kuiper μ invariant in dimension 7. Let M be a 7-dimensional closed oriented spin smooth manifold such that the 4-th cohomology group $H^4(M;\mathbb{R})$ vanishes. Since the spin cobordism group Ω_7^{Spin} is trivial [11], M bounds a compact oriented spin smooth manifold N. Then the first Pontrjagin class $p_1(N) \in H^4(N, M; \mathbb{Q})$ is well-defined. The Eells–Kuiper differential invariant $\mu(M) \in \mathbb{R}/\mathbb{Z}$ of M is given by

$$\mu(M) = \frac{p_1^2(N)}{2^7 \times 7} - \frac{\mathrm{Sign}(N)}{2^5 \times 7} \quad \mathrm{mod}(\mathbb{Z}),$$

where $p_1^2(N)$ denotes the corresponding Pontrjagin number and $\mathrm{Sign}(N)$ is the signature of N.

Theorem 2.9. *Let M be a closed smooth spin 7-manifold such that $H^4(M;\mathbb{R}) = 0$. Then the Θ_7-action on $\mathcal{C}^{PDiff}(M)$ of the form $M \mapsto M\#\Sigma$ is free and transitive. In particular, if N is a closed smooth manifold (oriented) PL-homeomorphic to M, then there is a unique homotopy 7-sphere $\Sigma^7 \in \Theta_7$ such that N is (oriented) diffeomorphic to $M\#\Sigma^7$.*

Proof. For any degree one map $f_M : M^7 \to \mathbb{S}^7$, we have a homomorphism

$$f_M^* : [\mathbb{S}^7, PL/O] \to [M^7, PL/O]$$

and in terms of the identifications

$$\Theta_7 = [S^7, PL/O] \text{ and } \mathcal{C}^{PDiff}(M) = [M^7, PL/O]$$

given by Theorem 2.7, f_M^* becomes $[\Sigma] \mapsto [M\#\Sigma]$. Therefore, to show that Θ_7 acts freely and transitively on $\mathcal{C}^{PDiff}(M)$, it is enough to prove that

$$f_M^* : [\mathbb{S}^7, PL/O] \to [M, PL/O]$$

is bijective. Let $M^{(6)}$ be the 6-skeleton of a CW-decomposition for M containing just one 7-cell. Such a decomposition exists by [16]. Let $f_M : M \to M/M^{(6)} = \mathbb{S}^7$ be the collapsing map. Now consider the Barratt–Puppe sequence for the inclusion $i : M^{(6)} \hookrightarrow M$ which induces the exact sequence of abelian groups on taking homotopy classes $[-, PL/O]$

$$\cdots \to [SM^{(6)}, PL/O] \to [S^7, PL/O] \xrightarrow{f_M^*} [M, PL/O] \xrightarrow{i^*} [M^{(6)}, PL/O] \cdots,$$

where SM is the suspension of M. As PL/O is 6-connected [1,7], it follows that any map from $M^{(6)}$ to PL/O is null-homotopic (see [2, Theorem 7.12]). Therefore $i^* : [M, PL/O] \to [M^{(6)}, PL/O]$ is the zero homomorphism and so $f_M^* : [\mathbb{S}^7, PL/O] \to [M, PL/O]$ is surjective. Since our assumption on M and using the additivity of the Eells–Kuiper differential invariant μ with respect to connected sums, if $\Sigma \in I(M)$, then

$$\mu(M) = \mu(M\#\Sigma) = \mu(M) + \mu(\Sigma).$$

Therefore $\mu(\Sigma) = 0$ in $\mathbb{R}/\mathbb{Z}$ would imply that Σ is diffeomorphic to $\mathbb{S}^7$, since Eells and Kuiper [3] showed that $\mu(\Sigma_M^{\#m}) = \frac{m}{28}$, where Σ_M is a generator of Θ_7, and $\Theta_7 \cong \mathbb{Z}_{28}$. Therefore $I(M) = 0$ and hence the homomorphism $f_M^* : [\mathbb{S}^7, PL/O] \to [M, PL/O]$ is injective, proving the first part of the theorem. The second part of the theorem follows easily from the first part. □

Remark 2.10. By Theorem 2.9, we can now prove the following.

(i) If a closed smooth manifold M is homotopy equivalent to $\mathbb{R}\mathbf{P}^7$, then M is a spin manifold with $H^4(M;\mathbb{R}) = 0$ and hence $I(M) = 0$.
(ii) If M is a closed 2-connected 7-manifold such that the group $H_4(M;\mathbb{Z})$ is torsion, then M is a spin manifold with $H^4(M;\mathbb{R}) = 0$ and hence $I(M) = 0$.

