Real Linear Operators and Factorization of Real Polynomials

Arindama Singh
Department of Mathematics
Indian Institute of Technology Madras
Chennai-6000036, India
Email: asingh@iitm.ac.in

Abstract: A linear operator on a finite dimensional nonzero real vector space may not have an eigenvalue. However, corresponding to each such operator T, there exists a pair of real numbers (α, β) and a nonzero vector v such that $[(T - \alpha I)^2 + \beta^2 I](v) = 0$. This can be proved by using the Fundamental theorem of algebra and Cayley-Hamilton theorem, and also by complexification. We construct an inductive proof of this fact, where we do not use even the complex numbers. From this we deduce that a polynomial with real coefficients can be written as a product of linear factors and quadratic factors with negative discriminant. It thus gives a proof of the latter fact about polynomials with real coefficients, which does not use complex numbers, thereby solving an open problem raised earlier.

Keywords: Linear operators, Eigenvalues, True-pair, Real polynomial, Factorization.

MSC: 00A05, 15A06

1 Introduction

Let V be a real vector space of dimension $n \geq 1$. Let $T : V \to V$ be a linear operator. Recall that a real number λ is called an eigenvalue of T if there exists a nonzero vector $u \in V$ such that $Tu = \lambda u$. In this case, any such vector u is called an eigenvector. The polynomial $\chi_T(t) := \det(tI - T)$ is called the characteristic polynomial of T, and it is a monic polynomial of degree n with real coefficients. Eigenvalues of T are precisely the real zeroes of $\chi_T(t)$. Thus, there are linear operators on real vector spaces having no eigenvalues. For instance, the linear operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(a, b) = (-b, a)$ does not have an eigenvalue. However, we see that $(T^2 + I)(a, b) = 0$ for any $(a, b) \in \mathbb{R}^2$.

Due to the fundamental theorem of algebra, $\chi_T(t)$ has n number of complex zeroes counting multiplicities. Since $\chi_T(t)$ has real coefficients, all its complex zeroes occur in conjugate pairs. That is, it can be written in the form

$$\mathcal{X}_T(t) = (t - a_1) \cdots (t - a_m)(t - b_1)(t - \overline{b}_1) \cdots (t - b_k)(t - \overline{b}_k)$$

where a_i are real numbers and b_j are complex numbers with nonzero imaginary parts. Further, if $\mathcal{X}_T(t)$ has no real zeroes, then $t - a_1, \ldots, t - a_m$ are absent in the above product; and similarly, if all zeroes of $\mathcal{X}_T(t)$ are real, then $t - b_1, \ldots, t - \overline{b}_k$ are absent. Writing each $b_j = \alpha_j + i\beta_j$ for real numbers α_j, β_j with $\beta_j \neq 0$, we see that $(t - b_j)(t - \overline{b}_j) = (t - \alpha_j)^2 + \beta_j^2$. Thus,

$$\mathcal{X}_T(t) = (t - a_1) \cdots (t - a_m)\big((t - \alpha_1)^2 + \beta_1^2\big) \cdots \big((t - \alpha_k)^2 + \beta_k^2\big). \tag{1}$$

Due to Cayley-Hamilton theorem, T satisfies its characteristic polynomial. That is,

$$(T - a_1 I) \cdots (T - a_m I)\big((T - \alpha_1 I)^2 + \beta_1^2 I\big) \cdots \big((T - \alpha_k I)^2 + \beta_k^2 I\big) = 0.$$

Multiply $(T - a_1 I) \cdots (T - a_m I)$ with this and rewrite $(T - a_i I)^2$ as $(T - \alpha_i I)^2 + 0^2 I$ to get

$$\begin{aligned}\big((T - a_1 I)^2 + 0^2 I\big) \cdots \ \big((T - a_m I)^2 + 0^2 I\big) \times \\ \big((T - \alpha_1 I)^2 + \beta_1^2 I\big) \cdots \big((T - \alpha_k I)^2 + \beta_k^2 I\big) = 0.\end{aligned}$$

