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Farrell–Jones spheres and inertia groups of complex projective spaces

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Abstract. We introduce and study a new class of homotopy spheres called Farrell–Jones spheres. Using Farrell–Jones sphere we construct examples of closed negatively curved manifolds M^{2n}, where $n = 7$ or 8, which are homeomorphic but not diffeomorphic to complex hyperbolic manifolds, thereby giving a partial answer to a question raised by C. S. Aravinda and F. T. Farrell. We show that every exotic sphere not bounding a spin manifold (Hitchin sphere) is a Farrell–Jones sphere. We also discuss the relationship between inertia groups of $\mathbb{CP}^n$ and Farrell–Jones spheres.

Keywords. Locally symmetric space, exotic smooth structure, complex hyperbolic, inertia groups.

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1 Introduction

Let Θ_m be the group of homotopy spheres defined by M. Kervaire and J. Milnor in [15].

Definition 1.1. We call $\Sigma^{2n} \in \Theta_{2n}$ $(n \geq 4)$ a Farrell–Jones sphere if $\mathbb{CP}^n \# \Sigma^{2n}$ is not concordant to $\mathbb{CP}^n$.

The following theorem gives an equivalent definition of Farrell–Jones spheres, which we prove in Section 3:

Theorem 1.2. *Let Σ^{2n} be an exotic sphere in Θ_{2n} $(n \geq 4)$. Then Σ^{2n} is a Farrell–Jones sphere if and only if $\mathbb{CP}^n \# \Sigma^{2n}$ is not diffeomorphic to $\mathbb{CP}^n$.*

By [10, Lemma 3.17], there exists a Farrell–Jones sphere $\Sigma^m \in \Theta_m$ for all $m = 8n + 2$ $(n \geq 1)$ and for $m = 8$. Also we prove the following theorem in Section 3:

Theorem 1.3. *The non-zero element of $\Theta_{2n} \cong \mathbb{Z}_2$ $(n = 7$ or $8)$ is a Farrell–Jones sphere.*

The study of Farrell–Jones spheres is motivated by the following result, which is a slight modification of [10, Theorem 3.20]:

Theorem 1.4. *Let M^{2n} be any closed complex hyperbolic manifold of complex dimension n. Let $\Sigma^{2n} \in \Theta_{2n}$ be a Farrell–Jones sphere. Given a positive real number ϵ, there exists a finite sheeted cover $\mathcal{N}^{2n}$ of M^{2n} such that the following is true for any finite sheeted cover N^{2n} of $\mathcal{N}^{2n}$.*

(i) *The smooth manifold N^{2n} is not diffeomorphic to $N^{2n} \# \Sigma^{2n}$.*

(ii) *The connected sum $N^{2n} \# \Sigma^{2n}$ supports a negatively curved Riemannian metric whose sectional curvatures all lie in the closed interval $[-4-\epsilon, -1+\epsilon]$.*

The proof of the above Theorem 1.4 follows from [10, Corollary 3.14 and Proposition 3.19]. By using Theorem 1.3 and Theorem 1.4, we also construct in Section 2 examples of closed negatively curved manifolds M^{2n}, where $n = 7$ or 8, which are homeomorphic but not diffeomorphic to complex hyperbolic manifolds, thereby giving a partial answer to a question raised by C. S. Aravinda and F. T. Farrell [5].

Another source for Farrell–Jones spheres is the class of the so-called Hitchin spheres. In [12], Hitchin showed that if Σ is a homotopy sphere with a metric of positive scalar curvature, then $\alpha(\Sigma) = 0$, where $\alpha : \Omega_*^{\text{spin}} \to \text{KO}_*$ is the ring homomorphism which associates to a spin bordism class the KO-valued index of the Dirac operator of a representative spin manifold. The following definition can be found in [19, Remark 3.4]:

Definition 1.5. An exotic sphere $\Sigma^m \in \Theta_m$ $(m \geq 1)$ is called a Hitchin sphere if $\alpha(\Sigma^m) \neq 0$.

