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On Regularization of a Source Identification Problem in a Parabolic PDE and its Finite Dimensional Analysis

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Abstract. We consider the inverse problem of identifying a general source term, which is a function of both time variable and the spatial variable, in a parabolic PDE from the knowledge of boundary measurements of the solution on some portion of the lateral boundary. We transform this inverse problem into a problem of solving a compact linear operator equation. For the regularization of the operator equation with noisy data, we employ the standard Tikhonov regularization, and its finite dimensional realization is done using a discretization procedure involving the space $L^2(0,\tau;L^2(\Omega))$. For illustrating the specification of an a priori source condition, we have explicitly obtained the range space of the adjoint of the operator involved in the operator equation.

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1 Introduction

Let $d\geq 1$ and Ω be a bounded domain in $\mathbb{R}^d$ with Lipschitz boundary. For a fixed $\tau>0$ we denote the cylindrical domain $\Omega\times[0,\tau]$ by Ω_τ and its lateral surface $\partial\Omega\times[0,\tau]$ by $\partial\Omega_\tau$. Let Σ be a relatively open subset of $\partial\Omega$. We denote the boundary surface $\Sigma\times[0,\tau]$ by Σ_τ. For

$$f\in L^2(0,\tau;L^2(\Omega)),\quad g\in L^2(0,\tau;L^2(\partial\Omega)),\quad h\in L^2(\Omega),$$

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we consider the parabolic PDE

$$\begin{cases} u_t - \nabla\cdot(Q(x)\nabla u) = f & \text{in } \Omega_\tau, \\ Q(x)\nabla u\cdot\vec{n} = g & \text{on } \partial\Omega_\tau, \\ u(\cdot,0) = h & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $Q \in (L^\infty(\Omega))^{d\times d}$ is a symmetric matrix with entries from $L^\infty(\Omega)$ satisfying the uniform ellipticity condition, that is, there exist a constant $\kappa_0 > 0$ such that

$$Q\xi\cdot\xi \geq \kappa_0|\xi|^2 \quad \text{a.e on } \Omega, \text{ and for all } \xi\in\mathbb{R}^d, \tag{1.2}$$

where $|\xi|^2 = {\xi_1}^2 + \ldots + {\xi_d}^2$ for $\xi = (\xi_1,\ldots,\xi_d)\in\mathbb{R}^d$ and $\vec{n}$ is the outward unit normal to $\partial\Omega$.

Throughout the paper, for a Banach space X, $\phi\in L^2(0,\tau;X)$ means ϕ is an X-valued function on $[0,\tau]$ such that $t\mapsto\|\phi(t)\|_X$ belongs to $L^2[0,\tau]$. Also, throughout we use the standard notations of the function spaces $L^2(\Omega)$ and the Sobolev spaces $H^1(\Omega)$ (see [1–3]).

For results related to existence and uniqueness of the classical solution corresponding to the forward problem associated with (1.1), namely, that of finding u satisfying (1.1) from the knowledge of f,g,h as considered above, one may refer to [4–6]. In certain cases, a classical solution may not exist for the forward problem, but we may have a *weak solution*. In [5, Theorem 2.4], the authors have given an existence result for a weak solution of (1.1). We first state the existence result precisely, whose proof follows along similar lines as in [5, Theorem 2.4].

Theorem 1.1. ([5, Theorem 2.4]) *Let $f\in L^2(0,\tau;L^2(\Omega))$, $g\in L^2(0,\tau;L^2(\partial\Omega))$ and $h\in L^2(\Omega)$. Also, let $Q\in(L^\infty(\Omega))^{d\times d}$ be symmetric satisfying the uniform ellipticity condition* (1.2)*. Then there exists a unique $u\in L^2(0,\tau;H^1(\Omega))$ with $u_t\in L^2(0,\tau;(H^1(\Omega))')$ satisfying*

$$\langle u_t(\cdot,t),\varphi\rangle + \int_\Omega Q\nabla u(\cdot,t)\cdot\nabla\varphi\,\mathrm{d}x = \int_\Omega f(\cdot,t)\,\varphi\,\mathrm{d}x + \int_{\partial\Omega} g\varphi\,\mathrm{d}x \tag{1.3}$$

for all $\varphi\in H^1(\Omega)$ and for a.a.(almost all) $t\in[0,\tau]$ with $u(\cdot,0)=h$ a.e. in Ω. Further, there exists a constant $C_1>0$, independent of f, such that

$$\begin{aligned} &\|u\|_{L^2(0,\tau;H^1(\Omega))} + \|u_t\|_{L^2(0,\tau;(H^1(\Omega))')} \\ \leq &C_1\Big(\|f\|_{L^2(0,\tau;L^2(\Omega))} + \|g\|_{L^2(0,\tau;L^2(\partial\Omega))} + \|h\|_{L^2(\Omega)}\Big). \end{aligned} \tag{1.4}$$

In (1.3), the notation $\langle\cdot,\cdot\rangle$ stands for the duality action between $H^1(\Omega)$ and $(H^1(\Omega))'$, where $(H^1(\Omega))'$ stands for the dual of $H^1(\Omega)$. Also, u_t denotes the distributional derivative of u with respect to t, that is, u_t is the unique element in $L^2(0,\tau;(H^1(\Omega))')$ such that

$$\int_0^\tau \varphi'(t)u(t)\,\mathrm{d}t = -\int_0^\tau \varphi(t)u_t(t)\,\mathrm{d}t \quad \text{for all } \varphi\in C_c^\infty(0,\tau).$$

Following [5], an element $u\in L^2(0,\tau;H^1(\Omega))$ with $u_t\in L^2(0,\tau;(H^1(\Omega))')$ is called a **weak solution** of the PDE (1.1), if it satisfies (1.3).

For a given $f\in L^2(0,\tau;L^2(\Omega)), g\in L^2(0,\tau;L^2(\partial\Omega))$ and $h\in L^2(\Omega)$, let u be the unique weak solution of (1.1). Let

$$z(x,t)=u(x,t) \quad \text{for} \quad (x,t)\in\Sigma_\tau, \quad \text{i.e.} \quad u_{|_{\Sigma_\tau}}=z.$$

Note that, here $u(x,t)$ on Σ_τ has to be understood in the sense of trace (see [1,2,7]). We are interested in the inverse problem of determining the *source function* f from the knowledge of z, i.e., from the knowledge of u on the portion Σ_τ of the boundary of Ω_τ. More precisely, our inverse problem at hand is the following:

> **(IP):** For the given boundary data $z\in L^2(0,\tau;L^2(\Sigma_\tau))$, determine the source function f in $L^2(0,\tau;L^2(\Omega))$ such that the corresponding unique weak solution u of (1.1) satisfies $u_{|_{\Sigma_\tau}}=z$.

In general, the solution of the inverse problem, if it exists need not be unique. To see this, we consider a simple example below.