Let v_k be a nonzero vector. Write $v_{k-1} =: \big((T - \alpha_k I)^2 + \beta_k^2 I\big)(v_k)$. From the above equation it follows that either $v_{k-1} = 0$ or

$$\begin{aligned}\big((T - a_1 I)^2 + 0^2 I\big) \cdots \ \big((T - a_m I)^2 + 0^2 I\big) \times \\ \big((T - \alpha_1 I)^2 + \beta_1^2 I\big) \cdots \big((T - \alpha_{k-1} I)^2 + \beta_{k-1}^2 I\big)(v_{k-1}) = 0.\end{aligned}$$

Proceeding inductively we see that there exist $\alpha, \beta \in \mathbb{R}$ and a nonzero vector $v \in V$ such that

$$[(T - \alpha I)^2 + \beta^2 I](v) = 0. \tag{2}$$

Notice that we have deduced (2) assuming that every polynomial with real coefficients can be factored into linear and/or quadratic factors with negative determinant as in (1). This statement about factorization of polynomials with real coefficients does not require complex numbers, but its proof uses complex numbers. Thus, an open problem has been raised in [5] asking for a proof of existence of such a factorization of a polynomial with real coefficients, which does not use complex numbers.

In order to construct such a proof, we proceed in the reverse direction. We first prove (2) without using complex numbers, and then deduce the result about the existence of the said factorization of polynomials with real coefficients. For this purpose, we use the following facts, whose proofs do not depend upon complex numbers, the fundamental theorem of algebra or Cayley-Hamilton theorem. By a real polynomial we mean a polynomial in a single variable t with real coefficients.

Fact 1: Any real polynomial of odd degree has a real zero.

Fact 2: Any real polynomial of degree four can be expressed as a product of two real polynomials of degree two each.

Fact 3: For each monic real polynomial $p(t)$ of degree n, there exists a matrix A of order n, called the companion matrix of $p(t)$, such that the characteristic polynomial $\det(tI - A)$ of A is $p(t)$.

Fact 1 follows from the Intermediate value property of continuous real valued functions. There are many methods to express a real polynomial of degree four as a product of two quadratic factors mentioned in Fact 2, which do not use complex numbers. For two such methods by Ferrari and Descartes, see articles 12-14 of Chapter XII in [1]. Fact 3 is discussed in almost all standard books on Linear Algebra; for instance, see Sect. 5.2 in [4].

Later, we will deduce the existence of a factorization of real polynomials as in (1) by using the division algorithm and Cayley-Hamilton theorem. It is pertinent to note that the division algorithm does not use complex numbers, and there are proofs of Cayley-Hamilton theorem which also do not use complex numbers; see for instance, Sect. 5.5 in [4] for such a proof.

2 Preliminary results

In this section, we introduce some terminology and prove some results, which will lead to our main result.

Let V be a finite dimensional nonzero real vector space. Let $T : V \to V$ be a linear operator on V. We say that T *has a true-pair vector* if there exist real numbers α, β and a nonzero vector $v \in V$ such that

$$[(T - \alpha I)^2 + \beta^2 I](v) = 0.$$

In such a case, we say that the pair of real numbers (α, β) is a *true-pair* of T and v is an associated *true-pair vector* of T.

We mention that the notion of a true-pair of a real linear operator has close connection with the spectrum $\sigma(a)$ of an element a of a real algebra A, which is defined as follows:

$$\sigma(a) = \{\alpha + i\beta : \alpha, \beta \in \mathbb{R},\ (a - \alpha)^2 + \beta^2 \text{ is not invertible in } A\}.$$

For more information on this, see [3].

If T has an eigenvalue λ with an associated eigenvector u, then $(T - \lambda I)(u) = 0$. It implies that $[(T - \lambda I)^2 + 0^2 I](u) = 0$; that is, T has a true-pair, namely $(\lambda, 0)$ with an associated true-pair vector u.