We prove the following theorem in Section 3:

Theorem 1.6. *Every Hitchin $(8n+2)$-sphere $(n \geq 1)$ is a Farrell–Jones sphere.*

Recall that the collection of homotopy spheres which admit an orientation preserving diffeomorphism $M \to M \# \Sigma$ form the inertia group of M, denoted by $I(M)$. There is a canonical topological identification $\iota : M \to M \# \Sigma$ which is the identity outside of the attaching region; the subset of the inertia group consisting of homotopy spheres that admit a diffeomorphism homotopic to ι is called the homotopy inertia group $I_h(M)$. Similarly, the concordance inertia group of M^m, $I_c(M^m) \subseteq \Theta_m$, consists of those homotopy spheres Σ^m such that M and $M \# \Sigma^m$ are concordant. By Theorem 1.2, we have that Σ^{2n} is a Farrell–Jones sphere iff $\Sigma^{2n} \notin I(\mathbb{CP}^n)$ iff $\Sigma^{2n} \notin I_c(\mathbb{CP}^n)$ iff $\Sigma^{2n} \notin I_h(\mathbb{CP}^n)$. In Section 4, we discuss the group $I(\mathbb{CP}^{4n+1})$.

2 Exotic smooth structures on complex hyperbolic manifolds

The negatively curved Riemannian symmetric spaces are of four types: $\mathbb{R}\mathbf{H}^n$, $\mathbb{C}\mathbf{H}^n$, $\mathbb{H}\mathbf{H}^n$ and $\mathbb{O}\mathbf{H}^2$, where $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ denote the real, complex, quaternion and Cayley numbers, i.e., the four division algebras $\mathbb{K}$ over the real numbers whose dimensions over $\mathbb{R}$ are $d = 1, 2, 4$ and 8 respectively. A Riemannian manifold M^{dn} is called a real, complex, quarternionic or Cayley hyperbolic manifold provided its universal cover is isometric to $\mathbb{R}\mathbf{H}^n$, $\mathbb{C}\mathbf{H}^n$, $\mathbb{H}\mathbf{H}^n$ and $\mathbb{O}\mathbf{H}^2$, respectively. (Note that we need to consider only the cases $n \geq 2$ and when $\mathbb{K} = \mathbb{O}$, $n = 2$.)

In [5, p. 2], C. S. Aravinda and F. T. Farrell ask the following:

Question 2.1. For each division algebra $\mathbb{K}$ over the reals and each integer $n \geq 2$ ($n = 2$ when $\mathbb{K} = \mathbb{O}$), does there exist a closed negatively curved Riemannian manifold M^{dn}(where $d = \dim_{\mathbb{R}} \mathbb{K}$) which is homeomorphic but not diffeomorphic to a $\mathbb{K}$-hyperbolic manifold.

Remark 2.2. For $\mathbb{K} = \mathbb{R}$ and $n = 2, 3$, this is impossible since homeomorphism implies diffeomorphism in these dimensions [17]. Also when $\mathbb{K} = \mathbb{R}$, it was shown in [11] that the answer is yes provided $n \geq 6$. When $\mathbb{K} = \mathbb{C}$, it was shown in [10] that the answer is yes for $n = 4m + 1$ for any integer $m \geq 1$ and for $n = 4$. When $\mathbb{K} = \mathbb{H}$, the answer is yes for $n = 2, 4$ and 5, see [5]. The answer to this question is yes for $\mathbb{K} = \mathbb{O}$ by [4] since only one dimension needs to be considered in this case. In this section, we consider the case $\mathbb{K} = \mathbb{C}$ and show that the answer is yes for $n = 7, 8$.

Since Borel [6] has constructed closed complex hyperbolic manifolds in every complex dimension $m \geq 1$ and by Theorem 1.3 and Theorem 1.4, we have the following result:

Theorem 2.3. *Let n be either 7 or 8. Given any positive number $\epsilon \in \mathbb{R}$, there exists a pair of closed negatively curved Riemannian manifolds M and N having the following properties:*

(i) *M is a complex n-dimensional hyperbolic manifold.*

(ii) *The sectional curvatures of N are all in the interval* $[-4 - \epsilon, -1 + \epsilon]$.

(iii) *The manifolds M and N are homeomorphic but not diffeomorphic.*

3 Farrell–Jones spheres and Hitchin sphere

In this section, we give proofs of the Theorems 1.2, 1.3 and 1.6.

Definition 3.1. Let M be a topological manifold. Let (N, f) be a pair consisting of a smooth manifold N together with a homeomorphism $f : N \to M$. Two such pairs (N_1, f_1) and (N_2, f_2) are concordant provided there exists a diffeomorphism $g : N_1 \to N_2$ such that the composition $f_2 \circ g$ is topologically concordant to f_1, i.e., there exists a homeomorphism $F : N_1 \times [0, 1] \to M \times [0, 1]$ such that $F_{|N_1 \times 0} = f_1$ and $F_{|N_1 \times 1} = f_2 \circ g$. The set of all such concordance classes is denoted by $\mathcal{C}(M)$.