Example 1.1. Let $\Omega=(0,1)$ and $h=0$ on $(0,1)$. For $t\in[0,1]$, let

$$g(t)=\begin{cases}\pi t+t & \text{if } x=0,\\ \pi t\cos\pi t & \text{if } x=1.\end{cases}$$

Let $Q=1$ on Ω, and $z(1,t)=\sin\pi t$ for $t\in[0,1]$. Then for

$$u_1(x,t)=\sin\pi xt+\frac{1}{3}t(x-1)^3, \quad u_2(x,t)=\sin\pi xt+\frac{1}{5}t(x-1)^5,$$

we have

$$u_1(1,t)=z=u_2(1,t), \quad t\in[0,1],$$

but the source functions corresponding to u_1 and u_2 are respectively,

$$\begin{aligned} f_1(x,t)&=\pi x\cos\pi xt+\frac{(x-1)^3}{3}+\pi^2t^2\sin\pi xt-2(x-1)t,\\ f_2(x,t)&=\pi x\cos\pi xt+\frac{(x-1)^5}{5}+\pi^2t^2\sin\pi xt-4(x-1)^3t.\end{aligned}$$

Thus, the source function f, if exists, from our considered boundary observation is not unique, in general. But, for certain specific cases of f and for different type of boundary measurements, the uniqueness results are well-established; see [8–10]. In [8], the author has considered the case when f depends only on the spatial variable, whereas in [9] the authors have considered the case, where f can be written as infinite sum of certain type of functions. Also, in [10], the authors have considered the case, where f can be written as $f(x,t) := \sigma(t)\phi(x)$ where $\sigma(t) = e^{-\lambda t}, \lambda > 0$ and considered an inverse problem of identifying the function ϕ. We carry out our analysis for a general source function f assuming only its existence, and obtain regularized approximations for that f which has minimum norm.

The inverse problems of source identification from boundary measurements have vast literature and they have real world applications. For inverse problems related to source identification, one may look into [8, 9, 11–15] and also the recent work in [16]. In fact, in [14], the authors have mentioned explicitly various inverse problems on source identifications from boundary measurements.

Usually the inverse source identification problems from boundary measurements or final time observations are ill-posed in nature (see [14]), that is, either the inverse problem do not have a unique solution, or even if the solution exist, it does not depend continuously on the data. In [16], the authors have considered the inverse problem of identifying source function from a boundary measurement for a parabolic PDE with Robin boundary conditions. For obtaining stable approximate solutions, they have used the output least square method combined with *Crank-Nicolson Galerkin method* to obtain numerical approximations for the source function.

In this paper, we convert the inverse problem **(IP)** into a linear operator equation first, where the associated operator is compact and is of infinite rank so that it is an ill-posed operator equation. We use Tikhonov regularization (see [17, 18]) in the infinite dimensional setting for obtaining stable approximate solutions corresponding to the noisy data. In order to obtain approximations in a finite dimensional setting, we employ Galerkin-projection method to the regularized operator equation, by using different projections corresponding to space variable and time variable. Making use of the fact that the operator involved in the infinite dimensional setting is compact, we derive order optimal error estimates by choosing the regularization parameter and the level of approximation appropriately. Thus, our method of obtaining regularized approximations is much simpler compared to the one considered in [16].

Also, we would like to mention that in [9], similar operator theoretic formulation has been adopted for the problem of identifying the function φ occurring in the source function $f(x,t) = \sqrt{t}\varphi(x)$, from the knowledge of a lateral boundary measurement $\tilde{z}$ defined on $\Sigma \times [0,\tau]$, where Σ is a relatively open subset of $\partial\Omega$ and the corresponding governing PDE is same as (1.1) with $g = 0$ and $h = 0$. But in this paper, we are considering the identification of a general source function which depends on both space and time variable, and g,h are not necessarily zero. Further, in [9], no finite dimensional analysis is done whereas in this paper we have given a finite dimensional analysis for obtaining stable approximations to the sought source function.

The rest of the paper is organized as follows: In Section 2, we introduce some notations that we shall use throughout and obtain some results related to continuity and compactness of the linear operator involved in the formulation of the inverse problem. Section 3 deals with the regularization analysis corresponding to the noisy data. In Section 4, we have considered the finite dimensional realization of our proposed method, which is one of the main objectives of this work. Finally, in Section 5, titled as appendix, we have explicitly obtained the range space of the adjoint of the linear operator.

2 Operator equation formulation

Let $\mathcal{X}=L^2(0,\tau;L^2(\Omega))$ and let

$$\mathcal{W}=\{u\in L^2(0,\tau;H^1(\Omega)):u_t\in L^2(0,\tau;(H^1(\Omega))')\}.$$

It is well known that $\mathcal{X}$ is a Hilbert space with the inner product

$$\langle u,v\rangle_{\mathcal{X}}:=\int_0^\tau \langle u(t),v(t)\rangle_{L^2}\mathrm{d}t \quad \text{for all } u,v\in\mathcal{X}.$$

It can be shown that $\mathcal{W}$ is a Banach space with respect to the norm defined by

$$\|u\|_{\mathcal{W}}:=\|u\|_{L^2(0,\tau;H^1(\Omega))}+\|u_t\|_{L^2(0,\tau;(H^1(\Omega))')} \quad \text{for all } u\in\mathcal{W}.$$

Also, it is known that, if X is a Banach space and $\phi\in L^2(0,\tau;X)$, then $\|\phi\|^2_{L^2(0,\tau;X)}:=\int_0^\tau\|\phi(t)\|_X^2\mathrm{d}t$. Recall that Σ is a relatively open subset of Ω and $\Sigma_\tau=\Sigma\times[0,\tau]$. Throughout we will use the notation $\mathcal{Y}=L^2(0,\tau;L^2(\Sigma))$.

Let $\gamma:H^1(\Omega)\to L^2(\partial\Omega)$ be the trace map (see [1–3]). It is known that γ is a continuous linear operator and the range of γ is $H^{1/2}(\partial\Omega)$. We define $\Gamma:\mathcal{W}\to\mathcal{Y}$ by

$$(\Gamma\phi)(t):=\gamma\phi(\cdot,t)_{|\Sigma} \quad \text{for all } t\in[0,\tau] \text{ and } \phi\in\mathcal{W}. \tag{2.1}$$

We first observe some properties of the map Γ.

Theorem 2.1. *The map $\Gamma:\mathcal{W}\to\mathcal{Y}$ defined as in* (2.1) *is a bounded linear operator.*

Proof. Linearity of Γ follows from the linearity of γ. Let $C_2>0$ be such that $\|\gamma\varphi\|_{L^2(\partial\Omega)}\le C_2\|\varphi\|_{H^1(\Omega)}$ for all $\varphi\in H^1(\Omega)$. Then, for all $\phi\in\mathcal{W}$, we have

$$\begin{aligned}\|\Gamma\phi\|^2_{\mathcal{Y}}&=\int_0^\tau\|\gamma\phi(\cdot,t)_{|\Sigma}\|^2_{L^2(\Sigma)}\mathrm{d}t\le\int_0^\tau\|\gamma\phi(\cdot,t)\|^2_{L^2(\partial\Omega)}\mathrm{d}t\\&\le C_2^2\int_0^\tau\|\phi(\cdot,t)\|^2_{H^1(\Omega)}\mathrm{d}t=C_2^2\|\phi\|^2_{L^2(0,\tau;H^1(\Omega))}\\&\le C_2^2\|\phi\|^2_{\mathcal{W}}.\end{aligned}$$

This shows that Γ is a bounded linear operator with $\|\Gamma\|\le C_2$. $\square$

For proving one more property of Γ, we first put on record some of the embedding results. For the proofs, one may refer to [19].