As shown in Sect.1, even if a real linear operator does not have an eigenvalue, it does have a true-pair vector. Further, this fact can also be proved by using the technique of complexification. In this method, one extends the underlying real vector space V of a linear operator T to a complex vector space V_c and considers the corresponding extended linear operator T_c on V_c. Then, one shows that a pair of real numbers (α, β) is a true-pair of T if and only if $\alpha + i\beta$ is an eigenvalue of T_c. Using the fact that T_c has an eigenvalue one concludes that T has a true-pair.

In this paper, our intention is to prove the existence of a true-pair of a real linear operator without using complex numbers. As usual, we write

$$\begin{aligned} N(T) &= \{x \in V : T(x) = 0\}, \text{ the null space of } T; \\ R(T) &= \{T(x) : x \in V\}, \text{ the range space of } T. \end{aligned}$$

We say that two linear operators S and T on V have a *common true-pair vector* if there exists a nonzero vector $v \in V$ and real numbers $\alpha, \beta, \gamma, \delta$ such that

$$[(S - \alpha I)^2 + \beta^2 I](v) = 0 \quad \text{and} \quad [(T - \gamma I)^2 + \delta^2 I](v) = 0.$$

In such a case, v is said to be a *common true-pair vector* of S and T.

Further, two linear operators S and T on V are said to be *commuting operators* if $S(T(x)) = T(S(x))$ for each $x \in V$.

Lemma 1. *Let S and T be two commuting operators on a finite dimensional nonzero real vector space V. Let (α, β) be a true-pair of S. Then the restrictions of S and T to $N\big((S - \alpha I)^2 + \beta^2 I\big)$ are commuting operators, and the restrictions of S and T to $R\big((S - \alpha I)^2 + \beta^2 I\big)$ are commuting operators.*

Proof. Since $(S-\alpha I)^2+\beta^2 I$ is a linear operator on V, write

$$\mathcal{N} = N\big((S-\alpha I)^2+\beta^2 I\big), \quad \mathcal{R} = R\big((S-\alpha I)^2+\beta^2 I\big).$$

These are subspaces of V. Let $x \in \mathcal{N}$. Then $[(S-\alpha I)^2+\beta^2 I](x) = 0$. Now,

$$\begin{aligned}
[(S-\alpha I)^2+\beta^2 I]S(x) &= S[(S-\alpha I)^2+\beta^2 I](x) = S(0) = 0,\\
[(S-\alpha I)^2+\beta^2 I]T(x) &= T[(S-\alpha I)^2+\beta^2 I](x) = T(0) = 0.
\end{aligned}$$

That is, $S(x) \in \mathcal{N}$ and $T(x) \in \mathcal{N}$. Hence, $\mathcal{N}$ is invariant under both S and T.

Next, let $y \in \mathcal{R}$. There exists $x \in V$ such that $y = [(S-\alpha I)^2+\beta^2 I](x)$. Then,

$$\begin{aligned}
S(y) &= S[(S-\alpha I)^2+\beta^2 I](x) = [(S-\alpha I)^2+\beta^2 I]S(x) \in \mathcal{R},\\
T(y) &= T[(S-\alpha I)^2+\beta^2 I](x) = [(S-\alpha I)^2+\beta^2 I]T(x) \in \mathcal{R}.
\end{aligned}$$

That is, $\mathcal{R}$ is invariant under both S and T.

Now that the subspaces $\mathcal{N}$ and $\mathcal{R}$ of V are invariant under both S and T, the conclusions follow. □

In [2], it has been shown that a finite number of commuting operators on a finite dimensional nonzero complex vector space have a common eigenvector. We apply a similar technique for obtaining information about true-pair vectors without using complex numbers.