We recall some terminology from [15]:

Definition 3.2. (a) A homotopy m-sphere Σ^m is an oriented smooth closed manifold homotopy equivalent to $\mathbb{S}^m$.

(b) A homotopy m-sphere Σ^m is said to be exotic if it is not diffeomorphic to $\mathbb{S}^m$.

(c) Two homotopy m-spheres Σ_1^m and Σ_2^m are said to be equivalent if there exists an orientation preserving diffeomorphism $f : \Sigma_1^m \to \Sigma_2^m$.

The set of equivalence classes of homotopy m-spheres is denoted by Θ_m. The equivalence class of Σ^m is denoted by $[\Sigma^m]$. When $m \geq 5$, Θ_m forms an abelian group with group operation given by the connected sum # and the zero element represented by the equivalence class of the round sphere $\mathbb{S}^m$. M. Kervaire and J. Milnor [15] showed that each Θ_m is a finite group; in particular, Θ_8, Θ_{14} and Θ_{16} are cyclic groups of order 2, Θ_{10} and Θ_{20} are cyclic groups of order 6 and 24 respectively and Θ_{18} is a group of order 16.

Start by noting that there is a homeomorphism $h : M^n \# \Sigma^n \to M^n$ $(n \geq 5)$ which is the inclusion map outside of the homotopy sphere Σ^n and well defined up to topological concordance. We will denote the class of $(M^n \# \Sigma^n, h)$ in $\mathcal{C}(M)$ by $[M^n \# \Sigma^n]$. (Note that $[M^n \# \mathbb{S}^n]$ is the class of (M^n, id_{M^n}).) Let $f_M : M^n \to \mathbb{S}^n$ be a degree one map. Note that f_M is well defined up to homotopy. Composition with f_M defines a homomorphism

$$f_M^* : [\mathbb{S}^n, \mathrm{Top}/O] \to [M^n, \mathrm{Top}/O],$$

and in terms of the identifications

$$\Theta_n = [\mathbb{S}^n, \mathrm{Top}/O] \quad \text{and} \quad \mathcal{C}(M^n) = [M^n, \mathrm{Top}/O]$$

given by [16, p. 25 and p. 194], f_M^* becomes $[\Sigma^m] \mapsto [M^m \# \Sigma^m]$.

Definition 3.3. If M is homotopy equivalent to $\mathbb{CP}^n$, we will call a generator of $H^2(M; \mathbb{Z})$ a c-orientation of M.

Hereafter g denotes the conjugation map

$$(z_0, z_1, z_2, z_3, z_4, \ldots, z_n) \mapsto (\bar{z}_0, \bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4, \ldots, \bar{z}_n)$$

(the complex conjugation) induces the diffeomorphism $g : \mathbb{CP}^n \to \mathbb{CP}^n$ such that $g^*(c_1) = -c_1$ where c_1 is the c-orientation of $\mathbb{CP}^n$.

Proof of Theorem 1.2. Assume that Σ^{2n} is a Farrell–Jones sphere. Suppose $\mathbb{CP}^n$ and $\mathbb{CP}^n \# \Sigma^{2n}$ are diffeomorphic. If $f : \mathbb{CP}^n \# \Sigma^{2n} \to \mathbb{CP}^n$ is a diffeomorphism, then f induces an isomorphism on cohomology

$$f^* : H^*(\mathbb{CP}^n, \mathbb{Z}) \to H^*(\mathbb{CP}^n \# \Sigma^{2n}, \mathbb{Z})$$