Lemma 2.1. (cf. [19]) *Let Ω be as considered and $0\le s<1$. Then we have the following*

(i) The space $H^1(\Omega)$ is compactly embedded in $H^s(\Omega)$ and $\mathcal{W}$ is compactly embedded in $L^2(0,\tau;H^s(\Omega))$.

(ii) For $s>\frac{1}{2}$, the trace map $\gamma_0:H^s(\Omega)\to H^{s-1/2}(\partial\Omega)$ is a continuous linear operator.

(iii) For $s>\frac{1}{2}$, $L^2(0,\tau;H^{s-1/2}(\partial\Omega))$ is continuously embedded in $L^2(0,\tau;L^2(\Sigma))$.

Remark 2.1. Let $0\le s<1$ and $\gamma_0:H^s(\Omega)\to H^{s-1/2}(\partial\Omega)$ be the trace map (see [19]). Then $\gamma\varphi=\gamma_0\varphi$ for all $\varphi\in H^s(\Omega)$. Let $\Gamma_0:L^2(0,\tau;H^s(\Omega))\to L^2(0,\tau;H^{s-1/2}(\partial\Omega))$ be defined by

$$(\Gamma_0\phi)(t)=\gamma_0\phi(\cdot,t)\quad\text{for all}\quad t\in[0,\tau],\quad \phi\in L^2(0,\tau;H^s(\Omega)).$$

Since $\gamma=\gamma_0$ on $H^s(\Omega)$, we have $\Gamma=\tilde{\Gamma}_0$ on $L^2(0,\tau;H^s(\Omega))$, where $(\tilde{\Gamma}_0\phi)(t)=(\Gamma_0\phi)(t)_{|_\Sigma}$.

Theorem 2.2. *Let Γ be as defined in* (2.1)*. Then Γ is a compact linear operator.*

Proof. Let $\frac{1}{2}<\theta<1$ and Γ_0 be as in Remark 2.1. Then by Lemma 2.1, $\mathcal{W}$ is compactly embedded in $L^2(0,\tau;H^\theta(\Omega))$, $\Gamma_0:L^2(0,\tau;H^\theta(\Omega))\to L^2(0,\tau;H^{\theta-1/2}(\partial\Omega))$ is continuous and $L^2(0,\tau;H^{\theta-1/2}(\partial\Omega))$ is continuously embedded in $\mathcal{Y}:=L^2(0,\tau;L^2(\Sigma))$. As mentioned in Remark 2.1, $\Gamma=\tilde{\Gamma}_0$ on $L^2(0,\tau;H^\theta(\Omega))$. Hence $\Gamma:\mathcal{W}\to\mathcal{Y}$ is compact. $\square$

Let $Q(x)=(q_{ij}(x))\in(L^\infty(\Omega))^{d\times d}$ be symmetric, i.e., $q_{ij}=q_{ji}$ a.e. on Ω for all $1\le i,j\le d$ and Q satisfies (1.2). Now consider the PDE

$$\begin{cases} v_t-\nabla\cdot(Q(x)\nabla v)=\Phi & \text{in } \Omega_\tau,\\ Q(x)\nabla v\cdot\vec{n}=0 & \text{on } \partial\Omega_\tau,\\ v(\cdot,0)=0 & \text{in } \Omega. \end{cases}\tag{2.2}$$

Then by Theorem 1.1, we know that for each $\Phi\in\mathcal{X}$, there exists a unique weak solution $v_\Phi\in\mathcal{W}$ of (2.2) satisfying

$$\|v_\Phi\|_{\mathcal{W}}\le C_1\|\Phi\|_{\mathcal{X}}\tag{2.3}$$

where C_1 is as in Theorem 1.1. We now define a map $S:\mathcal{X}\to\mathcal{W}$ by

$$S\Phi=v_\Phi,\quad \text{for all } \Phi\in\mathcal{X},\tag{2.4}$$

where $v_\Phi\in\mathcal{W}$. By the nature of PDE (2.2), it is clear that S is a linear operator. Also, by the estimate in (2.3), it follows that S is a continuous linear operator.

We now define the map $T:\mathcal{X}\to\mathcal{Y}$ by

$$T\Phi:=(\Gamma\circ S)(\Phi)\quad \text{for all } \Phi\in\mathcal{X}.\tag{2.5}$$

Theorem 2.3. *Let T be as defined in* (2.5)*. Then T is a compact linear operator and $\|T\|\le C_3$, where $C_3=C_1C_2$ with C_1,C_2 are constants as in Theorem* 1.1 *and Theorem* 2.1*, respectively.*

Proof. The linearity and continuity of T follows from the fact that Γ and S are both linear and continuous. Since T is a composition of a compact operator Γ (see Theorem 2.2) and a bounded linear operator S, the compactness of T follows. Finally, the estimate can be easily obtained by applying the estimates in Theorem 2.1 and (2.3). $\square$

Next we show that T is of infinite rank. In order to show this, we shall make use of the representation of T^* which is given in Section 4, as an appendix.

Theorem 2.4. *Let T be as defined in* (2.5). *Then $R(T)$, the range of T, is infinite dimensional.*

Proof. We know that T is a bounded linear operator from $\mathcal{X}$ to $\mathcal{Y}$. Therefore $\overline{R(T)} = N(T^*)^{\perp}$. But, T^* is an injective operator (see Theorem A.7). Therefore $\overline{R(T)} = N(T^*)^{\perp} = \mathcal{Y}$. Thus, $R(T)$ is dense in $\mathcal{Y}$, which is an infinite dimensional Hilbert space. $\square$

Let $z \in \mathcal{Y}$. Recall that our inverse problem is to determine an $f \in \mathcal{X}$ such that the corresponding unique weak solution u of (1.1) satisfies $u_{|\Sigma_\tau} = z$. As pointed out in Section 1, there may be more than one solution for this inverse problem. We assume that our inverse problem has a solution f_z. Let u_z be the corresponding unique weak solution of (1.1) for the source function f_z.