Applying Theorem 2.9, we immediately obtain

Corollary 2.11. *Let M be a closed smooth manifold homotopy equivalent to $\mathbb{R}\mathbf{P}^7$. Then M has, up to (oriented) diffeomorphism, exactly 28 distinct differentiable structures with the same underlying (oriented) PL structure of M.*

Remark 2.12. If a closed smooth manifold M is homotopy equivalent to $\mathbb{R}\mathbf{P}^n$, where $n = 5$ or 6, then M has exactly 2 distinct differentiable structures up to diffeomorphism [5,6,8,9].

3. The classification of smooth structures on a fake real projective space

The following theorem was proved in [13, Example 3.5.1] for $M = \mathbb{R}\mathbf{P}^7$. This proof works verbatim for an arbitrary manifold M as in Theorem 3.1.

Theorem 3.1. *Let M be a closed smooth manifold homotopy equivalent to $\mathbb{R}\mathbf{P}^7$. Then there is a closed smooth manifold $\widetilde{M}$ such that*

(i) *$\widetilde{M}$ is homeomorphic to M.*
(ii) *$\widetilde{M}$ is not (PL homeomorphic) diffeomorphic to M.*

Proof. Let $j_{TOP} : \mathcal{C}^{PL}(M) \to [M, TOP/PL] = H^3(M;\mathbb{Z}_2)$ be a bijection given by [8,9] and $j_F : \mathcal{S}^{PL}(M) \to [M, F/PL]$ be the normal invariant map defined by Sullivan, see [12,14]. Then the maps j_{TOP} and j_F can be included in the commutative diagram

$$\begin{array}{ccc}
\mathcal{C}^{PL}(M) & \xrightarrow{\;j_{TOP}\;} & [M, TOP/PL] \\
\downarrow{\scriptstyle \mathcal{F}} & & \downarrow{\scriptstyle a_*} \\
\mathcal{S}^{PL}(M) & \xrightarrow[\;j_F\;]{} & [M, F/PL]
\end{array}$$

where $\mathcal{F}$ is the obvious forgetful map and a_* is induced by the natural map $a : TOP/PL \to F/PL$. Consider an element $[\widetilde{M}, k] \in \mathcal{C}^{PL}(M)$, where $\widetilde{M}$ is a closed PL-manifold and $k : \widetilde{M} \to M$ is a homeomorphism such that

$$j_{TOP}([\widetilde{M}, k]) \neq 0 \in [M, TOP/PL] = H^3(M; \mathbb{Z}_2) \cong \mathbb{Z}_2. \tag{1}$$

Notice that the Bockstein homomorphism

$$\delta : \mathbb{Z}_2 = H^3(M; \mathbb{Z}_2) \to H^4(M; \mathbb{Z}[2]) = \mathbb{Z}_2$$

is an isomorphism, where $\mathbb{Z}[2]$ is the subring of $\mathbb{Q}$ consisting of all irreducible fractions with denominators relatively prime to 2. Hence

$$\delta(j_{TOP}([\widetilde{M}, k])) \neq 0.$$

So, by [13, Corollary 3.2.5], $a_*(j_{TOP}([\widetilde{M}, k])) \neq 0$. In view of the above commutativity of the diagram,

$$j_F(\mathcal{F}([\widetilde{M}, k])) = a_*(j_{TOP}([\widetilde{M}, k])),$$

i.e., $j_F(\mathcal{F}([\widetilde{M}, k])) \neq 0$. This implies that $\mathcal{F}([\widetilde{M}, k]) \neq 0$. Hence $[\widetilde{M}, k] \neq [M, Id]$ in $\mathcal{S}^{PL}(M)$. On the other hand, it follows from the obstruction theory that every orientation-preserving homotopy equivalence $h : M \to M$ is homotopic to the identity map. This shows that $\widetilde{M}$ is not PL homeomorphic to M. By an obstruction theory given by [6], every PL-manifold of dimension 7 possesses a compatible differentiable structure. This implies that $\widetilde{M}$ is smoothable such that $\widetilde{M}$ cannot be diffeomorphic to M. This proves the theorem. □

Theorem 3.2. *Let M be a closed smooth manifold homotopy equivalent to $\mathbb{R}\mathbf{P}^7$. Then*

$$\mathcal{C}^{Diff}(M) = \left\{[M\#\Sigma, Id], [\widetilde{M}\#\Sigma, k \circ Id] \;\mid\; \Sigma \in \Theta_7\right\},$$

where $\widetilde{M}$ is the specific closed smooth manifold given by Theorem 3.1 and $k : \widetilde{M} \to M$ is the homeomorphism as in Equation (1). In particular, M has exactly 56 distinct differentiable structures up to concordance.