such that $f^*(c_1) = \pm c_2$, where c_1, c_2 are c-orientations of $\mathbb{CP}^n$, $\mathbb{CP}^n \# \Sigma^{2n}$ respectively. If $f^*(c_1) = c_2$, then f is a c-orientation preserving diffeomorphism. If $f^*(c_1) = -c_2$, then $g \circ f$ is a c-orientation preserving diffeomorphism, where $g : \mathbb{CP}^n \to \mathbb{CP}^n$ is the conjugation map. In both cases, we have that $\mathbb{CP}^n \# \Sigma^{2n}$ is c-orientation diffeomorphic to $\mathbb{CP}^n$. By [18, Corollary 3, p. 97], $\mathbb{CP}^n \# \Sigma^{2n}$ is concordant to $\mathbb{CP}^n$. This is a contradiction since Σ^{2n} is a Farrell–Jones sphere. Thus $\mathbb{CP}^n \# \Sigma^{2n}$ and $\mathbb{CP}^n$ are not diffeomorphic. Conversely, suppose $\mathbb{CP}^n \# \Sigma^{2n}$ and $\mathbb{CP}^n$ are not diffeomorphic. Then, by [18, Corollary 3, p. 97], $\mathbb{CP}^n \# \Sigma^{2n}$ is not concordant to $\mathbb{CP}^n$. This shows that Σ^{2n} is a Farrell–Jones sphere. This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. Let Σ^{2n} be the generator of Θ_{2n} (with $n = 7$ or 8). Suppose Σ^{2n} is not a Farrell–Jones sphere. Then $\mathbb{CP}^n \# \Sigma^{2n}$ is concordant to $\mathbb{CP}^n$. By [18, Corollary 3, p. 97], $\mathbb{CP}^n \# \Sigma^{2n}$ is c-orientation diffeomorphic to $\mathbb{CP}^n$. Let $f : \mathbb{CP}^n \# \Sigma^{2n} \to \mathbb{CP}^n$ be a c-orientation diffeomorphism such that $f^*(c_1) = c_2$, where c_1 and c_2 are c-orientations of $\mathbb{CP}^n$ and $\mathbb{CP}^n \# \Sigma^{2n}$ respectively. Using properties of the cup product, we have $f^*(c_1^n) = c_2^n$. If $c_1 = c_2$ in $H^2(\mathbb{CP}^n, \mathbb{Z})$, then f is an orientation preserving diffeomorphism. If $c_1 \neq c_2$ in $H^2(\mathbb{CP}^n, \mathbb{Z})$, then $g \circ f$ is an orientation preserving diffeomorphism with the property that $(g \circ f)^*(c_1) = f^*(g^*(c_1)) = -c_2 = c_1$, where $g : \mathbb{CP}^n \to \mathbb{CP}^n$ is the conjugation map. In both cases, we have that $\mathbb{CP}^n \# \Sigma^{2n}$ is an orientation preserving diffeomorphic to $\mathbb{CP}^n$. This is a contradiction because, by [13, Theorem 1], $\mathbb{CP}^n \# \Sigma^{2n}$ cannot be orientation preserving diffeomorphic to $\mathbb{CP}^n$. Thus Σ^{2n} is a Farrell–Jones sphere. This completes the proof of Theorem 1.3. □

Recall that the α-invariant is the ring homomorphism $\alpha : \Omega_*^{\text{spin}} \to \mathrm{KO}_*$ which associates to a spin bordism class the KO-valued index of the Dirac operator of a representative spin manifold. We also write α for the corresponding invariant on a framed bordism:

$$\alpha : \Omega_*^f \to \Omega_*^{\text{spin}} \to \mathrm{KO}_*$$

Under the Pontryagin–Thom isomorphism $\Omega_*^f \cong \pi_*^s$, the α-invariant has the following interpretation as Adams d-invariant $d_{\mathbb{R}} : \pi_r^s \to \mathrm{KO}_*$, which was used already in [12, p. 44] and [9, Lemma 2.12].

Lemma 3.4. *Under the Pontryagin–Thom isomorphism* $\Omega_*^f \cong \pi_*^s$*, the* α*-invariant* $\alpha : \Omega_{8n+2}^f \to \mathrm{KO}_{8n+2}$ *may be identified with* $d_{\mathbb{R}} : \pi_{8n+2}^s \to \mathrm{KO}_{8n+2}$.

We start by recalling some facts from smoothing theory [7], which were used already in [10, Lemma 3.17]. There are H-spaces SF, F/O and Top/O and H-space maps $\phi : SF \to F/O$, $\psi : \mathrm{Top}/O \to F/O$ such that

$$\psi_* : \Theta_{8n+2} = \pi_{8n+2}(\mathrm{Top}/O) \to \pi_{8n+2}(F/O) \tag{3.1}$$

is an isomorphism for $n \geq 1$. The homotopy groups of SF are the stable homotopy groups of spheres π_m^s, i.e., $\pi_m(SF) = \pi_m^s$ for $m \geq 1$. For $n \geq 1$,

$$\phi_* : \pi_{8n+2}^s \to \pi_{8n+2}(F/O) \tag{3.2}$$

is an isomorphism. Since every homotopy sphere has a unique spin-structure, we obtain the α-invariant on $\pi_{8n+2}^s \cong \pi_{8n+2}(F/O) \cong \Theta_{8n+2}$:

$$\alpha : \pi_{8n+2}^s \xrightarrow{\phi_*} \pi_{8n+2}(F/O) \xrightarrow{\psi_*^{-1}} \Theta_{8n+2} \longrightarrow \Omega_{8n+2}^{\mathrm{spin}} \longrightarrow \mathrm{KO}_{8n+2},$$

where ψ_* and ϕ_* are the isomorphisms as in equation (3.1) and (3.2) respectively.