Lemma 2. *Let $k \in \mathbb{N}$. Assume that each linear operator on a finite dimensional nonzero real vector space whose dimension is not divisible by k has a true-pair vector. Then, any two commuting operators on a finite dimensional nonzero real vector space have a common true-pair vector.*

Proof. We use induction on the dimension of V. Suppose $\dim(V) = 1$. Then, $k = 1$. Let $\{v\}$ be a basis for V. Suppose S and T are linear operators on V. Then, $S(v) \in V$ implies that there exists $\alpha \in \mathbb{R}$ such that $S(v) = \alpha v$. Similarly, $T(v) = \gamma v$ for some $\gamma \in \mathbb{R}$. Now,

$$[(S-\alpha I)^2 + 0 \cdot I](v) = 0 = [(T-\gamma I)^2 + 0 \cdot I](v).$$

So, v is a common true-pair vector of S and T.

Assume the induction hypothesis that any two linear operators on a nonzero vector space of dimension $m < n$, where k does not divide m, have a true-pair vector. Let V be a nonzero real vector space of dimension n, where k does not divide n. Assume that each linear operator on V has a true-pair vector. Let S and T be two commuting operators on V. We need to show that S and T have a common true-pair vector.

Now that S and T have true-pair vectors, there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and nonzero vectors $u, v \in V$ such that

$$[(S - \alpha I)^2 + \beta^2 I](u) = 0, \quad [(T - \gamma I)^2 + \delta^2 I](v) = 0.$$

Write $\mathcal{N} = N\big((S-\alpha I)^2+\beta^2 I\big)$. Since u is a nonzero vector in $\mathcal{N}$, $\dim \mathcal{N} \geq 1$. Further, for each $x \in \mathcal{N}$, $[(S - \alpha I)^2 + \beta^2 I](x) = 0$.

If $\mathcal{N} = V$, then $v \in \mathcal{N}$ so that $[(S - \alpha I)^2 + \beta^2 I](v) = 0$. Now, S and T have a common true-pair vector, namely, v.

So, let $\mathcal{N}$ be a proper subspace of V; that is, $\dim(\mathcal{N}) < n$. By Lemma 1, the restrictions of S and T to $\mathcal{N}$ are commuting operators.

If k does not divide $\dim(\mathcal{N})$, then by the induction hypothesis, the restriction operators of S and T to $\mathcal{N}$ have a true-pair vector $y \in \mathcal{N}$. Then, y is a common true-pair vector of S and T as linear operators on V.

So, assume that k divides $\dim(\mathcal{N})$. Write $\mathcal{R} = R\big((S - \alpha I)^2 + \beta^2 I\big)$. By Lemma 1, the restrictions of S and T to $\mathcal{R}$ are commuting operators. Due to the Rank-nullity theorem, $\dim(\mathcal{N}) + \dim(\mathcal{R}) = n$. As k divides $\dim(\mathcal{N})$ and k does not divide n, it follows that k does not divide $\dim(\mathcal{R})$. Since $\dim(\mathcal{R}) < n$, by the induction hypothesis, the restrictions of S and T to $\mathcal{R}$ have a common true-pair vector y. Then, y is a common true-pair vector of S and T as linear operators on V. □

We slightly enlarge our vocabulary. Let $m \in \mathbb{N} \cup \{0\}$. A natural number n is called an *m-even* number if and only if 2^m divides n but 2^{m+1} does not divide n. By an *m-even dimensional* real vector space we mean a nonzero real vector space of dimension n, where n is an m-even number. Notice that a 0-even number is simply an odd number. Thus a 0-even dimensional real vector space is simply an odd dimensional real vector space.