Proof. Let $[N, f] \in \mathcal{C}^{Diff}(M)$, where N is a closed smooth manifold and $f : N \to M$ be a homeomorphism. Then (N, f) represents an element in

$$\mathcal{C}^{PL}(M) \cong H^3(M; \mathbb{Z}_2) = \mathbb{Z}_2 = \left\{[M, Id], [\widetilde{M}, k]\right\},$$

where $\widetilde{M}$ is the specific closed smooth manifold given by Theorem 3.1 and $k : \widetilde{M} \to M$ be a homeomorphism as in Equation (1). This implies that (N, f) is either equivalent to (M, Id) or $(\widetilde{M}, k)$ in $\mathcal{C}^{PL}(M)$. Suppose that (N, f) is equivalent to (M, Id) in $\mathcal{C}^{PL}(M)$, then there is a PL-homeomorphism $h : N \to M$ such that $Id \circ h : N \to M$ is topologically concordant to $f : N \to M$. Now consider a pair (N, h) which represents an element in $\mathcal{C}^{PDiff}(M)$. By Theorem 2.9, there is a unique homotopy sphere Σ such that (N, h) is PL-concordant to $(M\#\Sigma, Id)$. Hence there is a diffeomorphism $\phi : N \to M\#\Sigma$ such that $Id \circ \phi : N \to M$ is topologically concordant to $h : N \to M$. Note that $Id \circ h : N \to M$ is topologically concordant to $f : N \to M$. This implies that $Id \circ \phi : N \to M$ is topologically concordant to $f : N \to M$. Therefore, (N, f) and $(M\#\Sigma, Id)$ represent the same element in $\mathcal{C}^{Diff}(M)$.

On the other hand, suppose that (N, f) is equivalent to $(\widetilde{M}, k)$ in $\mathcal{C}^{PL}(M)$. This implies that there is a PL-homeomorphism $h : N \to \widetilde{M}$ such that $k \circ h : N \to M$ is topologically concordant to $f : N \to M$. By using the same argument as above, we have that there is a unique homotopy sphere Σ and a diffeomorphism $\phi : N \to \widetilde{M}\#\Sigma$ such that

$$k \circ Id \circ \phi : N \to \widetilde{M}\#\Sigma \to \widetilde{M} \to M$$

is topologically concordant to $f : N \to M$. Therefore, (N, f) and $(\widetilde{M}\#\Sigma, k \circ Id)$ represent the same element in $\mathcal{C}^{Diff}(M)$.

Thus, there is a unique homotopy sphere Σ such that (N, f) is either concordant to $(M\#\Sigma, Id)$ or $(\widetilde{M}\#\Sigma, k \circ Id)$ in $\mathcal{C}^{Diff}(M)$. This shows that

$$\mathcal{C}^{Diff}(M) = \left\{[M\#\Sigma, Id], [\widetilde{M}\#\Sigma, k \circ Id] \;\mid\; \Sigma \in \Theta_7\right\}.$$

In particular, M has exactly 56 distinct differentiable structures up to concordance. □

Theorem 3.3. *Let M be a closed smooth manifold homotopy equivalent to $\mathbb{R}\mathbf{P}^7$. Then Θ_7 acts freely on $\mathcal{C}^{Diff}(M)$.*

Proof. Suppose $[N\#\Sigma, f] = [N, f]$ in $\mathcal{C}^{Diff}(M)$. Then $N\#\Sigma \cong N$. Since by Theorem 3.2, there is a homotopy sphere Σ_1 such that $N \cong \overline{M}\#\Sigma_1$, where $\overline{M} = M$ or $\widetilde{M}$. This implies that

$$\overline{M}\#\Sigma_1\#\Sigma^{-1} \cong \overline{M}\#\Sigma_1$$

and hence $\Sigma_1\#\Sigma^{-1}\#\Sigma_1^{-1} \in I(\overline{M})$. But, by Remark 2.10(i), $I(\overline{M}) = 0$. This shows that $\Sigma_1\#\Sigma^{-1}\#\Sigma_1^{-1} \cong \mathbb{S}^7$. Hence $\Sigma \cong \mathbb{S}^7$. This proves that Θ_7 acts freely on $\mathcal{C}^{Diff}(M)$. □

Remark 3.4. Let M and $\widetilde{M}$ be as in Theorem 3.2. Then Θ_7 does not act transitively on $\mathcal{C}^{Diff}(M)$, since M and $\widetilde{M}$ are not PL-homeomorphic.