Let $\mathrm{Ker}(d_{\mathbb{R}})$ denote the kernel of the Adams d-invariant $d_{\mathbb{R}} : \pi_{8n+2}^s \to \mathbb{Z}_2$. By Lemma 3.4, $\mathrm{Ker}(d_{\mathbb{R}})$ consists of framed manifolds which bound spin manifolds.

Proof of Theorem 1.6. Consider the following commutative of diagram:

$$\begin{array}{ccc}
[\mathbb{S}^{2m}, \mathrm{Top}/O] = \Theta_{2m} & \xrightarrow{f_{\mathbb{CP}^m}^*} & [\mathbb{CP}^m, \mathrm{Top}/O] = \mathcal{C}(\mathbb{CP}^m) \\
\downarrow \psi_* & & \downarrow \psi_* \\
[\mathbb{S}^{2m}, F/O] & \xrightarrow{f_{\mathbb{CP}^m}^*} & [\mathbb{CP}^m, F/O] \\
\uparrow \phi_* & & \uparrow \phi_* \\
[\mathbb{S}^{2m}, SF] = \pi_{2m}^s & \xrightarrow{f_{\mathbb{CP}^m}^*} & [\mathbb{CP}^m, SF].
\end{array} \tag{3.3}$$

In this diagram, ϕ_* and ψ_* are induced by the H-space maps $\phi : SF \to F/O$ and $\psi : \mathrm{Top}/O \to F/O$ respectively and the homomorphism

$$\phi_* : [\mathbb{CP}^m, SF] \to [\mathbb{CP}^m, F/O]$$

is monic for all $m \geq 1$ by a result of Brumfiel [8, p. 77]. Recall that the concordance class $[\mathbb{CP}^m \# \Sigma] \in [\mathbb{CP}^m, \mathrm{Top}/O]$ of $\mathbb{CP}^m \# \Sigma$ is $f^*_{\mathbb{CP}^m}([\Sigma])$ when $m > 2$ and that $[\mathbb{CP}^m] = [\mathbb{CP}^m \# \mathbb{S}^{2m}]$ is the zero element of this group.

Let $\Sigma^{8n+2} \in \Theta_{8n+2}$ be a Hitchin $(8n+2)$-sphere (with $n \geq 1$) and further let $\eta \in \pi^s_{8n+2} = [\mathbb{S}^{8n+2}, SF]$ be such that

$$\psi_*^{-1}(\phi_*(\eta)) = \Sigma^{8n+2}.$$

Recall that $[X, SF]$ can be identified with the 0^{th} stable cohomotopy group $\pi^0(X)$. Let $h : \mathbb{S}^{q+8n+2} \to \mathbb{S}^q$ represent η. Since Σ^{8n+2} is a Hitchin sphere and by Lemma 3.4, we have

$$0 \neq \alpha(\Sigma^{8n+2}) = d_{\mathbb{R}}(h) = h^* \in \mathrm{Hom}(\widetilde{\mathrm{KO}}^q(\mathbb{S}^q), \widetilde{\mathrm{KO}}^q(\mathbb{S}^{q+8n+2})).$$

Also Adams and Walker [2] showed that $\Sigma^q f_{\mathbb{CP}^{4n+1}} : \Sigma^q \mathbb{CP}^{4n+1} \to \mathbb{S}^{q+8n+2}$ induces a monomorphism on $\widetilde{\mathrm{KO}}^q(\cdot)$. Consequently the composite map

$$h \circ \Sigma^q f_{\mathbb{CP}^{4n+1}} : \Sigma^q \mathbb{CP}^{4n+1} \to \mathbb{S}^q$$

induces a non-zero homomorphism on $\widetilde{\mathrm{KO}}^q(\cdot)$. This shows that

$$f^*_{\mathbb{CP}^{4n+1}}(\eta) = [h \circ \Sigma^q f_{\mathbb{CP}^{4n+1}}] \neq 0,$$

where

$$f^*_{\mathbb{CP}^{4n+1}} : [\mathbb{S}^{8n+2}, SF] \to [\mathbb{CP}^{4n+1}, SF].$$

Since the homomorphism $\phi_* : [\mathbb{CP}^m, SF] \to [\mathbb{CP}^m, F/O]$ is monic, by the commutative diagram (3.3) where $m = 4n+1$, we have

$$\psi_*(f^*_{\mathbb{CP}^{4n+1}}(\Sigma^{8n+2})) = \phi_*(f^*_{\mathbb{CP}^{4n+1}}(\eta)) \neq 0.$$