Lemma 3. *Let $m \in \mathbb{N}\cup\{0\}$. If each linear operator on an ℓ-even dimensional real vector space has a true-pair vector for $\ell = 0, 1, \ldots, m$, then each linear operator on an $(m + 1)$-even dimensional real vector space has a true-pair vector.*

Proof. Assume that each linear operator on an ℓ-dimensional real vector space has a true-pair vector for $\ell = 0, 1, \ldots, m$. Let $T : V \to V$ be a linear operator, where V is a real vector space of dimension n with n being an $(m+1)$-even number. Fix an ordered basis for V. Let A be the matrix representation of T with respect to this ordered basis. Then, A is an $n \times n$ matrix with real entries.

Now, $A = PTP^{-1}$, where P is the canonical basis isomorphism from V to $\mathbb{R}^{n\times 1}$. Then, for any $\alpha, \beta \in \mathbb{R}$,

$$P[(T-\alpha I)^2 + \beta^2 I]P^{-1} = P(T-\alpha I)P^{-1}P(T-\alpha I)P^{-1} + \beta^2 I = (A-\alpha I)^2 + \beta^2 I.$$

Thus, if u is a true-pair vector of A, then there exists a pair of real numbers (α, β) such that $\big((A-\alpha I)^2 + \beta^2 I\big)u = 0$. It gives $P[(T-\alpha I)^2 + \beta^2 I]\big(P^{-1}(u)\big) = 0$, which implies

$$[(T-\alpha I)^2 + \beta^2 I]\big(P^{-1}(u)\big) = 0.$$

That is, $P^{-1}(u)$ is a true-pair vector of T. Hence, it is enough to prove that A has a true-pair vector.

Consider $S_n = \{X \in \mathbb{R}^{n\times n} : X^t = X\}$, the set of all real symmetric matrices of order n. Then, S_n is a real vector space of dimension $n(n+1)/2$. Define

$$L_1(X) = AX + XA^t, \quad L_2(X) = AXA^t \quad \text{for } X \in S_n.$$

For $X, Y \in S_n$ and $b \in \mathbb{R}$, we have

$$\begin{aligned}
\big(L_1(X)\big)^t &= (AX + XA^t)^t = X^tA^t + AX^t = XA^t + AX = L_1(X),\\
\big(L_2(X)\big)^t &= (AXA^t)^t = AX^tA^t = AXA^t = L_2(X),\\
L_1(bX+Y) &= A(bX+Y) + (bX+Y)A^t = b(AX+XA^t) + (AY+YA^t)\\
&= bL_1(X) + L_1(Y),\\
L_2(bX+Y) &= A(bX+Y)A^t = bAXA^t + AYA^t = bL_2(X) + L_2(Y),\\
L_1(L_2(X)) &= L_1(AXA^t) = A(AXA^t) + (AXA^t)A^t = A^2XA^t + AX(A^t)^2,\\
L_2(L_1(X)) &= L_2(AX+XA^t) = A(AX+XA^t)A^t = A^2XA^t + AX(A^t)^2.
\end{aligned}$$

So, L_1 and L_2 are commuting operators on S_n.

Notice that $n = 2^{m+1}k$, where k an odd integer and $m \geq 0$. Thus, $n+1$ is odd so that $n(n+1)/2 = 2^m(n+1)k$, where $(n+1)k$ is odd. Thus, $\dim(S_n) = n(n+1)/2$ is an m-even number. Thus, L_1 and L_2 are commuting operators on a finite dimensional real vector space of dimension not divisible by 2^{m+1}. Our assumption that each linear operator on an ℓ-dimensional real vector space has a true-pair vector for $\ell = 0, 1, \ldots, k$ implies that each linear operator on a real vector space of dimension not divisible by 2^{m+1} has a true-pair vector. Due to Lemma 2, L_1

and L_2 have a common true-pair vector. So, let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and let $B \in S_n$ be a nonzero matrix such that

$$[(L_1 - \alpha I)^2 + \beta^2 I](B) = 0, \quad [(L_2 - \gamma I)^2 + \delta^2 I](B) = 0.$$