Theorem 3.5. *Let M be a closed smooth manifold which is homotopy equivalent to $\mathbb{R}\mathbf{P}^7$. Then M has exactly 56 distinct differentiable structures up to diffeomorphism. Moreover, if N is a closed smooth manifold homeomorphic to M, then there is a unique homotopy sphere $\Sigma \in \Theta_7$ such that N is either diffeomorphic to $M\#\Sigma$ or $\widetilde{M}\#\Sigma$, where $\widetilde{M}$ is the specific closed smooth manifold given by Theorem 3.1.*

Proof. Let N be a closed smooth manifold homeomorphic to M and let $f : N \to M$ be a homeomorphism. Then (N, f) represents an element in $\mathcal{C}^{Diff}(M)$. By Theorem 3.2, there is a unique homotopy sphere $\Sigma \in \Theta_7$ such that N is either concordant to $(M\#\Sigma, Id)$ or $(\widetilde{M}\#\Sigma, k \circ Id)$. This implies that N is either diffeomorphic to $M\#\Sigma$ or $\widetilde{M}\#\Sigma$. By Remark 2.10(i), $I(M) = I(\widetilde{M}) = 0$. Therefore there is a unique homotopy sphere $\Sigma \in \Theta_7$ such that N is either diffeomorphic to $M\#\Sigma$ or $\widetilde{M}\#\Sigma$. This implies that M has exactly 56 distinct differentiable structures up to diffeomorphism. □

References

[1] J. Cerf, Sur les difféomorphismes de la sphére de dimension trois ($\Gamma_4 = 0$), Lecture Notes in Mathematics, vol. 53, Springer-Verlag, Berlin–New York, 1968.

[2] J.F. Davis, P. Kirk, Lecture Notes in Algebraic Topology, Graduate Studies in Mathematics, vol. 35, American Mathematical Society, 2001.

[3] J. Eells, N.H. Kuiper, An invariant for certain smooth manifolds, Ann. Mat. Pura Appl. 60 (1962) 93–110.

[4] M.H. Freedman, F. Quinn, Topology of 4-Manifolds, Princeton Mathematical Series, vol. 39, Princeton University Press, Princeton, NJ, 1990.

[5] M.W. Hirsch, Obstruction theories for smoothing manifolds and maps, Bull. Amer. Math. Soc. 69 (1963) 352–356.

[6] M.W. Hirsch, B. Mazur, Smoothings of Piecewise Linear Manifolds, Annals of Mathematics Studies, vol. 80, Princeton University Press/University of Tokyo Press, Princeton, NJ/Tokyo, 1974.

[7] M.A. Kervaire, J.W. Milnor, Groups of homotopy spheres, I, Ann. of Math. 77 (2) (1963) 504–537.
[8] R.C. Kirby, L.C. Siebenmann, On the triangulation of manifolds and the Hauptvermutung, Bull. Amer. Math. Soc. 75 (1969) 742–749.
[9] R.C. Kirby, L.C. Siebenmann, Foundational Essays on Topological Manifolds, Smoothings, and Triangulations, with Notes by John Milnor and Michael Atiyah, Annals of Mathematics Studies, vol. 88, Princeton University Press/University of Tokyo Press, Princeton, NJ/Tokyo, 1977.
[10] S. López de Medrano, Involutions on Manifolds, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 59, Springer-Verlag, New York, 1971.
[11] J. Milnor, Spin structures on manifolds, Enseign. Math. (2) 9 (1963) 198–203.
[12] A. Ranicki, The Hauptvermutung Book, Kluwer, 1996.
[13] Yuli B. Rudyak, Piecewise Linear Structure on Topological Manifolds, World Scientific, New Jersey, 2016.
[14] D. Sullivan, On the Hauptvermutung for manifolds, Bull. Amer. Math. Soc. 73 (1967) 598–600.
[15] Z. Tang, W. Zhang, η-invariant and a problem of Bérard–Bergery on the existence of closed geodesics, arXiv:1302.2792, 2013.
[16] C.T.C. Wall, Poincaré complexes I, Ann. of Math. (2) 86 (1967) 213–245.
[17] C.T.C. Wall, Free piecewise linear involutions on spheres, Bull. Amer. Math. Soc. 74 (1968) 554–558.
[18] C.T.C. Wall, Surgery on Compact Manifolds, second edition, Mathematical Surveys and Monographs, vol. 69, American Mathematical Society, Providence, RI, 1999, Edited and with a foreword by A.A. Ranicki.