Let $f_0 \in \mathcal{X}$ be any *a priori* known function and $u_0 \in \mathcal{W}$ be the unique weak solution of (1.1) for $f = f_0$. Then it can be seen that $u_z - u_0$ is the unique weak solution of the PDE

$$\begin{cases} v_t - \nabla \cdot (Q(x)\nabla v) = f_z - f_0 & \text{in } \Omega_\tau, \\ Q(x)\nabla v \cdot \vec{n} = 0 & \text{on } \partial\Omega_\tau, \\ v(\cdot,0) = 0 & \text{in } \Omega. \end{cases} \tag{2.6}$$

Let $z_0 = u_{0|_{\Sigma_\tau}}$. Then it follows that $f_z - f_0$ is a solution of the operator equation

$$T\Phi = z - z_0. \tag{2.7}$$

Thus, our inverse problem has been transformed into the problem of solving the operator equation (2.7), which has a solution, namely $f_z - f_0$. Also, the solution of the operator equation (2.7) need not be unique. We would like to identify the unique $f^\dagger$, where

$$\|f^\dagger\| := \inf\big\{ \|f\| : Tf = z - z_0 \big\}. \tag{2.8}$$

In practical application, the exact data z may not be known. Instead, we may have a noisy measured data. But by Theorem 2.3 and Theorem 2.4, it follows that (2.7) is an ill-posed operator equation. Therefore solving (2.7) with perturbed right hand side may not give stable solutions, that is, small perturbation in the data may produce large deviation in the solution. So, as mentioned in the introductory section, we shall use the theory of Tikhonov regularization to obtain stable solutions, while dealing with noisy data.

3 Regularization with noisy data

Let $z\in\mathcal{Y}$ be the exact data as considered in our inverse problem **(IP)**. For $\delta>0$, let $z^\delta\in\mathcal{Y}$ be the measured noisy data satisfying

$$\|z-z^\delta\|_{\mathcal{Y}}\le\delta. \tag{3.1}$$

We now consider the perturbed operator equation

$$T\Phi=z^\delta-z_0. \tag{3.2}$$

As T is a linear compact operator of infinite rank, solving the operator equation is ill-posed. To obtain stable approximations with the help of the noisy data z^δ, we shall make use of the standard theory of Tikhonov regularization. For each $\alpha>0$, let f_α and f_α^δ be the unique elements in $\mathcal{X}$ such that

$$(T^*T+\alpha I)f_\alpha=T^*(z-z_0), \tag{3.3}$$

$$(T^*T+\alpha I)f_\alpha^\delta=T^*(z^\delta-z_0). \tag{3.4}$$

The following result is known in the literature ([17,18]).

Theorem 3.1. *For $\delta>0$, let z^δ be as in* (3.1) *and let f_α, f_α^δ be as in* (3.3) *and* (3.4)*, respectively. Let $f^\dagger$ be as defined in* (2.8)*. Then $\|f^\dagger-f_\alpha\|\to0$ as $\alpha\to0$ and*

$$\|f^\dagger-f_\alpha^\delta\|\le\|f^\dagger-f_\alpha\|+\frac{\delta}{2\sqrt{\alpha}}.$$

Remark 3.1. For $\delta>0$, if α_δ is chosen in such a way that $\alpha_\delta\to0$ and $\frac{\delta}{2\sqrt{\alpha_\delta}}\to0$ as $\delta\to0$, then it follows that $\|f^\dagger-f_{\alpha_\delta}^\delta\|\to0$ as $\delta\to0$. For example, taking $\alpha=\delta$, we have $\|f^\dagger-f_\delta^\delta\|\to0$ as $\delta\to0$.

Remark 3.2. In order to obtain an estimate for the quantity $\|f^\dagger-f_\alpha\|$, we need to assume some *a priori* condition on $f^\dagger$. It is well known in the theory of Tikhonov regularization that if $f^\dagger\in R(T^*)$, the range of T^*, then $\|f^\dagger-f_\alpha\|\le c\sqrt{\alpha}$ for some constant $c>0$, so that, in this case, we also have the rate

$$\|f^\dagger-f_{\alpha_\delta}^\delta\|=O(\sqrt{\delta})$$

for the choice $\alpha\sim\delta$. More generally, if $\varphi:(0,\infty)\to[0,\infty)$ is a monotonically increasing function such that

$$\sup_{\lambda>0}\frac{\alpha\varphi(\lambda)}{\alpha+\lambda}\le c_0\varphi(\alpha),\quad \alpha>0,$$

for some $c_0>0$ and if $f^\dagger\in R(\varphi(T^*T))$, then it is known that (see [18, pp. 195])

$$\|f^\dagger-f_\alpha\|=O(\varphi(\alpha)),$$

so that

$$\|f^\dagger - f^\delta_{\alpha_\delta}\| = O(\varphi(\alpha_\delta)),$$

where $\alpha_\delta > 0$ is such that $\sqrt{\alpha_\delta}\varphi(\alpha_\delta) \sim \delta$. Typical example of such functions are $\varphi(\lambda) := \lambda^\nu$ for some $\nu \in (0,1]$ or $\varphi(\lambda) = [\log(\frac{1}{\lambda})]^{-p}$ for some $p > 0$.

In practical situation, one would like to obtain finite dimensional approximations for f^δ_α in a stable way without losing the approximating property similar to that of f^δ_α. The finite dimensional analysis is one of the main purpose of this article.

4 Finite dimensional analysis

For each $n \in \mathbb{N}$, let $P_n : L^2(\Omega) \to L^2(\Omega)$ be an orthogonal projection of rank n and for each $m \in \mathbb{N}$, let $\Pi_m : L^2(0,\tau) \to L^2(0,\tau)$ be an orthogonal projection of rank m such that

$$P_n \to I \quad \text{and} \quad \Pi_m \to I$$

pointwise on $L^2(\Omega)$ and $L^2(0,\tau)$, respectively. Note that, these assumptions are natural in the context of numerical approximations: For instance, in numerical approximation of functions, one may have $R(P_n) \subset R(P_{n+1})$, $R(\Pi_m) \subset R(\Pi_{m+1})$ for all $n,m \in \mathbb{N}$, and $\overline{\bigcup_n R(P_n)} = L^2(\Omega)$, $\overline{\bigcup_m R(\Pi_m)} = L^2(0,\tau)$. In such case, we do have the above mentioned pointwise convergences.

Let $\{\varphi_1,\ldots,\varphi_n\}$ be an orthonormal basis of $R(P_n)$ and $\{g_1,\ldots,g_m\}$ be an orthonormal basis of $R(\Pi_m)$. Using the idea proposed in [20], for each n,m, we define $Q^m_n : \mathcal{X} \to \mathcal{X}$ as

$$(Q^m_n \Phi)(t) := \sum_{i=1}^{n} \Pi_m(\langle \Phi(t), \varphi_i \rangle)\varphi_i \quad \text{for all } \Phi \in \mathcal{X}. \tag{4.1}$$

Note that, for any $\Phi \in \mathcal{X}$ and $\varphi \in L^2(\Omega)$, the map $t \mapsto \langle \Phi(t), \varphi \rangle$ is an element of $L^2(0,\tau)$. Therefore, the map Q^m_n is well-defined. Also, it can be easily seen that Q^m_n is a linear operator, and it is of finite rank. Our next result shows that Q^m_n is also an orthogonal projection of finite rank. Here we would like to mention that the proof is similar to the proof of Theorem 4.5 in [20]. But, since the context is different, for the sake of completeness we are giving the detailed proof.