Then, $-\beta^2\delta^2 B = \delta^2(L_1 - \alpha I)^2(B) = \beta^2(L_2 - \gamma I)^2(B)$. It gives

$$[\delta(L_1 - \alpha I) + \beta(L_2 - \gamma I)][\delta(L_1 - \alpha I) - \beta(L_2 - \gamma I)](B) = 0. \tag{3}$$

Case 1: Suppose $[\delta(L_1 - \alpha I) - \beta(L_2 - \gamma I)](B) = 0$. Then,

$$\delta(AB + BA^t - \alpha B) - \beta(ABA^t - \gamma B) = 0.$$

It implies $(\delta I - \beta A)BA^t = \big(-\delta A + (\alpha\delta - \beta\gamma)I\big)B$. Using this, we obtain the following:

$$\begin{aligned}
(\delta I - \beta A)L_1(B) &= (\delta I - \beta A)AB + (\delta I - \beta A)BA^t \\
&= (\delta I - \beta A)AB + [-\delta AB + (\alpha\delta - \beta\gamma)B] \\
&= -\beta A^2 B + (\alpha\delta - \beta\gamma)B \\
&= \big(-\beta A^2 + (\alpha\delta - \beta\gamma)I\big)B,
\end{aligned}$$

$$\begin{aligned}
(\delta I - \beta A)^2 BA^t A^t &= (\delta I - \beta A)\big(-\delta A + (\alpha\delta - \beta\gamma)I\big)BA^t \\
&= \big(-\delta A + (\alpha\delta - \beta\gamma)I\big)(\delta I - \beta A)BA^t \\
&= \big(-\delta A + (\alpha\delta - \beta\gamma)I\big)^2 B,
\end{aligned}$$

$$\begin{aligned}
(\delta I - \beta A)^2 L_1^2(B) &= (\delta I - \beta A)^2 L_1\big(AB + BA^t\big) \\
&= (\delta I - \beta A)^2\big(A(AB + BA^t) + (AB + BA^t)A^t\big) \\
&= (\delta I - \beta A)^2(A^2B + 2ABA^t + BA^tA^t) \\
&= (\delta I - \beta A)^2 A^2 B + 2(\delta I - \beta A)^2 ABA^t + (\delta I - \beta A)^2 BA^tA^t \\
&= (\delta I - \beta A)^2 A^2 B + 2(\delta I - \beta A)A(\delta I - \beta A)BA^t \\
&\quad +(\delta I - \beta A)^2 BA^tA^t \\
&= (\delta I - \beta A)^2 A^2 B + 2(\delta I - \beta A)A\big(-\delta A + (\alpha\delta - \beta\gamma)\big)B \\
&\quad +\big(-\delta A + (\alpha\delta - \beta\gamma)I\big)^2 B \\
&= \big((\delta I - \beta A)A + (-\delta A + (\alpha\delta - \beta\gamma)I\big)^2 B \\
&= \big(-\beta A^2 + (\alpha\delta - \beta\gamma)I\big)^2 B.
\end{aligned}$$

Then,

$$\begin{aligned}
0 &= (\delta I - \beta A)^2\big((L_1 - \alpha I)^2 + \beta^2 I\big)(B) \\
&= (\delta I - \beta A)^2 L_1^2(B) - 2\alpha(\delta I - \beta A)^2 L_1(B) + (\delta I - \beta A)^2(\alpha^2 + \beta^2)B \\
&= \big(-\beta A^2 + (\alpha\delta - \beta\gamma)I\big)^2 B - 2\alpha(\delta I - \beta A)\big(-\beta A^2 + (\alpha\delta - \beta\gamma)I\big)B \\
&\quad +(\delta I - \beta A)^2(\alpha^2 + \beta^2)B \\
&= \big((-A^2 + \alpha A - \gamma I)^2 + (\delta I - \beta A)^2\big)B.
\end{aligned}$$

Due to Fact 2, there exist $a, b, c, d \in \mathbb{R}$ such that

$$(-t^2 + \alpha t - \gamma)^2 + (\delta - \beta t)^2 = (t^2 - at + b)(t^2 - ct + d).$$

Therefore,

$$(A^2 - aA + bI)(A^2 - cA + dI)B = 0. \tag{4}$$

Case 1A: Suppose $(A^2 - cA + dI)B = 0$. As $B \neq 0$, let x be a nonzero column of B. Then, $(A^2 - cA + dI)x = 0$.