This implies that

$$f^*_{\mathbb{CP}^{4n+1}}(\Sigma^{8n+2}) \neq 0$$

and hence $\mathbb{CP}^{4n+1} \# \Sigma^{8n+2}$ is not concordant to $\mathbb{CP}^{4n+1}$. This shows that Σ^{8n+2} is a Farrell–Jones sphere and this completes the proof of Theorem 1.6. □

Remark 3.5. (1) Let us note that the homotopy sphere Σ^{8n+2} $(n \geq 1)$ given by [10, Lemma 3.17] is the image of the Adams element μ_{8n+2} of order 2 under the composed isomorphism $\psi_*^{-1} \circ \phi_*$, where ψ_* and ϕ_* are the isomorphisms as in equations (3.1) and (3.2) respectively (see [10, equation (3.17.4)]). By [1, Theorem 1.2] and Lemma 3.4, we have

$$d_{\mathbb{R}}(\mu_{8n+2}) = \alpha(\Sigma^{8n+2}) = 1.$$

This shows that Σ^{8n+2} is a Hitchin sphere of order 2 in Θ_{8n+2}. By Theorem 1.6, Σ^{8n+2} is a Farrell–Jones sphere.

(2) Since $\Theta_{18} \cong \mathrm{Ker}(\alpha) \oplus \mathbb{Z}_2$, where the α-invariant $\alpha : \Theta_{18} \to \mathbb{Z}_2$ satisfies $\mathrm{Ker}(\alpha) = \mathbb{Z}_8$ (see [9, p. 12]), this shows that there are exotic spheres of order $\neq 2$ in Θ_{18} which are not in the kernel of α. This implies that there are Hitchin spheres of order $\neq 2$ in Θ_{18} which are all Farrell–Jones sphere by Theorem 1.6.

(3) In [3], Anderson, Brown and Peterson proved that one has $\alpha(\Sigma^m) \neq 0$ iff $m = 8k + 1$ or $8k + 2$ iff Σ^m is an exotic sphere not bounding a spin manifold, where $\alpha : \Theta_m \to \Omega_m^{\mathrm{spin}} \to \mathrm{KO}_m$ is the α-invariant. This implies that Σ^m is a Hitchin sphere in Θ_m iff Σ^m is an exotic sphere not bounding a spin manifold. By Theorem 1.6, every exotic sphere not bounding a spin manifold Σ^{8n+2} in Θ_{8n+2} is a Farrell–Jones sphere.

(4) By [1, Theorem 7.2], $\Theta_{10} \cong \mathrm{Ker}(d_{\mathbb{R}}) \oplus \mathbb{Z}_2$ such that $\mathrm{Ker}(d_{\mathbb{R}}) = \mathbb{Z}_3$. If Σ^{10} is a generator of $\mathrm{Ker}(d_{\mathbb{R}})$, then $d_{\mathbb{R}}(\Sigma^{10}) = \alpha(\Sigma^{10}) = 0$. This shows that Σ^{10} is not a Hitchin sphere. But, by [10, Lemma 3.17], Σ^{10} is a Farrell–Jones sphere.

4 The inertia groups of complex projective spaces

In this section, we discuss the relationship between inertia groups of $\mathbb{CP}^n$ and Farrell–Jones spheres.

Definition 4.1. Let M^m be a closed smooth, oriented m-dimensional manifold. Let $\Sigma \in \Theta_m$ and $g : \mathbb{S}^{m-1} \to \mathbb{S}^{m-1}$ be an orientation preserving diffeomorphism corresponding to Σ. Writing $M \# \Sigma$ as $(M^m \setminus \mathrm{int}(\mathbb{D}^{\mathrm{m}})) \cup_{\mathrm{g}} \mathbb{D}^{\mathrm{m}}$, let $\iota : M \to M \# \Sigma$ denote the PL homeomorphism defined by $\iota_{|M \setminus \mathrm{int}(\mathbb{D}^{\mathrm{m}})} = \mathrm{id}$ and $\iota_{|\mathbb{D}^m} = Cg$, where $Cg : \mathbb{D}^m \to \mathbb{D}^m$ is the cone extension of g.

The inertia group $I(M) \subset \Theta_m$ is defined as the set of $\Sigma \in \Theta_m$ for which there exists an orientation preserving diffeomorphism $\phi : M \to M \# \Sigma$.