Theorem 4.1. *For $n,m \in \mathbb{N}$, the linear operator $Q^m_n : \mathcal{X} \to \mathcal{X}$ defined as in* (4.1) *is an orthogonal projection of rank nm. In fact, if*

$$\Phi^{ij}(t) = g_j(t)\varphi_i, \quad t \in (0,\tau), \tag{4.2}$$

for $i = 1,\ldots,n$, $j = 1,\ldots,m$, then $\{\Phi^{ij} : i = 1,\ldots,n, j = 1,\ldots,m\}$ is an orthonormal set in $\mathcal{X}$ and

$$Q^m_n \Phi = \sum_{i=1}^{n}\sum_{j=1}^{m} \langle \Phi, \Phi^{ij} \rangle_{\mathcal{X}} \Phi^{ij}, \quad \Phi \in \mathcal{X}.$$

Proof. Since $\{g_1,\dots,g_m\}\subset L^2(0,\tau)$ is an orthonormal basis of $R(\Pi_m)$, we have

$$\Pi_m h=\sum_{j=1}^{m}\langle h,g_j\rangle_{L^2(0,\tau)}\, g_j,\quad h\in L^2(0,\tau).$$

Let $\Phi^{ij}(t)$ be as in (4.2). Then, we see that, $\Phi^{ij}\in\mathcal{X}$ for all $i=1,\dots,n, j=1,\dots,m$, and

$$\langle\Phi^{ij},\Phi^{rs}\rangle_{\mathcal{X}}=\langle\varphi_i,\varphi_r\rangle_{L^2(\Omega)}\langle g_j,g_s\rangle_{L^2(0,\tau)}=\delta_{ir}\delta_{js},$$

where $\delta_{pq}=1$ for $p=q$ and $\delta_{pq}=0$ for $p\neq q$. Therefore, $\{\Phi^{ij}:i=1,\dots,n,j=1,\dots,m\}$ is an orthonormal set in $\mathcal{X}$.

Next, we show that rank of Q_n^m is nm. For $\Phi\in\mathcal{X}$ we denote $\phi_i(t):=\langle\Phi(t),\varphi_i\rangle$ for all $i=1,\dots,n$ and $t\in(0,\tau)$. Then for $\Phi\in\mathcal{X}$ and $t\in(0,\tau)$,

$$(Q_n^m\Phi)(t)=\sum_{i=1}^{n}(\Pi_m\phi)(t)\varphi_i=\sum_{i=1}^{n}\Big(\sum_{j=1}^{m}\langle\phi_i,g_j\rangle_{L^2(0,\tau)}\, g_j(t)\Big)f_i=\sum_{i=1}^{n}\sum_{j=1}^{m}a_{ij}\Phi^{ij}(t),$$

where $a_{ij}=\langle\phi_i,g_j\rangle_{L^2(0,\tau)}$. Again by the definition of ϕ_i, we have

$$\begin{aligned}a_{ij}&=\int_0^\tau\phi_i(t)g_j(t)\,\mathrm{d}t=\int_0^\tau\langle\Phi(t),\varphi_i\rangle_{L^2(\Omega)}\, g_j(t)\,\mathrm{d}t\\&=\int_0^\tau\langle\Phi(t),g_j(t)\varphi_i\rangle_{L^2(\Omega)}\mathrm{d}t=\int_0^\tau\langle\Phi(t),\Phi^{ij}(t)\rangle_{L^2(\Omega)}\mathrm{d}t\\&=\langle\Phi,\Phi^{ij}\rangle_{\mathcal{X}}.\end{aligned}$$

Thus,

$$Q_n^m\Phi=\sum_{i=1}^{n}\sum_{j=1}^{m}\langle\Phi,\Phi^{ij}\rangle_{\mathcal{X}}\Phi^{ij},\quad\Phi\in\mathcal{X}.$$

Since $\{\Phi^{ij}:i=1,\dots,n,j=1,\dots,m\}$ is an orthonormal set, therefore Q_n^m is an orthogonal projection of rank nm. □

Recall that our aim is to solve (3.4) in a finite dimensional setting. We are now in a position to do so. We shall use the *Galerkin method* for obtaining solutions in a finite dimensional space. We first observe that f_α^δ satisfies (3.4) if and only if

$$\langle(T^*T+\alpha I)f_\alpha^\delta,\Phi\rangle=\langle T^*(z^\delta-z_0),\Phi\rangle\quad\text{for all }\Phi\in\mathcal{X}.\tag{4.3}$$

Now, we obtain approximate solution in the finite dimensional space $\mathcal{X}_n^m:=R(Q_n^m)$ by varying Φ in $\mathcal{X}_n^m:=R(Q_n^m)$. That is, we would like to obtain a unique $\tilde f\in\mathcal{X}_n^m$ satisfying the equation

$$\langle(T^*T+\alpha I)\tilde f,\Phi\rangle=\langle T^*(z^\delta-z_0),\Phi\rangle\quad\text{for all }\Phi\in\mathcal{X}_n^m.\tag{4.4}$$

Since Q_n^m is an orthogonal projection, the equation (4.4) is same as

$$\langle (Q_n^m T^* T Q_n^m + \alpha I)\tilde{f}, \Phi\rangle = \langle Q_n^m T^*(z^\delta - z_0), \Phi\rangle \quad \text{for all } \Phi \in \mathcal{X}_n^m; \tag{4.5}$$

equivalently,

$$(Q_n^m T^* T Q_n^m + \alpha I)\tilde{f} = Q_n^m T^*(z^\delta - z_0).$$

Since $Q_n^m T^* T Q_n^m$ is a positive self adjoint operator, $Q_n^m T^* T Q_n^m + \alpha I$ is invertible for each $\alpha > 0$, and hence, there exists a unique $\tilde{f} \in \mathcal{X}_n^m$ satisfying (4.5). We shall denote this $\tilde{f}$ by $f^\delta_{n,m,\alpha} \in \mathcal{X}_n^m$. Thus,

$$(Q_n^m T^* T Q_n^m + \alpha I) f^\delta_{n,m,\alpha} = Q_n^m T^*(z^\delta - z_0) \tag{4.6}$$

and

$$\langle (Q_n^m T^* T Q_n^m + \alpha I) f^\delta_{n,m,\alpha}, \Phi\rangle = \langle T^*(z^\delta - z_0), \Phi\rangle \quad \text{for all } \Phi \in \mathcal{X}_n^m,$$

equivalently

$$\langle T f^\delta_{n,m,\alpha}, T\Phi^{ij}\rangle + \alpha \langle f^\delta_{n,m,\alpha}, \Phi^{ij}\rangle = \langle (z^\delta - z_0), T\Phi^{ij}\rangle \quad \text{for all } i = 1,\ldots,n, j = 1,\ldots,m. \tag{4.7}$$