If $c^2 - 4d \geq 0$, then write $\gamma = \big(c + \sqrt{c^2 - 4d}\,\big)/2$ and $\delta = \big(c - \sqrt{c^2 - 4d}\,\big)/2$. Now, $\gamma, \delta \in \mathbb{R}$ and

$$(A - \gamma I)(A - \delta I)x = (A^2 - cA + dI)x = 0.$$

If $(A - \delta I)x = 0$, then

$$[(A - \delta I)^2 + 0 \cdot I]x = (A - \delta I)[(A - \gamma I)(A - \delta I)x] = 0$$

shows that x is a true-pair vector of A.

If $(A - \delta I)x \neq 0$, then $[(A - \gamma I)^2 + 0 \cdot I](A - \delta I)x = 0$ shows that $(A - \delta I)x$ is a true-pair vector of A.

If $c^2 - 4d < 0$, then write $\gamma = c/2$ and $\delta = \big(\sqrt{4d - c^2}\,\big)/2$. Now, $\gamma, \delta \in \mathbb{R}$ and

$$[(A - \gamma I)^2 + \delta^2 I]x = (A^2 - cA + dI)x = 0.$$

Thus, x is a true-pair vector of A.

Case 1B: Write $C := (A^2 - cA + dI)B \neq 0$. Equation 4 gives

$$(A^2 - aA + bI)C = 0.$$

This case reduces to Case 1A with a, b, C in place of c, d, B, respectively.

Case 2: Suppose $D := [\delta(L_1 - \alpha I) - \beta(L_2 - \gamma I)](B) \neq 0$. Equation 3 yields

$$[\delta(L_1 - \alpha I) + \beta(L_2 - \gamma I)]D = 0.$$

This case is reduced to Case 1 with $-\beta$ in place of β and D in place of B. □

3 Main results

In this section we prove our main result and then derive the existence of the intended factorization of a polynomial wih real coefficients.

Theorem 1. *Every linear operator on a finite dimensional nonzero real vector space has a true-pair vector.*

Proof. We use induction on the evenness of the dimension of the underlying vector space of a linear operator. In the basis step, let T be a linear operator on a 0-even dimensional real vector space V. Then, $\dim(V)$ is odd so that the characteristic polynomial of T is of odd degree. By Fact 1, it has a real zero, say, λ. Then, λ is an eigenvalue of T with an associated eigenvector v. As remarked earlier, v is a true-pair vector of T.

Assume the induction hypothesis that each linear operator on an ℓ-even dimensional real vector space has a true-pair vector for $\ell = 0, 1, \ldots, m$. Let $T : V \to V$ be a linear operator, where V is an $(m+1)$-even dimensional real vector space. By Lemma 3, T has a true-pair vector. □

As a corollary to Theorem 1, we obtain the following result about real polynomials.

Theorem 2. *Every non-constant real polynomial in a single variable t has either a real linear factor in the form $t - \alpha$ or a real quadratic factor in the form $(t-\beta)^2 + \gamma^2$ for real numbers α, β and γ.*