Define the homotopy inertia group $I_h(M)$ to be the set of all $\Sigma \in I(M)$ such that there exists a diffeomorphism $M \to M \# \Sigma$ which is homotopic to ι.

Define the concordance inertia group $I_c(M)$ to be the set of all $\Sigma \in I_h(M)$ such that $M \# \Sigma$ is concordant to M. Clearly, $I_c(M) \subseteq I_h(M) \subseteq I(M)$.

Note that for $M = \mathbb{CP}^n$, Theorem 1.2 can be restated as:

Theorem 4.2. *A sphere $\Sigma^{2n} \in \Theta_{2n}$ is a Farrell–Jones sphere iff $\Sigma^{2n} \notin I(\mathbb{CP}^n)$.*

Remark 4.3. Since $I_c(\mathbb{CP}^n) \subseteq I_h(\mathbb{CP}^n) \subseteq I(\mathbb{CP}^n)$ and by Theorem 4.2, we have that

$$I_c(\mathbb{CP}^n) = I_h(\mathbb{CP}^n) = I(\mathbb{CP}^n).$$

The proof of Theorem 1.6 leads one to the following question.

Question 4.4. Let $f : \mathbb{CP}^{4n+1} \to \mathbb{S}^{8n+2}$ be any degree one map ($n \geq 1$). Does there exist an element $\eta \in \mathrm{Ker}(d_{\mathbb{R}}) \subset \pi^s_{8n+2} = \Theta_{8n+2}$ such that the following is true?

($\star$) If any map $h : \mathbb{S}^{q+8n+2} \to \mathbb{S}^q$ represents η, then

$$h \circ \Sigma^q f : \Sigma^q \mathbb{CP}^{4n+1} \to \mathbb{S}^q$$

is not null homotopic.

Remark 4.5. (1) By [14, Lemma 9.1], $I(\mathbb{CP}^{4n+1}) \subseteq \mathrm{Ker}(d_{\mathbb{R}})$. If the answer to Question 4.4 is yes, then we have $I(\mathbb{CP}^{4n+1}) \neq \mathrm{Ker}(d_{\mathbb{R}})$, i.e., there exists an exotic sphere Σ bounding spin manifold in Θ_{8n+2} such that $\Sigma \notin I(\mathbb{CP}^{4n+1})$. This can be seen as follows: Let $\eta \in \mathrm{Ker}(d_{\mathbb{R}})$ and let $h : \mathbb{S}^{q+8n+2} \to \mathbb{S}^q$ represent η such that $h \circ \Sigma^q f : \Sigma^q \mathbb{CP}^{4n+1} \to \mathbb{S}^q$ is not null homotopic. This implies that

$$f^*_{\mathbb{CP}^{4n+1}}(h) = [h \circ \Sigma^q f_{\mathbb{CP}^{4n+1}}] \neq 0,$$

where $f^*_{\mathbb{CP}^{4n+1}} : \pi^0(\mathbb{S}^{8n+2}) \to \pi^0(\mathbb{CP}^{4n+1})$. A similar argument given in the proof of Theorem 1.6 using the commutative diagram (3.3) shows that there exists an exotic sphere $\Sigma \in \Theta_{8n+2}$ such that $\psi_*^{-1} \circ \phi_*(\eta) = \Sigma$, $d_{\mathbb{R}}(\eta) = \alpha(\Sigma) = 0$ and $\mathbb{CP}^{4n+1} \# \Sigma$ is not concordant to $\mathbb{CP}^{4n+1}$, where ψ_* and ϕ_* are the isomorphisms as in (3.1) and (3.2) respectively. This implies that Σ is a Farrell–Jones sphere such that $\Sigma \in \mathrm{Ker}(d_{\mathbb{R}})$. By Theorem 4.2, $I(\mathbb{CP}^{4n+1}) \neq \mathrm{Ker}(d_{\mathbb{R}})$.

(2) If all non-zero elements in $\mathrm{Ker}(d_{\mathbb{R}})$ satisfy the condition ($\star$) in Question 4.4, then, by the above remark (1), $\Sigma \notin I(\mathbb{CP}^{4n+1})$ for all exotic sphere $\Sigma \in \mathrm{Ker}(d_{\mathbb{R}})$ and hence $I(\mathbb{CP}^{4n+1}) = 0$.