Let $f^\delta_{n,m,\alpha} = \sum_{i=1}^n \sum_{j=1}^m c_{ij}\Phi^{ij}$ for some scalars $c_{ij}, i = 1,\ldots,n, j = 1,\ldots,m$. Then (4.7) takes the form

$$\begin{aligned} &\sum_{i=1}^n \sum_{j=1}^m c_{ij}\langle T\Phi^{ij}, T\Phi^{pq}\rangle + \alpha \sum_{i=1}^n \sum_{j=1}^m c_{ij}\langle \Phi^{ij}, \Phi^{pq}\rangle \\ &= \langle z^\delta - z_0, T\Phi^{pq}\rangle \qquad \text{for all } 1 \le p \le n, 1 \le q \le m. \end{aligned}$$

Therefore, we have the matrix equation

$$(A + \alpha D)\vec{c} = \vec{b}, \tag{4.8}$$

where

$$A = [\vec{a}_{11}, \ldots, \vec{a}_{nm}]^t, \quad D = [\vec{d}_{11}, \ldots, \vec{d}_{nm}]^t, \quad \vec{b} = [b_{11}, \ldots, b_{nm}]^t$$

with

$$\vec{a}_{pq} := (\langle T\Phi^{11}, T\Phi^{pq}\rangle, \ldots, \langle T\Phi^{nm}, T\Phi^{pq}\rangle), \quad \vec{d}_{pq} := (\langle \Phi^{11}, \Phi^{pq}\rangle, \ldots, \langle \Phi^{nm}, \Phi^{pq}\rangle)$$

and $b_{pq} = \langle z^\delta - z_0, \Phi^{pq}\rangle$, for $1 \le p \le n, 1 \le q \le m$. Note that, since (4.5) has a unique solution, the matrix equation (4.8) also has a unique solution. Thus, we have obtained

$$f^\delta_{n,m,\alpha} = \sum_{i=1}^n \sum_{j=1}^m c_{ij}\Phi^{ij}$$

where $\vec{c} = [c_{11}, \ldots, c_{nm}]^t$ is the unique solution of (4.8).

Having obtained the finite dimensional solution $f^\delta_{n,m,\alpha}$, we want to know how good it is as an approximation for the sought source function $f^\dagger$. For that we first obtain an estimate for $\|f^\dagger - f^\delta_{n,m,\alpha}\|$.

Theorem 4.2. *Let z^δ be as in* (3.1) *and f_α, f_α^δ and $f_{n,m,\alpha}^\delta$ be as in* (3.3), (3.4) *and* (4.6)*, respectively, Then*

$$\|f^\dagger - f_{n,m,\alpha}^\delta\| \le \|f^\dagger - f_\alpha\| + \frac{\|T - TQ_n^m\|}{\sqrt{\alpha}}\|f^\dagger\| + \frac{\delta}{2\sqrt{\alpha}}.$$

Proof. Since $f^\dagger$ is a solution of (2.7) and $f_{n,m,\alpha}^\delta$ satisfies (4.6), we have

$$\begin{aligned}
&\|f^\dagger - f_{n,m,\alpha}^\delta\| \\
\le&\|f^\dagger - f_\alpha\| + \|f_\alpha - f_{n,m,\alpha}^\delta\| \\
\le&\|f^\dagger - f_\alpha\| + \|(T^*T+\alpha I)^{-1}T^* - (Q_n^m T^* T Q_n^m + \alpha I)^{-1} Q_n^m T^*\|\|z - z_0\| \\
&\quad + \|((Q_n^m T^* T Q_n^m + \alpha I)^{-1} Q_n^m T^*\|\|z - z^\delta\| \\
\le&\|f^\dagger - f_\alpha\| + \|(Q_n^m T^* T Q_n^m + \alpha I)^{-1} Q_n^m T^*(T - TQ_n^m) T^* T (T^*T + \alpha I)^{-1}\|\|f^\dagger\| \\
&\quad + \alpha\|(Q_n^m T^* T Q_n^m + \alpha I)^{-1}(Q_n^m T^* - T^*)(T^*T+\alpha I)^{-1} T\|\|f^\dagger\| \\
&\quad + \|((Q_n^m T^* T Q_n^m + \alpha I)^{-1} Q_n^m T^*\|\|z - z^\delta\|.
\end{aligned}$$

Now, using the estimates

$$\begin{aligned}
&\|(Q_n^m T^* T Q_n^m + \alpha I)^{-1}\| \le \frac{1}{\alpha}, \quad \|T^*T(T^*T+\alpha I)^{-1}\| \le 1, \\
&\|(Q_n^m T^* T Q_n^m + \alpha I)^{-1} Q_n^m T^*\| \le \frac{1}{2\sqrt{\alpha}}, \qquad \|(TT^* + \alpha I)^{-1} T\| \le \frac{1}{2\sqrt{\alpha}},
\end{aligned}$$

and the fact that $\|Q_n^m T^* - T^*\| = \|TQ_n^m - T\|$, we have

$$\|f^\dagger - f_{n,m,\alpha}^\delta\| \le \|f^\dagger - f_\alpha\| + \frac{\|T - TQ_n^m\|}{\sqrt{\alpha}}\|f^\dagger\| + \frac{\delta}{2\sqrt{\alpha}}.$$

This completes the proof. □

Our next attempt is to show that the quantity $\|T - TQ_n^m\|$ can be made small enough for some large $n,m \in \mathbb{N}$. For that, we require a few results whose proof uses similar arguments as given in [20]. But, since the context is different, for the sake of completeness we give the proofs also.

Theorem 4.3. *Let Q_n^m be as defined in* (4.1)*. Then, for each $\Phi \in \mathcal{X}$,*

$$\lim_{n\to\infty}\lim_{m\to\infty}\|Q_n^m \Phi - \Phi\|_{\mathcal{X}} = 0.$$

Proof. Let $\Phi \in \mathcal{X}$ and for $i = 1,\dots,n$, let $\phi_i(t) = \langle \Phi(t), \varphi_i\rangle$ for $t \in (0,\tau)$. Then, it can be easily seen that for each $t \in (0,\tau)$, $\phi_i(t)\varphi_i \in L^2(\Omega)$ so that the function $t \mapsto \phi_i(t)\varphi_i$ is an element of $\mathcal{X}$. Since $\{\varphi_i : i = 1,\dots,n\}$ is an orthonormal set, we have

$$\|Q_n^m \Phi - \Phi\|_{\mathcal{X}} = \Big\|\sum_{i=1}^n (\Pi_m \phi_i)(\cdot)\varphi_i - \Phi\Big\|_{\mathcal{X}}$$

$$
\begin{aligned}
&\le \Big\| \sum_{i=1}^{n} [(\Pi_m \phi_i)(\cdot) - \phi_i(\cdot)]\varphi_i \Big\|_{\mathcal{X}} + \Big\| \sum_{i=1}^{n} \phi_i(\cdot)\varphi_i - \Phi \Big\|_{\mathcal{X}} \\
&\le \sum_{i=1}^{n} \|\varphi_i\|_{L^2(\Omega)} \|\Pi_m \phi_i - \phi_i\|_{L^2(0,\tau)} + \Big\| \sum_{i=1}^{n} \phi_i(\cdot)\varphi_i - \Phi \Big\|_{\mathcal{X}} \\
&= \sum_{i=1}^{n} \|\Pi_m \phi_i - \phi_i\|_{L^2(0,\tau)} + \Big\| \sum_{i=1}^{n} \phi_i(\cdot)\varphi_i - \Phi \Big\|_{\mathcal{X}}.
\end{aligned}
$$