Proof. Without loss in generality, consider monic polynomials with degree at least 3. So, let

$$p(t) = a_1 + a_2 t + a_3 t^2 + \cdots + a_n t^{n-1} + t_n,$$

where $n \in \mathbb{N}$, $n \geq 3$ and $a_1, a_2, \ldots, a_n \in \mathbb{R}$. In view of Fact 3, let A be a matrix of order n such that $\det(tI - A) = p(t)$. By Theorem 1, there exist $\beta, \gamma \in \mathbb{R}$ and a nonzero vector $v \in \mathbb{R}^{n\times 1}$ such that $[(A - \beta I)^2 + \gamma^2 I](v) = 0$. Let $r(t) = (t-\beta)^2 + \gamma^2$. Now, $r(A)(v) = 0$. By the Division algorithm,

$$\text{either } \; p(t) = q(t)r(t) \; \text{ or } \; p(t) = q(t)r(t) + (at - b)$$

for some real polynomial $q(t)$ of degree $n-2$ and some $a, b \in \mathbb{R}$. In the former case, we are through. In the latter case, $p(A)(v) = q(A)r(A)(v) + (aA - bI)(v)$. By Cayley-Hamilton theorem, $p(A) = 0$. So, $(aA - bI)(v) = 0$.

If $a = 0$, then $bv = 0$; and $v \neq 0$ implies $b = 0$. Hence, $p(t) = q(t)r(t)$.

If $a \neq 0$, then b/a is an eigenvalue of A. Thus, $p(b/a) = 0$. Then, $p(t) = (t - b/a)q_1(t)$ for some real polynomial $q_1(t)$ of degree $n - 1$. □

Observe that a quadratic factor $(t - \alpha)^2 + \beta^2$ of $p(t)$ includes two cases. If $\beta = 0$, then the real linear polynomial $t - \alpha$ divides $p(t)$. And, if $\beta \neq 0$, then the discriminant of $(t - \alpha)^2 + \beta^2$ is $4\alpha^2 - 4(\alpha^2 + \beta^2) = -4\beta^2 < 0$; so that a real quadratic polynomial with negative discriminant, namely, $(t - \alpha)^2 + \beta^2$ divides $p(t)$. Thus, applying Theorem 2 repeatedly, we obtain the required factorization proving the following statement.

Theorem 3. *Every real polynomial $p(t)$ of degree $n \geq 1$ can be factorized as*

$$p(t) = a \cdot L_1(t) \cdots L_m(t) \cdot Q_1(t) \cdots Q_k(t)$$

where $a \in \mathbb{R}$, $L_i(t)$ are linear real polynomials and $Q_j(t)$ are quadratic real polynomials with negative discriminant.

Notice that the factorization in Theorem 3 is unique since $\mathbb{R}[t]$ is a unique factorization domain. Further, in Theorem 3, $n = m + 2k$ for $n, m, k \in \mathbb{N} \cup \{0\}$.

4 Conclusions

As shown earlier, use of complex numbers leads to simple proofs of Theorem 1. However, our intention here is to construct a proof that does not use complex numbers. We have constructed such a proof. During this construction we have introduced the notions of a true-pair vector and the evenness of a natural number. Though these definitions are not standard, they helped us to present the proof in a comprehensible manner. It demonstrates that both Theorem 1 and Theorem 3 could be proved without using complex numbers, solving the Open Problem 2 mentioned in [5]. In this connection, we remark that perhaps the author in [5] is asking for a proof of the factorization of real polynomials which uses the techniques of real analysis only. In that sense, the problem is still open.

Acknowledgements: The author thanks cheerfully the unknown referees and S. H. Kulkarni of IIT Pallakad for their constructive suggestions which greatly improved the presentation.

References

[1] S. Bernard, J. M. Child, *Higher Algebra*, Macmillan & Co Ltd, London, 1967.

[2] H. Derksen, *The fundamental theorem of algebra and linear algebra*, Amer. Math. Monthly 110 (2003), pp. 620–623.

[3] S. H. Kulkarni, B. V. Limaye, *Real Function Algebras*, Marcel Dekker, New York, 1992.

[4] M. T. Nair, A. Singh, *Linear Algebra*, Springer, 2018.

[5] V. K. Srinivasan, *Open problems in analysis*, Int. J. Math. Educ. Sci. Technol. 28 (1997) pp. 117–121.