Theorem 4.6. *Let n be a positive integer such that Θ_{8n+2} is a cyclic group of order 2. Then $I(\mathbb{CP}^{4n+1}) = 0$.*

Proof. Let Σ^{8n+2} be the generator of $\Theta_{8n+2} \cong \mathbb{Z}_2$. Let

$$\psi_* : \Theta_{8n+2} \to \pi_{8n+2}(F/O) \quad \text{and} \quad \phi_* : \pi^s_{8n+2} \to \pi_{8n+2}(F/O)$$

be the isomorphisms as in (3.1) and (3.2). By [1, Theorem 1.2], there exists an element μ_{8n+2} of order 2 in π^s_{8n+2}. This shows that

$$\phi_*^{-1} \circ \psi_*(\Sigma^{8n+2}) = \mu_{8n+2}.$$

By [1, Theorem 1.2] and Lemma 3.4, $d_{\mathbb{R}}(\mu_{8n+2}) = \alpha(\Sigma^{8n+2}) = 1$. This implies that Σ^{8n+2} is a Hitchin sphere. By Theorems 1.6 and 1.2, $\mathbb{CP}^{4n+1} \# \Sigma^{8n+2}$ is not diffeomorphic to $\mathbb{CP}^{4n+1}$. This implies that $I(\mathbb{CP}^{4n+1}) = 0$. □

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Bibliography

[1] J. F. Adams, On the groups $J(X)$. IV, *Topology* **5** (1966), 21–71.

[2] J. F. Adams and G. Walker, On complex Stiefel manifolds, *Proc. Cambridge Philos. Soc.* **61** (1965), 81–103.

[3] D. W. Anderson, E. H. Brown and F. P. Peterson, The structure of the spin cobordism ring, *Ann. of Math. (2)* **86** (1967), 271–298.

[4] C. S. Aravinda and F. T. Farrell, Exotic negatively curved structures on Cayley hyperbolic manifolds, *J. Differential Geom.* **63** (2003), 41–62.

[5] C. S. Aravinda and F. T. Farrell, Exotic structures and quaternionic hyperbolic manifolds, in: *Algebraic Groups and Arithmetic,* Tata Institute of Fundamental Research Studies in Mathematics, Mumbai (2004), 507–524.

[6] A. Borel, Compact Clifford–Klein forms of symmetric spaces, *Topology* **2** (1963), 111–122.

[7] G. Brumfiel, Homotopy equivalences of almost smooth manifolds, *Comment. Math. Helv.* **46** (1971), 381–407.

[8] G. W. Brumfiel, Homotopy equivalence of almost smooth manifolds, in: *Algebraic Topology*, Proc. Symp. Pure Math. 22, American Mathematical Society, Providence (1971), 73–79.

[9] D. Crowley and T. Schick, The Gromoll filtration, KO-characteristic classes and metrics of positive scalar curvature, *Geom. Topol.* **17** (2013), 1773–1789.

[10] F. T. Farrell and L. E. Jones, Complex hyperbolic manifolds and exotic smooth structures, *Invent. Math.* **117** (1994), 57–74.

[11] F. T. Farrell, L. E. Jones and P. Ontaneda, Hyperbolic manifolds with negatively curved exotic triangulations in dimensions larger than five, *J. Differential Geom.* **48** (1998), 319–322.

[12] N. Hitchin, Harmonic spinors, *Adv. Math.* **14** (1974), 1–55.

[13] K. Kawakubo, Inertia groups of low dimensional complex projective spaces and some free differentiable actions on spheres I, *Proc. Japan Acad.* **44** (1968), 873–875.

[14] K. Kawakubo, Smooth structures on $\mathbb{S}^p \times \mathbb{S}^q$, *Osaka J. Math.* **6** (1969), 165–196.

[15] M. Kervaire and J. Milnor, Groups of homotopy spheres: I, *Annals of Math. (2)* **77** (1963), 504–537.

[16] R. C. Kirby and L. C. Siebenmann, *Foundational Essays on Topological Manifolds, Smoothings, and Triangulations. With Notes by John Milnor and Michael Atiyah*, Ann. of Math. Stud. 88, Princeton University Press, Princeton, 1977.

[17] E. E. Moise, *Geometric Topology in Dimensions* 2 *and* 3, Grad. Texts in Math. 47, Springer-Verlag, New York, 1977.

[18] D. P. Sullivan, Triangulating and smoothing homotopy equivalences and homeomorphisms, in: *The Hauptvermutung Book. A Collection of Papers of the Topology of Manifolds*, *K*-Monogr. Math. 1, Kluwer Academic Publishers, Dordrecht (1996), 69–103.

[19] Exotic spheres on the Manifold Atlas, `http://www.map.mpim-bonn.mpg.de/Exotic_spheres`.

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