Since $\Pi_m \to I$ pointwise in $L^2(0,\tau)$, for each $n \in \mathbb{N}$, we have

$$
\lim_{m\to\infty} \|Q_n^m \Phi - \Phi\|_{\mathcal{X}} \le \Big\| \sum_{i=1}^{n} \phi_i(\cdot)\varphi_i - \Phi \Big\|_{\mathcal{X}}.
$$

Also, since $\{\varphi_1,\ldots,\varphi_n\}$ is an orthonormal basis of $R(P_n)$, for each $t \in (0,\tau)$, we have

$$
P_n \Phi(t) = \sum_{i=1}^{n} \langle \Phi(t), \varphi_i \rangle_{L^2(\Omega)} \varphi_i = \sum_{i=1}^{n} \phi_i(t)\varphi_i.
$$

Since $P_n \to I$ pointwise in $L^2(\Omega)$ and $\|P_n\Phi(t) - \Phi(t)\| \le 2\|\Phi(t)\|$, by dominated convergence theorem we have

$$
\lim_{n\to\infty} \Big\| \sum_{i=1}^{n} \phi_i(\cdot)\varphi_i - \Phi \Big\|_{\mathcal{X}}^2 = \lim_{n\to\infty} \int_0^\tau \|P_n \Phi(t) - \Phi(t)\|_{L^2(O)}^2 \mathrm{d}t = 0.
$$

Therefore

$$
\lim_{n\to\infty} \lim_{m\to\infty} \|Q_n^m \Phi - \Phi\|_{\mathcal{X}} = 0
$$

for every $\Phi \in \mathcal{X}$. This completes the proof. □

From the above theorem we obtain the following corollary, by simply using the definition of double limits as obtained in the theorem.

Corollary 4.1. *Let Q_n^m be as defined in* (4.1)*. Then, for every $\varepsilon > 0$ and for each $\Phi \in \mathcal{X}$, there exist $N \in \mathbb{N}$ and $m_n \in \mathbb{N}$ for every $n \ge N$ such that*

$$
\|Q_n^m \Phi - \Phi\|_{\mathcal{X}} < \varepsilon \quad \forall m \ge m_n, \quad n \ge N.
$$

Theorem 4.4. *Let $T : \mathcal{X} \to \mathcal{Y}$ and $Q_n^m : \mathcal{X} \to \mathcal{X}$ be as defined in* (2.5) *and* (4.1)*, respectively, and let $\varepsilon > 0$ be given. Then there exists $N \in \mathbb{N}$ and $m_n \in N$ for every $n \ge N$ such that*

$$
\|T Q_n^m - T\| < \varepsilon \quad \forall m \ge m_n, \quad n \ge N.
$$

Proof. Since $T:\mathcal{X}\to\mathcal{Y}$ is a compact operator, $T^*:\mathcal{Y}\to\mathcal{X}$ is also a compact operator. Therefore $S=\mathrm{cl}\{T^*y:y\in\mathcal{Y},\|y\|\le 1\}$ is a compact subset of $\mathcal{X}$. Hence

$$\|TQ_n^m-T\|=\|Q_n^mT^*-T^*\|=\sup_{\|y\|\le 1}\|(Q_n^mT^*-T^*)y\|=\sup_{\xi\in S}\|Q_n^m\xi-\xi\|.$$

Let $\xi\in S$ and $\varepsilon>0$. Since S is a compact in $\mathcal{X}$, there exists $\xi_1,\dots,\xi_k\in S$ such that $S\subset\cup_{i=1}^k B(\xi_i,\dfrac{\varepsilon}{4})$, where $B(\xi_i,\frac{\varepsilon}{4})$ denotes the open ball in $\mathcal{X}$ centered at ξ_i and radius $\frac{\varepsilon}{4}$. Let $j\in\{1,\dots,k\}$ be such that $\|\xi-\xi_j\|_{\mathcal{X}}<\frac{\varepsilon}{4}$. Now, by Corollary 4.1, we have for each $i\in\{1,\dots,k\}$ there exists $N_i\in\mathbb{N}$ and $m_{i,n}\in\mathbb{N}$ for each $n\ge N_i$ such that

$$\|Q_n^m\xi_i-\xi_i\|<\frac{\varepsilon}{2}\quad\text{for all }\ m\ge m_{i,n},\quad n\ge N_i.$$

Let $N=\max\{N_i:i=1,\dots,k\}$ and for $n\ge N$, let $m_n=\max\{m_{i,n}:i=1,\dots,k\}$. Then for every $n\ge N$ and $m\ge m_n$, we have

$$\|Q_n^m\xi_i-\xi_i\|<\frac{\varepsilon}{2}\quad\text{for all }\ i\in\{1,\dots,k\}.$$

Thus, for every $n\ge N$, there exists $m_n\in\mathbb{N}$ such that for all $m\ge m_n$

$$\begin{aligned}\|Q_n^m\xi-\xi\|&\le\|Q_n^m\xi-Q_n^m\xi_j\|+\|Q_n^m\xi_j-\xi_j\|+\|\xi_j-\xi\|\\&\le 2\|\xi_j-\xi\|+\|Q_n^m\xi_j-\xi_j\|\\&<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

Thus, the proof is completed. □

In view of Theorem 4.4 and Theorem 4.2 we have the following.

Theorem 4.5. *For $\delta>0$, let n_δ,m_δ be in $\mathbb{N}$ such that $\|T-TQ_{n_\delta}^{m_\delta}\|\le\delta$. Then*

$$\|f^\dagger-f^\delta_{n_\delta,m_\delta,\alpha}\|\le\|f^\dagger-f_\alpha\|+\frac{2\delta}{\sqrt{\alpha}}C_{f^\dagger},$$

where $C_{f^\dagger}=\max\{\frac{1}{2},\|f^\dagger\|\}$.

The above theorem along with Remark 3.2 leads to the following theorem.

Theorem 4.6. *For $\delta>0$, let $\hat{n}:=n_\delta,\hat{m}:=m_\delta$ be as in Theorem 4.5. Then we have the following.*

(i) *If $f^\dagger\in R(T^*)$ and $\alpha:=c\delta$ for some $c>0$, then*

$$\|f^\dagger-f^\delta_{n_\delta,m_\delta,\delta}\|=O(\sqrt{\delta}).$$

(ii) *If $f^\dagger \in R(\varphi(T^*T))$, where φ is as in Remark* 3.2 *and if $\alpha_\delta > 0$ is such that $\sqrt{\alpha}_\delta \varphi(\alpha_\delta) = \delta$, then*
$$\|f^\dagger - f^\delta_{n_\delta,m_\delta,\alpha_\delta}\| = O(\varphi(\alpha_\delta)).$$

Remark 4.1. From the characterization of $R(T^*)$ as obtained in Theorem A.7 in the Appendix, we give a procedure to check whether an apriori known source function is in the range of T^* or not. Following is the condition:

Suppose $f \in \mathcal{W}$. Then $f \in R(T^*)$ if and only if

1. $f(\cdot,\tau) = 0$ in Ω,
2. $Q\nabla f \cdot \vec{n} = 0$ on $(\partial\Omega \setminus \Sigma) \times [0,\tau]$,
3. $f(\cdot,t) = \int_t^\tau \nabla \cdot Q \nabla f(\cdot,s) \mathrm{d}s$ in Ω for a.a $t \in [0,\tau]$,

and in that case $T^*(\hat{\phi}) = f$, where $\hat{\phi} = Q\nabla f \cdot \vec{n}_{|\Sigma_\tau}$.

Remark 4.2. In Theorem 4.6, we have considered a procedure of choosing the regularization parameter α. There is a vast literature regarding various parameter strategies both *a priori* and *a posteriori*. For, instance in [21, 22], the authors have given a procedure of adaptive choice of parameters by balancing principle. Those parameter choice strategies of choosing α_δ can be employed here, leading to order optimal rate for $\|f^\dagger - f^\delta_{n_\delta,m_\delta,\alpha_\delta}\|$. Also, one can employ the idea of dynamic regularization algorithm for choosing the regularizing parameter, see [23, 24].

Appendix

In this section, we shall find explicitly the range space of T^*. Recall that for $\Phi \in \mathcal{X}$ and $\phi \in \mathcal{Y}$, we have
$$\langle T\Phi, \phi \rangle = \int_0^\tau \langle (T\Phi)(t), \phi(t) \rangle_{L^2(\Sigma)} \mathrm{d}t = \int_0^\tau \langle \gamma(S\Phi(t)), \phi(t) \rangle_{L^2(\Sigma)} \mathrm{d}t$$
where γ is the trace map and S is the linear map defined in Section 2, by assigning each $\Phi \in \mathcal{X}$, the unique weak solution $v_\Phi := S\Phi$ of (2.2) .

For $\phi \in \mathcal{Y}$, we consider the PDE
$$\begin{cases} -w_t - \nabla \cdot (Q(x)\nabla w) = 0 & \text{in } \Omega_\tau, \\ Q(x)\nabla w \cdot \vec{n} = \phi & \text{on } \Sigma \times [0,\tau], \\ Q(x)\nabla w \cdot \vec{n} = 0 & \text{on } (\partial\Omega \setminus \Sigma) \times [0,\tau], \\ w(\cdot,\tau) = 0 & \text{in } \Omega, \end{cases} \tag{A.1}$$

where $Q\in(L^\infty(\Omega))^{d\times d}$ is as in (2.2). Then reversing the time direction and following the lines of the proof of Theorem 1.1 and following Part 3 of the proof of Theorem 3 in [3, pg. 357], (A.1) has a unique weak solution $w_\phi\in\mathcal{W}$, that is,

$$-\langle (w_\phi)_t(\cdot,t),\varphi\rangle+\int_\Omega Q(x)\nabla w_\phi(\cdot,t)\cdot\nabla\varphi \mathrm{d}x=\int_\Sigma \phi\varphi\mathrm{d}x \tag{A.2}$$

for all $\varphi\in H^1(\Omega)$ for a.a. $t\in[0,\tau]$.

Theorem A.7. *For a given $\phi\in\mathcal{Y}$, let $w_\phi\in\mathcal{W}$ be the unique weak solution of* (A.1). *Let T be as defined in* (2.5). *Then $T^*\phi=w_\phi$,*

$$R(T^*)=\{w_\phi\in\mathcal{W}:\phi\in L^2(0,\tau;L^2(\Sigma))\}$$

and T^* is one-one.

Proof. Let v_Φ be the unique weak solution of (2.2). Since $v_\Phi\in L^2(0,\tau;H^1(\Omega))$, by (A.2), we have

$$-\langle (w_\phi)_t(\cdot,t),v_\Phi(\cdot,t)\rangle+\int_\Omega Q(x)\nabla w_\phi(\cdot,t)\cdot\nabla(v_\Phi)(\cdot,t)\mathrm{d}x=\int_\Sigma \phi\gamma(v_\Phi)(\cdot,t)\mathrm{d}x$$

for a.a. $t\in[0,\tau]$. Integrating the above with respect to t, we have

$$\begin{aligned}&-\int_0^\tau\langle (w_\phi)_t(\cdot,t),v_\Phi(\cdot,t)\rangle\mathrm{d}t+\int_0^\tau\int_\Omega Q(x)\nabla w_\phi(\cdot,t)\cdot\nabla(v_\Phi)(\cdot,t)\mathrm{d}x\mathrm{d}t\\&=\int_0^\tau\int_\Sigma\phi\gamma(v_\Phi)(\cdot,t)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Thus, by applying integration by parts for the first integral of the left hand side in the above, we have

$$\begin{aligned}&\int_0^\tau\langle (w_\phi)(\cdot,t),(v_\Phi)_t(\cdot,t)\rangle\mathrm{d}t+\int_0^\tau\int_\Omega Q(x)\nabla w_\phi(\cdot,t)\cdot\nabla(v_\Phi)(\cdot,t)\mathrm{d}x\mathrm{d}t\\&=\int_0^\tau\int_\Sigma\phi\gamma(v_\Phi)(\cdot,t)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Since v_Φ is the weak solution of (2.2) and $w_\phi\in L^2(0,\tau;H^1(\Omega))$, from the above equation we have

$$\int_0^\tau\int_\Omega\Phi w_\phi=\int_0^\tau\int_\Sigma\phi\gamma(v_\Phi)(\cdot,t)\mathrm{d}x\mathrm{d}t=\int_0^\tau\langle\phi(t),\gamma(S\Phi(t))\rangle_{L^2(\Sigma)}\mathrm{d}t=\langle T\Phi,\phi\rangle.$$

Thus,

$$\langle T\Phi,\phi\rangle_{L^2(0,\tau;L^2(\Sigma))}=\langle\Phi,w_\phi\rangle_{L^2(0,\tau;L^2(\Omega))}$$

for all $\Phi\in\mathcal{X}$ and $\phi\in\mathcal{Y}$. Hence, $T^*:L^2(0,\tau;L^2(\Sigma))\to L^2(0,\tau;L^2(\Omega))$, the adjoint of T, is defined by

$$T^*\phi=w_\phi\quad\text{for all }\ \phi\in L^2(0,\tau;L^2(\Sigma)),$$

and $R(T^*)=\{w_\phi\in\mathcal{W}:\phi\in L^2(0,\tau;L^2(\Sigma))\}$. The fact that T^* is one-one, follows from the representation of T^* as obtained above together with the second equation of (A.1). □

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